

Smoothed Particles Hydrodynamics for Impact Simulation

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February 23, 2015

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Part I

Bibliographic Revision

Chapter 1

Liu & Liu

1.1 Introduction

In the SPH method, the state of a system is represented by a set of particles, which possess individual material properties and move according to the governing conservation equations.

The principal characteristics of the SPH method are:

- Adaptive Nature: Any domain can be described by an arbitrary set of particles. The Sph Approximation does not get effected by the particle distribution.
- Meshfree: SPH does not uses a any mesh to provide interactions between particles. The field values are evaluated for all particles at each time step, which makes SPH esay to track the properties of particles in time. This property tracking is specially difficult for othr numerical methods that use Eulerian mesh.

SPH simmulation involves two major steps: particle representation and particle approximation. The particle representation is related to only the initial creation fo the particles. Sph is important because:

- SPH have recently reached an acceptable level for practical engineering applications
- Applicable in CFD, CSM, from micro-scale to macro-scale and astronomical-scale, fromd discrete to continuum systems.
- SPH have been introduced in commercial codes [1]

1.2 SPH Concept and Essential Formulation

The general solution procedure of SPH is:

1. The problem domain is represented by a set of arbitrarily distributed particles. NO connectivity of these particles is needed (*Meshfree*)
2. The *integral representation method* is used for field function representation. *Kernel approximation*. (*Integral function representation*)
3. The kernel approximation is the further approximated using particles (*Particle Approximation*). It is done by replacing the integration in the integral representation of the field functions and its derivatives with summations over all the corresponding values at the neighboring in a local domain called *Support domain*. (*Compact Support*)
4. The particle approximation is performed at every time step, and hence the use of the particle depends on the current local distribution of the particles. *Adaptive*)
5. The particle approximations are performed to all the terms related to the fields function in the PDEs to produce a set of ODEs in discretized form with respect to time only. (*Lagrangian*)
6. The ODEs are solved using an *explicit* integration algorithm to achieve fast time stepping, and to obtain the time history of all the field variables for all the particles. *Dynamic*

The combination of the 6-point strategy makes the SPH method be a mesh-free, adaptive, stable and Lagrangian solver for Dynamic problems.

1.2.1 Essential Formulation of SPH

Integral Representation of a function and derivative

A function can be represented in an integral form (Eq. 2.1 in [1]):

$$f(x) = \int_{\Omega} f(x') W(x - x', h) dx'$$

- f is a function of the three-dimensional position vector x

- $W(x - x', h)$ Smoothing function (*kernel*), h is the *smoothing length* defining the influence area of the smoothing function.

The derivative of a function also can be approximated as (Eq. 2.15 in [1]):

$$\nabla \cdot f(x) = - \int_{\Omega} f(x') \cdot \nabla W(x - x', h) dx'$$

The differential operation on a function is transmitted to a differential operation on the smoothing function. The SPH integral representation of the derivative of a function allows the spatial gradient to be determined from the values of the function and the derivatives of the smoothing function W .

Particle Approximation

The continuous integral representations concerning the kernel approximation (Equations above) can be converted to discretized forms of summation over all the particles in the support domain. This process is known as *Particle approximation*.

The particle approximation for function at particle i can be written as (Eq 2.18 in [1]):

$$f(x_i) = \sum_{j=1}^N \frac{m_j}{\rho_j} f(x_j) W_{ij}$$

$$W_{ij} = W(x_i - x_j, h)$$

The value of a function at particle i is approximated using the average of those values of the function at all the particles in the support domain of particle i weighted by the smoothing function. The function derivative can be approximated as:

$$\nabla \cdot f(x_i) = \sum_{j=1}^N \frac{m_j}{\rho_j} f(x_j) \cdot \nabla_i W_{ij}$$

$$\nabla_i W_{ij} = \frac{x_i - x_j}{r_{ij}} \frac{\partial W_{ij}}{\partial r_{ij}} = \frac{x_{ij}}{r_{ij}} \frac{\partial W_{ij}}{\partial r_{ij}}$$

- r_{ij} is the distance between particle i and j .

The gradient $\nabla_i W_{ij}$ is taken with respect to the particle i .

The value of gradient of a function at particle i is approximated using the average of those values of the function at all the particles in the support

domain of particle i weighted by the gradient of the smoothing function. Those equations convert the continuous integral representation of a function and its derivatives to the discretized summations based on an arbitrary set of particles. The relation of particle i and the particles j in its support domain Ω is shown in

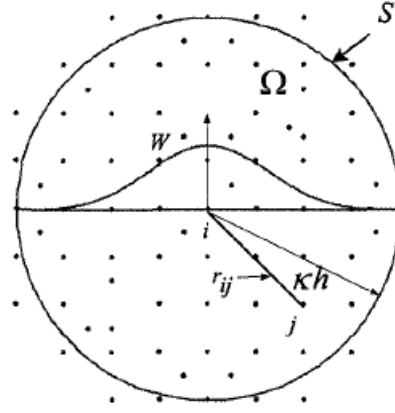


Figure 1.1: Particle approximation using particles within the support domain of the smoothing function W for particle i . The support domain is circular with a radius of kh . Image 2.3 in reference [1]

Concluding remarks

The computational frame in SPH are the moving particles in space.

