HW 5

Questions

The next 6 questions refer to the following problem:

An oil refinery in the US produces gasoline for external clients. In addition, it maintains five gas stations itself: S1, S2, S3, S4, S5. It can supply these gas station from two depots: D1, D2. The distance (in miles) between each depot and gas station is presented in the following table:

Distance	S1	S2	S3	S4	S5
D1	25	50	40	30	10
D2	40	25	40	60	75

The weekly demand (in barrels) for each station is as follows:

	S1	S2	S3	S4	S5
Demand	220	350	600	440	180

Lastly, the weekly capacity for depot D1 is 750, while the weekly capacity for depot D2 is 1250 (both in barrels).

The unit shipment cost is \$1 per mile per barrel. We need to allocate the available gasoline from the depots to the stations, such that the total cost is minimized and the capacity and demand requirements are satisfied. To this end, we introduce nonnegative variables x_{ij} representing the total amount to be shipped from depot $i \in \{\mathrm{D1}, \mathrm{D2}\}$ to station $j \in \{\mathrm{S1}, \mathrm{S2}, \mathrm{S3}, \mathrm{S4}, \mathrm{S5}\}$. We can then formulate this problem as the following linear program:

$$
\begin{aligned}
\text{minimize} \quad & 25x_{\mathrm{D1,S1}} + 50x_{\mathrm{D1,S2}} + 40x_{\mathrm{D1,S3}} + 30x_{\mathrm{D1,S4}} + 10x_{\mathrm{D1,S5}} + \\
& 40x_{\mathrm{D2,S1}} + 25x_{\mathrm{D2,S2}} + 40x_{\mathrm{D2,S3}} + 60x_{\mathrm{D2,S4}} + 75x_{\mathrm{D2,S5}} \\
\text{subject to} \quad & x_{\mathrm{D1,S1}} + x_{\mathrm{D2,S1}} \geq 220 \\
& x_{\mathrm{D1,S2}} + x_{\mathrm{D2,S2}} \geq 350 \\
& x_{\mathrm{D1,S3}} + x_{\mathrm{D2,S3}} \geq 600 \\
& x_{\mathrm{D1,S4}} + x_{\mathrm{D2,S4}} \geq 440 \\
& x_{\mathrm{D1,S5}} + x_{\mathrm{D2,S5}} \geq 180 \\
& x_{\mathrm{D1,S1}} + x_{\mathrm{D1,S2}} + x_{\mathrm{D1,S3}} + x_{\mathrm{D1,S4}} + x_{\mathrm{D1,S5}} \leq 750 \\
& x_{\mathrm{D2,S1}} + x_{\mathrm{D2,S2}} + x_{\mathrm{D2,S3}} + x_{\mathrm{D2,S4}} + x_{\mathrm{D2,S5}} \leq 1250 \\
& x_{ij} \geq 0
\end{aligned}
$$

An implementation of this LP in Excel is depicted on the next page (top), together with the associated sensitivity report (bottom). Note that the value of cell `E24` has been erased.

	A	B	C	D	E	F	G	H	I
1	**Quiz 5 - Question 1**								
2									
3	**Distance**	**S1**	**S2**	**S3**	**S4**	**S5**			
4	**D1**	**25**	**50**	**40**	**30**	**10**			**Total cost**
5	**D2**	**40**	**25**	**40**	**60**	**75**			54600
6									
7									
8		**S1**	**S2**	**S3**	**S4**	**S5**			**Capacity**
9	**D1**	130	0	0	440	180	750	<=	**750**
10	**D2**	90	350	600	0	0	1040	<=	**1250**
11		220	350	600	440	180			
12		>=	>=	>=	>=	>=			
13	**Demand**	**220**	**350**	**600**	**440**	**180**			
14									

	A	B	C	D	E	F	G	H
6	Adjustable Cells							
7				Final	Reduced	Objective	Allowable	Allowable
8		Cell	Name	Value	Cost	Coefficient	Increase	Decrease
9		B9	D1 S1	130	0	25	15	15
10		C9	D1 S2	0	40	50	1E+30	40
11		D9	D1 S3	0	15	40	1E+30	15
12		E9	D1 S4	440	0	30	15	45
13		F9	D1 S5	180	0	10	50	25
14		B10	D2 S1	90	0	40	15	15
15		C10	D2 S2	350	0	25	40	25
16		D10	D2 S3	600	0	40	15	40
17		E10	D2 S4	0	15	60	1E+30	15
18		F10	D2 S5	0	50	75	1E+30	50
19								
20	Constraints							
21				Final	Shadow	Constraint	Allowable	Allowable
22		Cell	Name	Value	Price	R.H. Side	Increase	Decrease
23		G9	D1	750	-15	750	90	130
24		G10	D2	1040		1250	1E+30	210
25		B11	S1	220	40	220	210	90
26		C11	S2	350	25	350	210	350
27		D11	S3	600	40	600	210	600
28		E11	S4	440	45	440	130	90
29		F11	S5	180	25	180	130	90

1. Is the optimal solution unique?

   Yes; the Allowable Decrease and Allowable Increase values for the adjustable cells are all non-zero.

2. Is the optimal solution degenerate?

   No; the Allowable Decrease and AllowableIncrease values for the constraint cells are all non-zero.

