TA manual Motion Analysis

Joris Busink

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Contents

1 Introduction

This manual is designed to assist Teacher Assistants (TA's) in effectively teaching the Motion Analysis (MA) experiment to second-year bachelor students in the MNW2 practical and NSP2 practical series. This manual aims to provide detailed guidance on essential aspects, including proficient handling of scientific equipment, implementing an optimal data acquisition strategy and we present several example experiments that the students can study within the given time constraint. The first part of the manual was originally written by Munib Ozdemir as part of an education project. It was translated and adapted in this manual. Finally, we provide a preliminary daily schedule. We value your input and suggestions to enhance this manual further.

2   Hardware & software

In this section, we will provide an overview of the scientific equipment utilized in the MA experiment, along with technical details and helpful tips and tricks for both students and TA's. For a more comprehensive understanding of the hardware, we direct readers to the student manual. While these camera fundamentals may appear elementary to some, it is worth noting that students generally have limited prior experience with cameras and photography.

2.1   Hardware: Camera and zoom lens

During the MA experiment, students will use the Lumenera Lt225C high-speed camera to capture a dynamical process. The camera features a 2/3 inch CMOS sensor with a resolution of 2048x1088 pixels and has a fully Electronic Global Shutter, it will capture an image by turning all the pixels on and off electronically. The camera is capable of recording at approximately 700 frames per second (fps), but this depends on the resolution and color depth of the image. The image is displayed on the camera using an already affixed zoom lens. The zoom lens consists of three components (from left to right): 1) a focus ring, 2) a zoom ring and 3) an aperture ring. The function of each of these elements is the following:

- Aperture (ring): the aperture refers to the adjustable opening in the camera lens that controls the amount of light entering the camera. It is often represented by the f-number, such as f/2.8, f/4, f/5.6, and so on. The f-number is the ratio of focal length f to the effective aperture diameter D, with symbol N. $N = f/D$. The numbers (**ranging from 1.4 to 22**) on the aperature ring provide the f-number (**symbol N, excersise 1**). The aperture plays a vital role in the MA experiment as it influences two main aspects: the exposure and the depth of field. Regarding depth of field, the aperture size affects the range of distances that appear in sharp focus within an image. A wider aperture (lower f-number) creates a shallow depth of field, i.e. the background appears blurred. On the other hand, a narrower aperture (higher f-number) increases the depth of field.

- Zoom ring: a zoom ring is a control ring on a camera or lens that allows you to adjust the focal length (**symbol f, excersise 1**), enabling you to zoom in or out and control the magnification and field of view of the captured image. The focal length of the camera is second list of engraved marking, **ranging from 11.5 to 69 mm**.

- focus (ring): the focus ring in a camera is a manual control mechanism located on the camera lens. It allows the user to manually adjust the focal distance of the lens. By rotating the focus ring, the lens elements move back and forth, altering the focal length. This allows the user to bring the object into sharp focus. A sharp focus allows to study finer details of the motion of the dynamical process.

Besides the terminology of the camera and zoom-lens, students will also encounter some terminology regarding the data-acquisition. This includes concepts such as frame rate, depth of field, color depth, and more, may be unfamiliar to them. Therefore, it is the TA's responsibility to acquaint the students with these terms and provide practical guidance on their utilization throughout the experiment.

The list below shows common terminology used in the MA experiment. It is important to, not only, understand the terminology, but also to use it correctly in the experiments. A

good understanding of the terminology and concepts saves a tremendous amount of time in the processing of the data!

- Plane of focus: the plane of focus is the distance in the scene where objects appear sharpest and in focus when captured by the camera.

- Depth of Field (Dof): the distance between the nearest and the farthest objects that are in acceptably sharp focus in an image captured with a camera. The precise definition of "sharp focus" is subjective, as it relies on individual perception. Nonetheless, an empirical understanding of DoF can be gained through an experiment using a paper with a basic text message. By manipulating the camera's focus to capture the text and observing the resultant image, one can establish the boundaries of the DoF. By systematically adjusting the position of the paper, an intuitive sense of optimal focus can be developed. The DoF is quantified by the extent of the range between the closest and farthest objects that results in a sharp focus.

- Circle of confusion (CoC): the circle of confusion, denoted by c, is the maximum allowable blur circle size that still appears as a sharp point in an image. It depends on the sensor size, lens characteristics, aperture setting, and viewing conditions. In the experiment we will use a static value of c = 0.008 10^{-3}m. Students may ask questions about the CoC, you can refer them to a quick google (image) search. It is a rather technical term that is difficult to explain and we do not use it any further in the experiments, apart from the introduction experiment (**excersise 1**).

- Frames per second (fps): Frames per second (fps) refers to the number of frames or images the camera can capture per second, equivalent to the framerate. While the term may seem self-explanatory, it is crucial to optimize the fps setting during the experiment. Although the Lumenera Lt225C camera can reach a maximum of approximately 700 fps (and even higher if desired), it is important to consider the limitations of the hardware's data acquisition rate. When working with high-resolution settings, the hardware's bandwidth becomes the critical limiting factor. Futher note that a high fps results in a low amount of light collected on the CMOS chip. The image can becomes very dark and the lighting needs modification!

  To ensure efficient data collection, it is advisable to align the fps with the timescale of the experiment. For example, let's consider a mathematical pendulum with a length of 1 meter. Its period, denoted as T, can be calculated using the formula $T = 2\pi\sqrt{1/9.81} \approx 2$ seconds. If our aim is to capture 50 frames per swing, it is sufficient to limit ourselves to 25 fps. By doing so, we significantly reduce the amount of time required for data analysis, compared to capturing at the maximum 700 fps.

  By carefully selecting an appropriate fps setting that balances data acquisition needs with practical considerations, we can optimize the efficiency of the experiment and streamline subsequent data analysis efforts.

- Pixel: picture element that is the smallest addressable element in a raster image. A pixel can be on or off (1 bit per pixel, 1 bpp) or have multiple colors e.g. 8bpp = 256 colours.

