HW 10 Solutions

Questions

The next three questions refer to the following problem:

Consider the traveling salesman problem instance from HW9 (questions 3–6 in HW9); the distance matrix can be found also in `HW9.xlsx`. We will apply the subtour constraint generation method to find an optimal solution to this problem.

We let V denote the set of locations including the depot, and we let E denote the set of all possible edges. Note that we assume edges to be undirected, i.e., edge (i, j) is equivalent to edge (j, i) for two locations $i, j \in V$.

Following the model in the slides, we introduce a binary variable x_e for each edge $e \in E$. Furthermore, we denote the distance of edge $e = (i, j)$ connecting two locations $i, j \in V$ by d_{ij}. The initial integer linear programming model then becomes

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} d_e x_e \\ \text{subject to} \quad & \sum_{e \in L(i)} x_e = 2 \quad \text{for all locations } i \in V, \end{aligned}$$

where $L(i)$ is the subset of edges linked to location $i \in V$, i.e., $L(i) = \{(i, j) \mid j \in V\}$.

1. Implement the initial integer linear program above (without any subtour constraints) in Excel, and find the optimal objective value.

   See `HW10Sol.xlsx` for the optimal solution. The optimal objective value is 278.

2. Assume that the solution found under 1. contains three subtours, on the following sets of locations: $\{1, 2, 4\}$, $\{5, 6, 10\}$, and $\{3, 7, 8, 9\}$. What are the subtour elimination constraints that need to be added for these sets of locations?

   These constraints are:

$$x_{(1,2)} + x_{(1,4)} + x_{(2,4)} \leq 2$$
$$x_{(5,6)} + x_{(5,10)} + x_{(6,10)} \leq 2$$
$$x_{(3,7)} + x_{(3,8)} + x_{(3,9)} + x_{(7,8)} + x_{(7,9)} + x_{(8,9)} \leq 3$$

3. Continue the constraint generation process until your solution does not contain subtours anymore. What is the objective value for the optimal solution? [Note: you may need several 'rounds' of adding subtour constraints to find the optimal solution.]

   The optimal tour is 1-2-4-9-3-7-8-10-5-6-1. The optimal objective value is 298.

The next four questions refer to the following problem:

Consider the linear demand function $D(x) = M - ax$, where x is the price, M and a are positive constants and $x \leq M/a$. Thus demand for a free good is M, and demand drops linearly until the price reaches M/a, at which point demand is zero.

4. Find the price x for which the derivative of the revenue function $R(x) = x \cdot D(x)$ is equal to zero.
   $R'(x) = M - 2ax$. Setting $R'(x) = 0$ gives $x^* = \frac{M}{2a}$.
5. Use the second derivative of $R(x)$ to determine whether the price found under 4. is a local minimum or a local maximum.
   $R''(x) = -2a < 0$, and therefore x^* is a local maximum.
6. Determine whether this demand function is concave, convex, or neither on the domain $0 \leq x \leq M/a$.
   Since $R''(x) < 0$ for all $0 \leq x \leq M/a$, $R(x)$ is concave on this domain.
7. Determine the global maximum of $R(x)$ on the domain $0 \leq x \leq M/a$.
   $x^* = \frac{M}{2a}$ is the global maximum on this domain, and $R(x^*) = \frac{M^2}{4a}$.

The next four questions refer to the following problem:

Consider the function $f(x) = x^3 - 7.5x^2 + 18x - 8$ on the domain $0 \leq x \leq 6$. We will minimize and maximize $f(x)$ on this domain using Excel Solver.

Recall that a 'starting point' for Excel Solver is a value that you give to the changing cell representing variable x before you solve the model.

8. Use Excel Solver to *maximize* $f(x)$ using the starting point 0 in your variable cell. What are the optimal solution and the optimal objective value?
   The optimal solution is $x = 2$, with objective value 6.
9. Use Excel Solver to *maximize* $f(x)$ using the starting point 4 in your variable cell. What are the optimal solution and the optimal objective value?
   The optimal solution is $x = 6$, with objective value 46.
10. Use Excel Solver to *minimize* $f(x)$ using the starting point 1.5 in your variable cell. What are the optimal solution and the optimal objective value?
    The optimal solution is $x = 0$, with objective value -8.
11. Use Excel Solver to *minimize* $f(x)$ using the starting point 4 in your variable cell. What are the optimal solution and the optimal objective value?
    The optimal solution is $x = 3$, with objective value 5.5.