3. What is the Excel formula that goes in cell `I5` of the LP model?

   `=SUMPRODUCT(B4:F5,B9:F10)`

4. What is the value for cell `E24` of the sensitivity report?

   0; the constraint is non-binding and therefore its shadow price must be zero.

5. Suppose the weekly demand of station S2 increases to 450 barrels. By how much would the total cost increase?

   By \$2,500: The right hand side of the demand constraint for S2 would increase by 100, within the allowed range. Therefore, we can multiply the increase with the shadow price (25) and obtain 2500 as increase in the objective.

6. We have the option of increasing the capacity at depot D1 to 800. This capacity increase would incur an additional weekly maintainance cost of \$600. Should the refinery expand the capacity at D1 to 800 in order to save costs?

   Yes. Increasing capacity at D1 by 50 units is within the allowed range, and would therefore impact the objective by -15*50 = -750. These savings are more than the additionally incurred maintainance costs of 600.

The next 8 questions refer to the following problem:

Given the current economic situation, a client of a retirement fund company wishes to revisit her portfolio. She wants to invest \$100,000, and has selected six potential bonds to invest in. Each bond has different characteristics (return rate, maturity (in years), and rating), as indicated the following table.

	Return	Years to Maturity	Rating
Bond1	8.50%	10	1 (excellent)
Bond2	7.50%	12	1 (excellent)
Bond3	9.50%	7	3 (good)
Bond4	8.00%	11	2 (very good)
Bond5	10.00%	8	4 (fair)
Bond6	9.00%	9	3 (good)

The goal of the client is to maximize total return. However, she imposes the following restrictions to have a more balanced portfolio. First, we can spend no more than 30% of the total amount of money in a single bond. Second, at least 50% of the total amount of money must be invested in long-term bonds, having maturity of 10 years or more. Finally, she wants the rating of the portfolio to be at most 2 (very good or better). The rating of the portfolio is defined as the weighted average rating of the bonds:

$$\frac{\sum_{i=1}^{6}(\text{rate of bond } i) \cdot (\text{amount invested in bond } i)}{100,000}.$$

The client uses a spreadsheet linear programming model to determine the optimal portfolio. The solved spreadsheet model and sensitivity report are as follows.

	A	B	C	D	E	F	G
1							
2			Years to		Amount		
3		Return	Maturity	Rating	Invested		
4	Bond1	0.085	10	1	30000	<=	30000
5	Bond2	0.075	12	1	26666.67	<=	30000
6	Bond3	0.095	7	3	30000	<=	30000
7	Bond4	0.08	11	2	0	<=	30000
8	Bond5	0.1	8	4	13333.33	<=	30000
9	Bond6	0.09	9	3	0	<=	30000
10		8733.333	56666.667	2	100000		
11			>=	<=	=		
12			50000	2	100,000		

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
E4	Bond1 Invested	30000	0.01	0.085	1E+30	0.01
E5	Bond2 Invested	26666.66667	0	0.075	0.01	0.005
E6	Bond3 Invested	30000	0.003333333	0.095	1E+30	0.003333333
E7	Bond4 Invested	0	-0.003333333	0.08	0.003333333	1E+30
E8	Bond5 Invested	13333.33333	0	0.1	0.005	0.0025
E9	Bond6 Invested	0	-0.001666667	0.09	0.001666667	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E10	Invested	100000	0.066666667	100000	2500	5000
C10	Maturity	56666.66667	0	50000	6666.666667	
D10	Rating	2	833.3333333	2	0.2	0.1

Answer the following questions based on the sensitivity report. Do *not* implement and solve the spreadsheet model. For each question "Cannot determine based on the sensitivity report" is a possible answer.

7. Write down the Excel formula that should go in cell `C10` of the Excel LP model.

   `=SUMIF(C4:C9,">=10",E4:E9)`
   Alternative: `=E4+E5+E7`

8. One of the entries in the sensitivity report has been deleted (the Allowable Decrease for Maturity). What is its numeric value?

   1E+30 or infinity, because the constraint is nonbinding.

9. Is the optimal solution unique?

   Yes, because the Allowable Increase and Decrease for the adjustable cells are all non-zero.

10. Is the optimal solution degenerate?

    No, because the Allowable Increase and Decrease for the constraint cells are all non-zero.

11. Suppose the client decides to invest at least 65% of the total amount in long-term bonds (having maturity of at least 10 years). What would be the new value of the optimal total return?

    The right hand side of that constraint would be increased from 50,000 to 65,000 for an increase of 15,000. The maximum allowed increase is 6,666.66. Therefore, we cannot determine the answer from the sensitivity report.

12. What should be the minimum return rate for bond 4 in order to be part of the portfolio?

    8.33333% (the current objective coefficient plus the amount needed to make reduced cost positive)

13. Suppose the client is willing to allow the portfolio rating to be at least 2.1 instead of the current 2. How would this affect the optimal total return of the portfolio?

    The right hand side increase of 0.1 is within allowed range, so the total return increases with 0.1*833.3333 to 8,816.66.

14. (Bonus question) After inspecting the optimal portfolio, the retirement company suggests a new bond to the client. This bond has a return rate of 9.25%, a maturity of 11 years, and a rating of 3 (good). Should this new bond be part of the portfolio?

    The reduced cost of the new variable corresponding to the amount invested in this bond is 0.0925 - (0.06666 + 0 + (3/100,000)*833.3333) = 0.0008333. This is positive, so the new bond should be part of the portfolio.