- Resolution: the resolution refers to the level of detail and clarity in an image or display. It is commonly expressed in terms of the number of pixels (picture elements)

that make up the image, typically presented as width x height (e.g., 1920 x 1080 pixels). A higher resolution corresponds to a greater number of pixels, resulting in finer details and enhanced visual fidelity. However, it is essential to consider whether these additional details are necessary for the specific analysis at hand. In cases where fine-grained details are not crucial, opting for a lower resolution can be advantageous, as it reduces the volume of data to be processed. This can significantly streamline the data-processing workflow. The maximum resolution of the camera is 2048x1088 = 2228224 bits (or 0.278528 MB). This specification provides an estimation of the amount of data required to store each frame at maximum resolution, note that this is not the equivalent to the data stored on e.g. a harddrive.

- Colour Depth: the Chrominance/Luminance information per pixel. For tracking we use typically 8bpp grayscale, so we have $2^8 = 256$ gray tones.

- Bitrate: the number of bits that are processed per unit of time, e.g. 1 kbit/s = 1000 bits per second. The bitrate of the camera is the product of the resolution, the framerate and the colordepth. For a given resolution of 2048x1088, at a framerate of 50 hz and a colordepth of 8bpp we have 111,669,760 bits per second! That is 111.67 MB per second. A typical USB2 port has a bandwidth of around 50 MB per second. So we cannot use these in this scenario.

- Bandwidth: maximum rate of data transfer across a given path. The cameras need to be connected to an USB3.0 port (or higher).

By controlling the aperture, zoom ring and/or the focus ring, and by optimizing the frame rate, resolution, etc. the students can optimize the raw footage of the dynamical process. It can take several attempts to create an optimized image-sequence, it is part of the experiment and should not be skipped at any point. An un-optimized image-sequence is very hard to analyze. It is therefore essential that the raw footage of the dynamical process is optimized. The students may use different lamps, supports, etc. to optimize their footage, be creative!

2.2   Tips and tricks

- Use LED lights, black/white cardboard, theater lights to optimize contrast.

- Use clamps to create a stable configuration. It is possible to rotate the camera by 90 (or any amount) degrees to create a canvas of 1088x2048 (instead of 2048x1088). You can correct the data during the data-processing.

- Choose a framerate that is comparable to the timescale of the dynamical process.

- Use a resolution that captures the *essential* details of the experiment. It is often more efficient to use a smaller resolution, since the data processing is much faster. In almost all cases, the limiting factor of an experiment is not the resolution.

- Before you make several movies, do a single analysis from start to finish. This avoids wasting time analyzing un-optimized footage.

2.3   Software

The software that is used during the experiment is the following:

- **LuCam Capture**. LuCam Capture is a very basic software to test the camera. LuCam Capture can only capture images, hence it cannot be used to make a movie. However, LuCam Capture is easy to use, and you can check the camera options and play around with them.

- **TroublePix**. The video capturing is done with TroublePix. You can use the same camera options as LuCam Capture. You can change the camera settings in *view*. Using TroublePix is straightforward and the settings can be found in the student manual. To install the software on a new computer, please contact Gerrit Kuik.

- **MaxTRAQ**. Lastly, we use MaxTRAQ software. MaxTRAQ is the motion-analysis software that we will use. MaxTRAQ is able to track the position of points, the angle between the points, distances, etc. MaxTRAQ tracks objects by fitting an ellipsoid to the object of interest, the centre of the ellipsoid is the mean-position, thus the coordinates of objects. Furthermore, MaxTRAQ estimates, based on several frames, the velocity and the direction of the object, this enables us to track multiple objects without assignment problems (a very typical problem in multiple object tracking is that the software cannot distinguish the individual objects, this results in a messy data file.). To enable tracking the object of interest needs to have a high contrast with respect to the background. Furthermore, the object needs to have a well-defined shape (e.g. circle, square, etc.). The shape of the object may vary between frames but it should not vanish or break-up between the frames. To optimize the shape of the object you can use LED lights, clamps, theater lights and markers.

  At the start of the tracking analysis you have to select the objects to track. To track multiple object or when tracking angles, you have to select the appropriate settings and select each of the objects. For small objects, it can be useful to adjust the tracking region and sensitivity in the settings page. Before you track a whole video, try if you can see if the object is fully visible in the time-region of interest. If the object vanishes a lot it is most likely not worth investing the time to manually track every frame; you should make a better video. If the object is fully visible and does not break up, you can start the tracking software. The exact details to track the object are provided in the student manual.

  If the tracking fails, you can manually continue the tracking by re-selecting the (missed) object. If you have a very difficult object to track you can do the following things: 1) make a better video, 2) scroll with you mouse and select the missed objects during tracking (scrolling is much faster than checking the frames frame to frame.). Every time the tracking fails MaxTRAQ will pause the program. In the bottom graphs you can see the results of the object tracking (s-t, v-t or a-t).

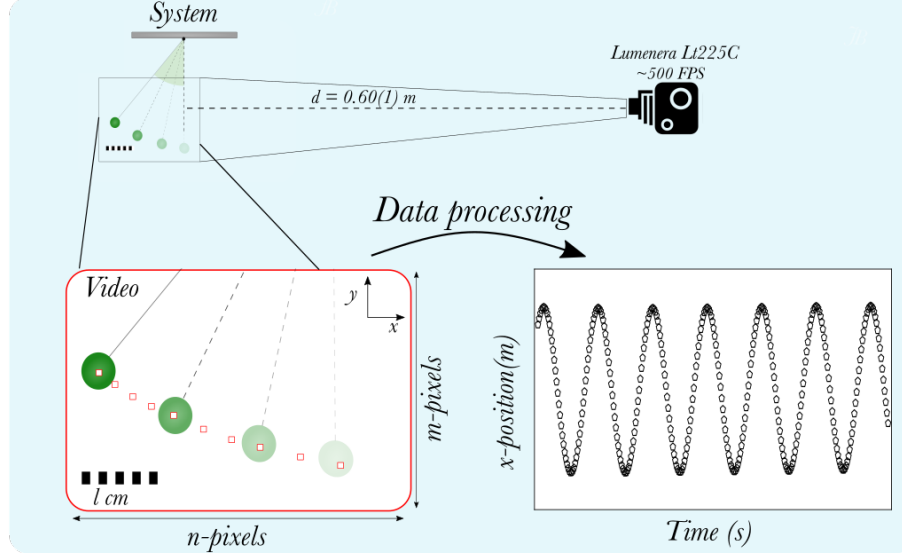# 3   Introduction experiment: Simple Pendulum

Figure 1: The setup of the pendulum experiment. The LumeneraLi255C captures the motion of the simple pendulum. The x-position of the mass of the pendulum is tracked using MaxTRAQ, the resulting oscillatory movement is shown in the right bottom of the graph.

3.0.1   Theory

The simple (mathematical) pendulum consists of a mass m attached to a massless cord of length l. The mass is constrained to move in the azimuthal direction θ and cannot stretch $\dot{l} = 0$. When the mass is pulled out of equilibrium at an angle θ it wil make an arc of length s, the net force it will experience is a gravitation force $F_g = -mg\sin\theta$. Hence, Newton's equation for the movement along the arc s is given by:

$$m\ddot{s} = -mg\sin(\theta). \tag{1}$$

The relationship between the arc length s and the angle θ is $s = l\theta$. The acceleration in the displacement direction s is therefore

$$ml\ddot{\theta} = -mg\sin\theta \tag{2}$$

$$\ddot{\theta} = -\omega_0^2 \sin(\theta). \tag{3}$$

Here we defined the angular (eigen)frequency $\omega_0^2 = g/l$. Eq. 3 has no analytic solution for arbitrary angle θ (the solutions are of the form of elliptic integrals). But, for a small angle $\theta \approx 0$, we can Taylor expand the sinusoidal equation $\sin(\theta) \approx \theta$ and we arrive to

$$\ddot{\theta} + \omega_0^2 \theta = 0. \tag{4}$$

The solution of Eq. 4 is an oscillation, hence

$$\theta(t) = \theta_0 \cos(\omega_0 t + \phi_0). \tag{5}$$

θ_0 is the amplitude of the oscillation and ϕ_0 is an arbitrary phase. If we choose the x-axis along the displacement axis, with $x = l\theta$ we arrive to:

$$x(t) = x_0 \cos(\omega_0 t + \phi_0). \tag{6}$$

The period T is straightforward derived from the eigen-frequency

$$T = \frac{2\pi}{\omega}. \tag{7}$$

3.0.2 Setup and results

The mathematical pendulum experiment is offered in the introduction of the Motion Analysis experiment. The students should be able to build the experiment from scratch using the simple tools that are available in the practical room. The setup consist of a cord with length l with a mass m at the end. The linear frequency $\omega_0 = \sqrt{g/l}$, can be altered by varying the length of the cord. In Fig. 1 the setup is shown. The setup does not offer a lot of freedom in adjusting the parameters, but the analysis can be extended. To track the motion of mass of the pendulum is relatively straightforward. By connecting a simple LED light to the cord it is possible to track the x-position and y-position.
Although this experiment is offered as an introduction experiment, it is possible to investigate different pendulum experiments as a main experiment. A few example experiments can be done using this setup:

3.0.3 Gravitational acceleration (easy)

Calculate the gravitational acceleration g using the pendulum model.

$$\omega_0 = \sqrt{\frac{g}{l}} \tag{8}$$

Students can calculate the gravitational acceleration by fitting the oscillator model to the x-direction of the pendulum. A possible extension is to use different lengths of the cord l and linearize the data. A more robust approach to calculate the gravitational acceleration is to include a third order term in the Taylor expansion (or even more) $\sin(\theta) \approx \theta - \frac{1}{6}\theta^3 + \ldots$ and solve the equation. The resulting period depends on the angle of release an can be derived using a Legendre Polynomial solution in θ_0

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \ldots\right). \tag{9}$$

3.0.4 Compound pendulum (hard)

The mathematical pendulum equation assumes a point mass m and a massless cord of length l. This is not very realistic, a physical pendulum however, corrects for the finite moment of

inertia of the cord and mass. The physical model uses the torque $\tau = -mgL\sin(\theta)$ of the mass and calculates the resulting angular acceleration $\tau = I\alpha \approx I\ddot{\theta}$. L is the distance from the pivot point to the centre of mass and I is the moment of inertia around the pivot point. The resulting period T, again assuming $\sin(\theta) \approx \theta$, is therefore

$$T = 2\pi\sqrt{\frac{I}{mgL}}. \tag{10}$$

Students can calculate the moment of inertia for simple object like rulers and a string of point masses, etc. It is in general difficult to calculate, but it can be very useful to discuss the differences found in the experiment versus the theoretical prediction. This extension can also be used to understand the double pendulum details.

3.0.5 Phase-space (hard)

Show that the phase-space plot x-v (or p = mv) consist of inward spiraling circles. The staring point of this project is to investigate the dynamical system:

$$x(t) = Ae^{-\zeta\omega_0 t}\cos(\omega_1 t) \tag{11}$$

$$v(t) = -A\zeta\omega_0 e^{\zeta\omega_0 t} - A\omega_1 e^{\zeta\omega_0 t}\sin(\omega_1 t). \tag{12}$$

ω_1 is the new bare frequency, corrected for the dissipation, but in this project the dissipation is very small and the frequency of the pendulum approximately the eigenfrequency $\omega_1 = \omega_0\sqrt{1-\zeta^2} \approx \omega_1$. If we plot the position against the velocity the phase-space plot shows a circle. What is the interpretation of the circle (Energy). Why is the circle inward spiraling and can we show the theoretical curve on top of the phase-space picture? To do this, the students have to calculate the frequency ω_0 and the damping coefficient ζ. Then, you can make a theoretical prediction of x(t) and v(t) and compare these with the experimental results. This project is complete analogous to the damped harmonic oscillator experiment.

4 Example Experiments

4.1 Driven (Damped) Harmonic Motion

4.1.1 Theory Driven Damped Harmonic Motion

In this experiment we have a spring with stiffness k and natural frequency ω_0, connected to a periodic (sinusoidal) driving force of amplitude $A(t) = F_0 \cos(\omega_d t)$ and frequency ω_d. Furthermore, the spring is connected to an (adjustable) mass m and the end of the spring is inside a tube that contains a viscous liquid. The (periodic) motion inside the viscous liquid results in a damping (counter) force $F_y = -cv$, with c a viscous damping coefficient and v the velocity of the tube inside the liquid. The strenght of the viscious damping can be controlled by hand. The Equation of Motion (EoM) in the y-direction (denoted by x) is:

$$m\ddot{x} = -c\dot{x} - kx + A(t) \tag{13}$$

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega_r t, \tag{14}$$

here we have introduced the damping ratio $\zeta = \frac{c}{2\sqrt{mk}}$. The solution of Eq. 14 has the form:

$$x(t) = \frac{F_0}{mZ(\omega_0, \omega_r, \zeta)\omega_r}\cos(\omega_r t + \phi), \tag{15}$$

with $Z(\omega_0, \omega_r, \zeta)$ is the linear response function:

$$Z(\omega_0, \omega_r, \zeta) = \sqrt{(2\omega_0\zeta)^2 + (\omega_0 - \omega_r)^2}. \tag{16}$$

The linear response function has a maximum on resonance $\omega_0 = \omega_r$, with amplitude $Z = 2\omega_0\zeta$. In the absence of damping $\zeta = 0$, the response function is zero, hence the amplitude is infinite. For a finite damping $\zeta > 0$ the amplitude of the oscillation is finite. Furthermore, the phase of the response oscillation $x(t)$ has the form:

$$\phi = \tan^{-1}\left(\frac{\omega_0\omega_r\zeta}{\omega_0^2 - \omega_r^2}\right) + n\pi. \tag{17}$$

On resonance the phase between the driving force and the natural frequency ϕ is $\pi/2$ out of phase. Hence, the sinusoidal force A is exactly out of phase with the natural frequency of the system, this, ofcourse, results in maximum energy transfer. This can be verfied in the experiment.

4.1.2 Theory Damped Harmonic Motion

In the absence of a driving force $A(t) = 0$, the EoM of the system is a damped harmonic oscillator:

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0. \tag{18}$$

The solution Eq. 18 can be obtained by using a trial solution $x(t) = Ae^{\lambda t}$ and solving the characteristic equation for λ:

$$\lambda^2 + 2\zeta\omega_0\lambda + \omega_0^2 = 0 \tag{19}$$

The solutions of the characteristic equation are the normal modes of the system, hence $\lambda_\pm = -\zeta\omega_0 \pm i\omega_0\sqrt{1-\zeta^2}$. In the absence of damping $\zeta = 0$ the frequency of the system is the natural frequency ω_0. For small damping $\zeta > 0$ the oscillation is damped by $\tau = \zeta\omega_0$ and the frequency of the system is almost the natural frequency $\omega_1 \approx \omega_0$. If the oscillation is subjected to a (very) strong damping force $\lambda > 1$ the system is overdamped. The eigenmodes are purely real valued and the solutions are exponentially decaying. The timescale associated with the strong damping $\tau_{\lambda>0}$ is larger than the critically damped oscillator $\zeta = 1$, with $\tau_{\lambda=0} = -\omega_0$. Stronger damping does not imply faster damping.

Using the solutions of the characteristic equation the x-position and the velocity as a function of time is given by:

$$x(t) = Ae^{-\zeta\omega_0 t}(e^{i\omega_1 t}) + e^{-i\omega_1 t}, \tag{20}$$

$$x(t) = Ae^{-\zeta\omega_0 t}\cos(\omega_1 t) \tag{21}$$

$$v(t) = -A\zeta\omega_0 e^{\zeta\omega_0 t} - A\omega_1 e^{\zeta\omega_0 t}\sin(\omega_1 t). \tag{22}$$

Again, in the absence of damping $\zeta = 0$, the equations reduce to a more familiar form:

$$x(t) = A\cos(\omega_0 t) \tag{23}$$

$$v(t) = -A\omega_0 \sin(\omega_0 t). \tag{24}$$

The total energie of the system is given by $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$, in the absence of coupling, this is a conserved quantity.

$$\frac{d}{dt}E = \frac{d}{dt}(\frac{1}{2}mv^2 + \frac{1}{2}kx^2) \tag{25}$$

$$= \frac{d}{dt}(\frac{A^2 k}{2}) = 0 \tag{26}$$

Lastly, the above analysis can be extended to a 2D-parameter space by using the tools of dynamical system theorie. I define the vector $\vec{A} = \begin{pmatrix} x \\ v \end{pmatrix}$ and decouple the differential equation, hence:

$$\frac{d}{dt}\vec{A} = M\vec{A} \tag{27}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix}. \tag{28}$$

We solve this equation by solving for the eigenvalues $|A - \lambda I| = 0$, the solution has the form:

$$A(t) = A_0 e^{\lambda_\pm t} \tag{29}$$

$$A(t) = A_0 e^{-\zeta\omega_0 t} e^{\pm i\omega_1 t}, \tag{30}$$

with $\lambda_\pm$ the eigenvalues of the matrix M. Since the real part of eigenvalues of $\lambda_\pm$ are smaller than zero $s\mathcal{R}e(\lambda_\pm) < 0$ and the imaginary part is non-zero $\mathcal{I}m(\lambda_\pm) \neq 0$, the phase-space picture (x-v plane) will show inward spiraling $\zeta > 0$ circles. In the absence of a restoring force $c = 0$, the phase space plot is a perfect circle that shows that the position and the velocity of the harmonic oscillator are coupled.

An equivalent analysis can be done in a simple (damped) pendulum, in that case the interpretation is simple: the phase diagram shows the transfer of potential energy to kinetic energy and vice versa. A similar argument can be made here.

4.1.3 Setup and results

The driven damped harmonic oscillator is a pre-build setup. If students want to use this setup, the should contact the technical support team.

The setup consist of a spring with variable stiffness k connected with an external mass m. The coupled system of the spring and the mass have an eigenfrequency $\omega_0 = \sqrt{k/m}$, which can be varied by choosing a different mass or spring stiffness. The spring and mass are connected via a rod with a viscous damper. The damping force can be adjusted within the setup, the combined system can act as a damped oscillator. Furthermore, the damped oscillator can be connected to an electrical motor with adjustable frequency ω_r. The schematic setup is shown in Fig. 2a. The setup offers a lot of freedom in adjusting the parameters (driving/no-driving, low damping/high damping, etc.).

To track the y-motion of the oscillator is relatively straightforward. By connecting a simple LED light to the mass it is possible to track the y-position. If students are interested in the relative phase ϕ between the spring an the driving force A_y, you need multiple lED lights. Preferably at the mass and at the end of the rotor.

A few example experiments can be done using this setup:

4.1.4 Eigenfrequencies (easy)

Show that eigenfrequency depends on the mass:

$$\omega_0 = \sqrt{\frac{k}{m}}. \tag{31}$$

By varying the mass m (or the spring stiffness k), the eigenfrequency changes. The effect can be derived using a force balance. An increasing mass will results in a lower frequency. Possible extensions include: 1) linearization of the data, 2) include the mass of the spring, 3) effect of damping.

4.1.5 Resonance curve (hard)

Show a resonance curve (A - ω_0):

$$A(\omega_0) = \frac{F_0}{m\omega_r \sqrt{(2\omega_0\zeta)^2 + (\omega_0 - \omega_r)^2}}. \tag{32}$$

By varying the driving frequency ω_r the amplitude of the oscillation changes. The amplitude A is the amplitude in the y-position and the frequency can be extracted from the oscillation frequency in the y-direction. The frequency at the maximum amplitude corresponds to the

eigenfrequency of the system ω_0. The width of the resonance is a measure of the damping and can be extracted from the fit. A possible (very hard) extension to this experiment, is to calculate the relative phase between the driving force and the resulting force ϕ, on resonance the relative phase is 90 degrees out of phase.

4.1.6   Damping (both easy and hard)

Calculate the damping coefficient c or ζ of the combined system:

$$x(t) = e^{-\zeta\omega_0 t} \cos(\omega_1 t + \phi) \tag{33}$$

In this experiment the students can calculate the damping coefficient c or ζ of the damped oscillator model. The damping coefficient can be extracted from a simple x(t)-t graph. Note that the fit function contains two frequencies ω_0 and $\omega_1 = \omega_0\sqrt{(1-\zeta^2)}$. If the damping is very weak, we can just assume $\omega_0 \approx \omega_1$ (easy). However, for strong damping, the frequency of the oscillation starts to shift, investigate this phenomena (hard). An example (tracking) analysis of damped harmonic motion is shown in Fig.2b/c.

4.1.7   Phase-space (hard)

Show that the phase-space plot x-v (or p = mv) consist of inward spiraling circles. The staring point of this project is to investigate a dynamical system:

$$x(t) = Ae^{-\zeta\omega_0 t} \cos(\omega_1 t) \tag{34}$$
$$v(t) = -A\zeta\omega_0 e^{\zeta\omega_0 t} - A\omega_1 e^{\zeta\omega_0 t} \sin(\omega_1 t). \tag{35}$$

If we plot the position against the velocity the phase-space plot shows a circle. What is the interpretation of the circle (Energy). Why is the circle inward spiraling and can we show the theoretical curve on top of the phase-space picture? To do this, the students have to calculate the frequency ω_0 and the damping coefficient c. Then, you can make a theoretical prediction of x(t) and v(t) and compare these with the experimental results. This project can be seen as an extension of the previous experiment on simple damping.
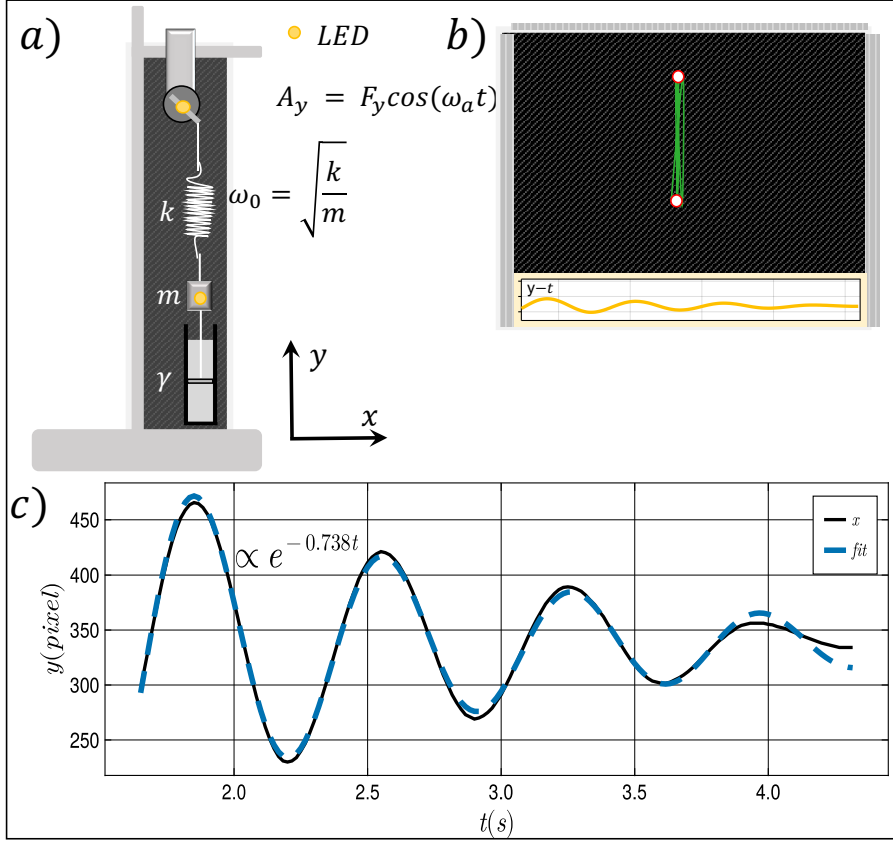
Figure 2: (a) The experimental setup of the driven harmonic oscillator. The periodic force A_y stretches the spring, which results in a (counter) force $F_y = -ky$. The result is a resonance system with viscous damping γ. (b) The result of the object tracking using MaxTRAQ 2D. (c) A plot of the y-coordinate of the object as a function of time. Note that there is not external driving force $A_y = 0$, so we have damped harmonic motion.

4.2 Falling objects

4.2.1 Theory

4.2.2 Setup and results

4.3 Rotating Wheel

This section is originally developed by T.W. Hijmans.

4.3.1 Theory

In this experiment, we examine a transparent plexiglass hollow wheel containing a viscous fluid. This problem finds its application in cycling. Mountain bikes, as well as road bikes increasingly, use tubeless tires. These tires are devoid of inner tubes and provide an airtight seal with the rim, containing a latex-based fluid. These tires are lighter than ones with inner tubes, and the vulcanizing fluid automatically seals small leaks. In this experiment, we investigate the influence of the internally present fluid on the rolling resistance of such a wheel.
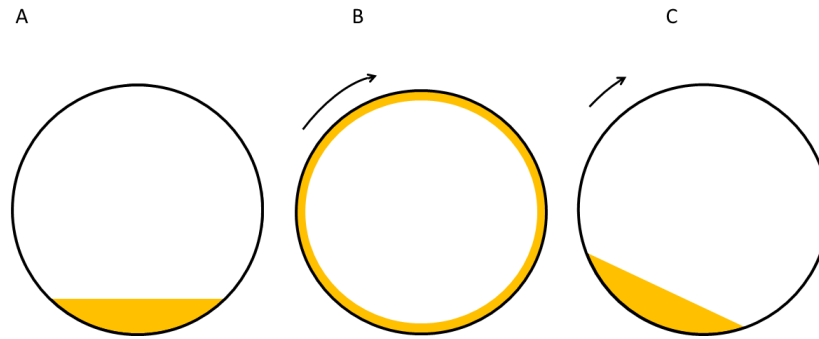

Figure 3: A hollow wheel filled with fluid. In situation A, the wheel is not rotating, and the fluid collects at the bottom. In situation B, the wheel rotates rapidly, causing the fluid to spread as a film across the entire inner wall due to centrifugal force. In situation C, the wheel rotates slowly. In this case, the slightly off-center puddle of fluid induces a frictional force that slows down the rotation of the wheel.

We will replace the "tire" mentioned in the introduction with a hollow plastic wheel filled with fluid made of transparent plastic (see Fig. 3). Let's first consider three situations: A) a stationary wheel with the fluid collecting at the bottom, B) a wheel rotating at high speed, where the fluid forms a film on the inner side due to centrifugal force, and finally, C) a situation where the wheel rotates slowly, causing the fluid to move but not fully coat the entire inner side. In situation B (high rotational speed), we expect the fluid film to move along with the rotating wheel. The only difference from a wheel without fluid is that the mass and moment of inertia of the wheel slightly increase due to the presence of the fluid film. The angular frequency at which the wheel rotates will decrease over time due to air resistance and friction in the wheel bearings, but these forces are practically small.

In situation C, the rotational speed is so low that the fluid remains at the bottom of the wheel under the influence of gravity. Due to the friction between the fluid and the wheel, we

can expect the fluid to be partially dragged along by the wheel, creating a situation similar to panel C in Fig. 3. Two forces act on the fluid, balancing each other: gravity, which tries to bring the puddle of fluid to the lowest point of the wheel, and the frictional force with the wall that drags the fluid. Thus, the equilibrium position of the fluid is determined by the rotational speed of the wheel.

The rotational speed v_o that determines the transition between regimes B and C in Fig. 3 is given by:

$$\frac{v_o^2}{R} = g, \tag{36}$$

where R is the radius of the wheel, and g is the acceleration due to gravity. The speed v_0 is related to the angular velocity ω of the wheel by:

$$v_o = \omega R. \tag{37}$$

There are various possibilities for the resistance force that slows down the wheel. First, let's consider the situation without fluid in the wheel. If there is a "classical" frictional force, i.e., a force (or more accurately, an angular acceleration) that is independent of the rotational speed, we have:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\lambda_0, \tag{38}$$

where λ_0 is a constant. Alternatively, we can assume that the frictional resistance experienced by the empty wheel is proportional to the angular velocity:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\lambda_1\omega, \tag{39}$$

where λ_1 is a new constant describing this case. Lastly, we can assume that air resistance is the main factor, in which case we have:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\lambda_2\omega^2, \tag{40}$$

where λ_2 is a constant. It is possible that multiple types of resistance forces are simultaneously active.

When we fill the wheel with fluid and rotate it rapidly (situation B in Fig. 3), the angular acceleration is given by equations (38) to (40), but with slightly different values for the constants λ_0, λ_1, and λ_2 due to the increased mass and moment of inertia.

The last, and most interesting for us, case is $v < v_0$, corresponding to panel C in Fig. 3. Without proof, we mention that in this case, we can expect the frictional force to be quadratic in velocity. In this case, equation (40) applies again, but now with a much larger value of λ_2 due to the stronger interaction between the wheel and the viscous fluid.

4.3.2   Setup and results

In this experiment the student will use a rotating wheel. A schematic setup is shown in Fig. 4a). The positioning of the LED lights is essential and one LED needs to be attached to the outer surface, to maximize contrast we place the setup behid a black cardboard sheet. The second LED can be positioned in the center of the wheel. Obtaining a clear footage is difficult and it is worth to invest the time in optimizing the camera position. During the analysis the LED light can move behind the wheel pole, this is inevitable and it is still
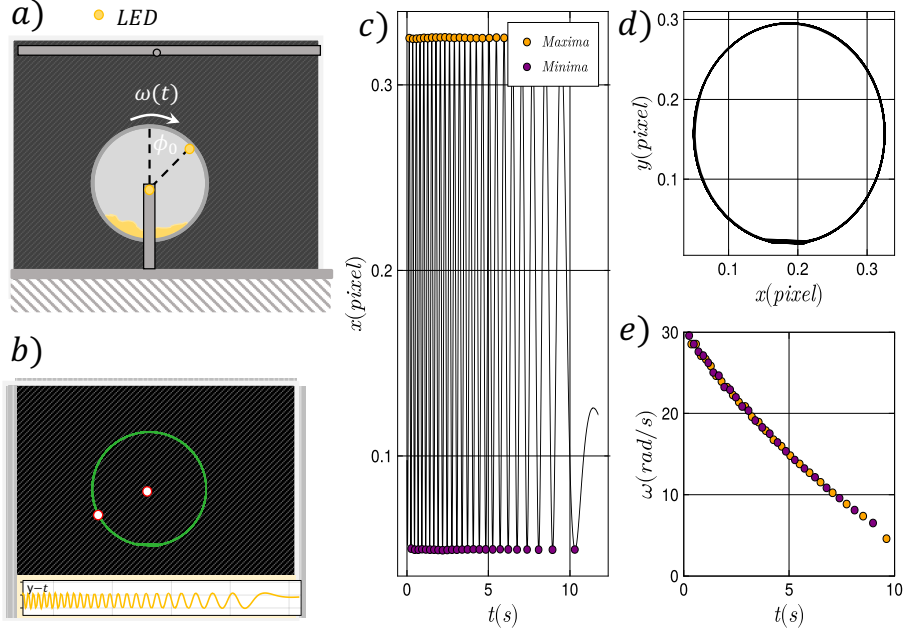
Figure 4: The rotating wheel experiment. a) The schematic setup of the rotating wheel experiment. The wheel is rotating at an angular velocity ω, but it experiences an unknown frictional force that slows the rotational motion down. The LED light capture the slowing down of the angular frequency. In b) the the tracked position is shown using MaxTRAQ 2D. In c) the tracked x-position is shown and in d) the x-y plane is shown. In c) the yellow and purple dots correspond to the maxima and minima of the position. They were found using a peak finding algorithm. Using the difference between each of the maxima the angular frequency is calculated. It is clear that the angular frequency experiences a friction forces.

feasible to do the tracking. In Fig. 4 b) we show the tracked path of the LED lights. In Fig. 4 c) the x-position of the tracked position is shown, a peek-finding algorithm was used to detect the maxima and minima.

The x-position (or y-position) of the LED light shows a dissipating oscillating signal. However, the type of dissipation is unknown in this case. That means that we do not have a closed-form analytical solution for the x-position as a function of time. To calculate the angular velocity ω we have to calculate it at different time intervals. The procedure is as follows: find the time interval $T_{a-b} = T_a - T_b$ between the neighboring maxima (or minima). Convert this to an angular frequency by $\omega_{a-b} = \frac{2\pi}{T_{a-b}}$ and denote the center time t_{a-b} between T_a and T_b. This procedure should be done with all the minima and maxima. The procedure is tedious and labour intensive (you can use a programming language to speed this up), but the result is a graph of ω-t. The time derivative can also be visualized $\frac{d\omega}{dt} - t$ and should be compared to the theoretical prediction. An example analysis of a empty wheel is shown in Fig. 4 e).

19

4.3.3   Friction of a empty rotating wheel (hard)

Try to analyze the camera footage to observe how the angular velocity of the wheel changes due to the frictional effect. Start the experiment by examining a wheel without fluid. Pay close attention to visually observe what happens around this transition point.
By analyzing the camera footage, you can track the changes in angular velocity as the wheel rotates. It is important to visually observe the wheel's behavior to gain a comprehensive understanding of the dynamics involved.
Try to find how the angular frequency changes in time. The frictional forces that are dominant depend on the angular velocity as well, try to maximize the angular velocity and observe the behavior.

4.3.4   Friction of an oil filled rotating wheel (hard)

Try to analyze the camera footage to observe how the angular velocity of the wheel changes due to the different friction effects. It would be helpful to start by examining a wheel without fluid. When observing the filled wheel, we expect that when we start with a rapidly rotating wheel, at a certain point (i.e., when the velocity drops below the critical threshold of Eq. 37), a new regime with significantly increased friction will be reached. Pay close attention to visually observe what happens around this transition point.
By analyzing the camera footage, you can track the changes in angular velocity as the wheel rotates. Look for any distinct patterns or behaviors that occur during the transition from one regime to another. It is important to visually observe the wheel's behavior to gain a comprehensive understanding of the dynamics involved.

4.4   Rolling Cylinder

4.4.1   Theory

4.4.2   Setup and results

4.5 Chaotic pendulum

4.5.1 Theory

A chaotic system is a mathematical model that exhibits sensitive dependence on initial conditions, a property known as "chaotic behavior." Such systems are typically described by nonlinear equations and demonstrate complex, intricate, and seemingly random patterns in their dynamics.

One defining characteristic of chaotic systems is their sensitivity to initial conditions, often referred to as the "butterfly effect." Even small changes in the starting state of a chaotic system can lead to dramatically different trajectories and outcomes. This sensitivity arises from the amplification of initial perturbations through the nonlinear interactions within the system, ultimately resulting in diverging behaviors.

Despite being deterministic, chaotic systems display behavior that appears random and unpredictable over long time scales. This unpredictability arises from the exponential divergence of trajectories in phase space, rendering long-term predictions highly challenging. There are a few conditions that defines a chaotic system:

- Sensitive to initial conditions;

- Dense periodic orbits;

- Topological transitive.

The first condition is referred to as the "Butterfly effect". The second condition is less trivial. A linear system has a single predictive orbit (in for example phase space) that is constant, a chaotic system however can have a very dense orbit structure. By changing for example, the driving amplitude of a mathematical pendulum we can have a cascade structure of orbits (a period double sequence). The last condition, topological transitivity, refers to a property of dynamical systems where the trajectories or paths of the system can visit every part of its phase space. In simpler terms, it means that no matter where you start within the system, you can eventually reach any desired location.

The second and third conditions are very hard to observe in an experimental system and a way beyond the level of a second year student. The first condition however, is relatively easy and can be characterized by a Lyapunov exponent.

Given two starting trajectories in the phase-space with initial seperation $\delta d_0| = |d_1 - d_2|$, the separation grows exponentially according to

$$|\delta(t)| \approx |\delta d_0| e^{\lambda t}. \tag{41}$$

λ is referred to as the Lyapunov exponent. If $\lambda < 0$ the trajectories converge and the orbit is exponentially stable. If $\lambda = 0$ the trajectories do not converge, nor will they diverge, they are stable. Stability does not imply that their distance stays constant, but that they are not attracted to a stable manifold.

The last scenario, $\lambda > 0$, implies instability. The nature of the chaotic system results in diverging trajectories. This is independent of the initial separation.

For a linear pendulum we can derive the Lyapunov exponent by hand. We start by defining two trajectories $\theta(t)$ and $\theta'(t)$ that are separated by a small difference in their phase ϵ. The time-evolution of θ is given by the linear oscillator equation

$$\theta(t) = A\cos(\omega t + \phi), \tag{42}$$

with A the amplitude, ω the eigen frequency of the pendulum and ϕ the starting angle. The separation between the oribits is now given by:

$$|\delta\theta(t)| = \theta(t) - \theta'(t) \tag{43}$$

$$= A\cos(\omega t + \phi) - A\cos(\omega t + \phi + \epsilon) \tag{44}$$

$$= A\cos(\omega t + \phi) - A\cos(\omega t + \phi)\cos(\epsilon) + A\sin(\omega t + \phi)\sin(\epsilon) \tag{45}$$

$$\approx A\cos(\omega t + \phi) - A\cos(\omega t + \phi) + A\sin(\omega t + \phi)\epsilon \tag{46}$$

$$\approx |\epsilon A\sin(\omega t + \phi)|, \tag{47}$$

where I used that $\epsilon \ll 1$ and Taylor expanded the cosine and sine up to first order $\cos(\epsilon) \approx 1$ and $\sin(\epsilon) \approx \epsilon$. Furthermore I used the sum and difference formule of the cosine function. The separation $|\delta\theta(t)|$ follows a sine function with amplitude $A\epsilon$, hence the separation is not constant, but periodic in nature. However, the separation may not be constant, but the orbits do not diverge. This implies that the Lyapunov exponent is zero $\lambda = 0$.

4.5.2 Setup and results

In this experiment the student will use a pre-build double pendulum. A schematic setup is shown in Fig. 5a). The LED lights should be attached at the end of each of the pendulum. To maximize contrast the experiment was placed behind a black cardboard sheet. During the analysis the LED light can move behind the first pendulum, this is should be avoided. In Fig. 5 b) we show the tracked path of the LED lights. In Fig. 5 c) the separation of the trajectories is shown, the black line is a fit using Eq. 41 with $\lambda > 0$.

The x-position (or y-position) of the LED light shows a dissipating oscillating signal. However, the type of dissipation is unknown in this case. That means that we do not have a closed-form analytical solution for the x-position as a function of time. To calculate the angular velocity ω we have to calculate it at different time intervals. The procedure is as follows: find the time interval $T_{a-b} = T_a - T_b$ between the neighboring maxima (or minima). Convert this to an angular frequency by $\omega_{a-b} = \frac{2\pi}{T_{a-b}}$ and denote the center time t_{a-b} between T_a and T_b. This procedure should be done with all the minima and maxima. The procedure is tedious and labour intensive (you can use a programming language to speed this up), but the result is a graph of ω-t. The time derivative can also be visualized $\frac{d\omega}{dt} - t$ and should be compared to the theoretical prediction. An example analysis of a empty wheel is shown in Fig. 4 e).
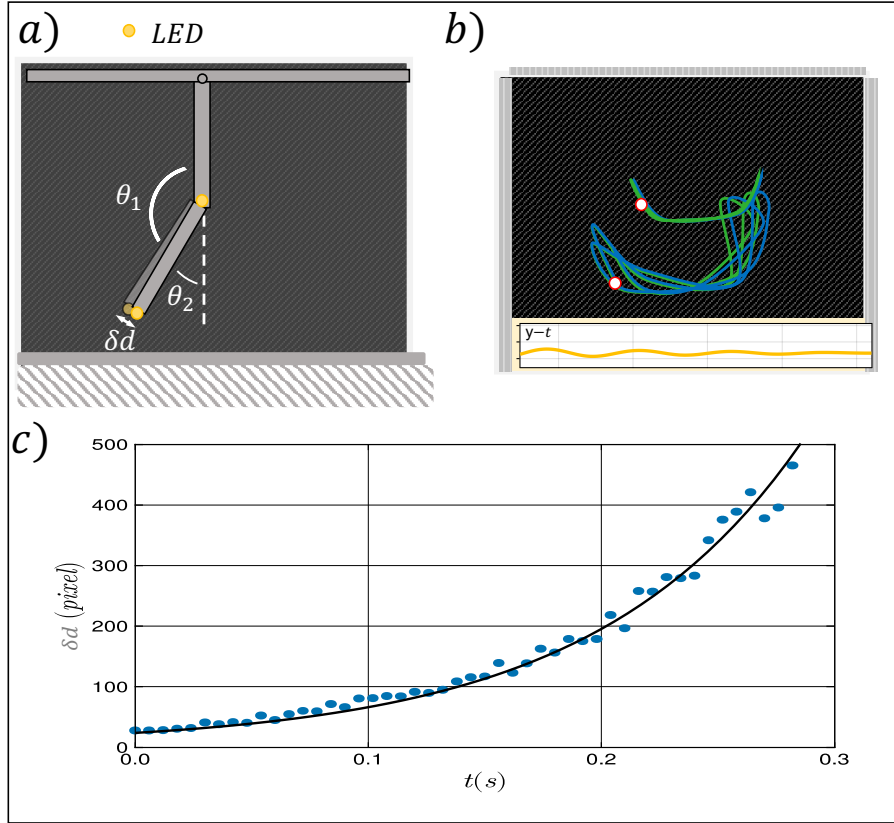
23

Figure 5: The double pendulum experiment. a) A schematic representation of the double pendulum. The initial separation δd is determined visually and must be minimized. The LED lights track the position of both pendula, this enables to construct a displacement vector of the second pendulum. In b) the tracked position of the LED lights is shown. The experiment is repeated twice, the individual trajectories are shown in blue and green. Obtained using MaxTRAQ 2D. c) The separation between the two trajectories shown in the first 0.3 seconds. The black line is a fit using Eq. 41, with $\lambda > 0$.

4.5.3 Lyapunov exponent (easy & hard)

4.6 Buoyancy

4.6.1 Theory

4.6.2 Setup and results

4.7 Elastic Collisions

4.7.1 Theory

4.7.2 Setup and results