The particle approximation in the SPH method is performed at every time step with particles in the current support domain, and it is done for the governing PDEs in Lagrangian description.

The special remarks are:

1. The SPH method employs particles to represent the material and form the computational frame. There is no need for predefined connectivity between these particles, all one needs is the initial particle distribution.
2. The SPH approximation consists of kernel approximation and particle approximation. The kernel approximation of a function and its derivative are carried out in the continuum domain, and the particle approximation of a function and its derivative are carried out using discretized particles in the support domain at the current time step.

3. Each particle in the SPH method is associated with a support domain and influence domain.

1.3 Construction of Smoothing functions

A smoothing function is required to accomplish the following properties:

1. The smoothing function must be normalized (*Unity*) over its support domain.

$$\int_{\Omega} W(x - x', h) dx' = 1$$

2. The smoothing function must be compactly supported (*Compact Support*)

$$W(x - x') = 0, \text{ for } |x - x'| > kh$$

The dimension of the compact support is defined by the smoothing function length h and a scaling factor k , where h is the smoothing length, and k determines the spread of the specified smoothing function.

3. $W(x - x') \geq 0$ for any point at x' within the support domain of the particle at point x (*Positivity*) This property ensures a physically meaningful representation.
4. The smoothing function value for a particle should be monotonically decreasing with the increase of the distance away from the particle (*Decay*)
5. The smoothing function should satisfy the Dirac delta function condition

$$\lim_{h \rightarrow 0} W(x - x', h) = \delta(x - x')$$

6. Even function
7. Smoothing function should be sufficiently smooth

An example of a smoothing function is the Johnson kernel.

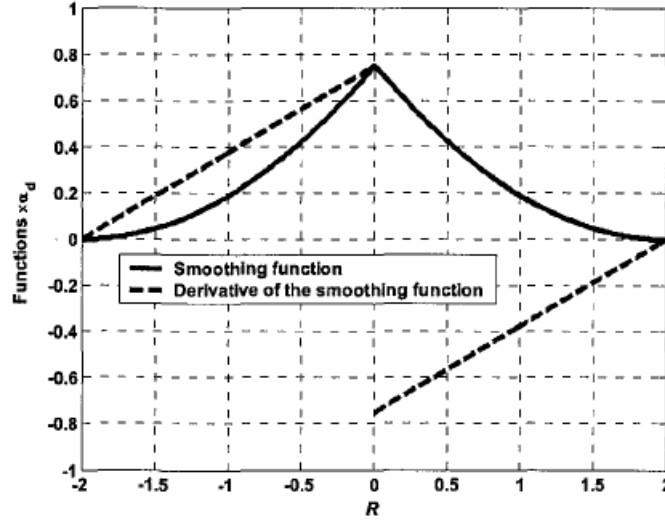


Figure 1.2: The quadratic smoothing function and its first derivative used by Jhonson et al. (1996b). α_d is $1/h$, $2/h$ and $5/4\pi h^3$ in one-, tow- and three-dimnesional space, respectively. Image from reference [1]

1.4 SPH for general dynamic Fluid Flows

a

4.4.9 Time integration

The discrete SPH equations can be integrated with standard methods such as the second order accurate Leap-Frog, predictor-corrector and Runge-Kutta schemes and so on.

The explicit time integration schemes are subject to the CFL (*Courant-Friedrich-levy*) condition for stability. The CFL condition states that the computational domain of dependence in a numerical simulation should include the physical domain of dependence, or the maximum speed of numerical propagation must exceed the maximum speed of physical propagation. This CFL condition requieres the time step to be proportional to the smallest spatial particle resolution, which in SPH applications is represented by the

smallest smoothing length[1]

$$\Delta t = \min \left(\frac{h_i}{c} \right)$$

Chapter 2

Luna

2.1 Introduction

Fragil materials used as armour present a failure by Spalling, it is the fragil fragmentation of the material when it is being perforated. The sparced fragments are dangerous because of its high velocity.

Constitutive Model of Material

When adapting the SPH formulation to solid mechanics, the stress tensor have to include the effect of the deviatoric stress. The stress tensor has two parts:

- Hidrostatic Pressure
- Deviatoric Stress

$$\sigma^{\alpha\beta} = -P\delta^{\alpha\beta} + \tau^{\alpha\beta}$$

The deviatoric stress are defined with small displacements

$$\tau^{\dot{\alpha}\beta} = G \left(\epsilon^{\alpha\beta} - \frac{1}{3} \delta^{\alpha\beta} \epsilon^{\gamma\gamma} \right)$$

- G is the shear modulus
- $\tau^{\alpha\beta}$ is the constant stress rate

- $\epsilon^{\alpha\beta}$ is the deformation tensor

Jaumann constitutive equation is used to describe the deviatoric stress in a the reference system of the material:

$$\tau^{\dot{\alpha}\beta} = G \left(\epsilon^{\alpha\beta} - \frac{1}{3} \delta^{\alpha\beta} \epsilon^{\gamma\gamma} \right) + \tau^{\alpha\beta} R^{\beta\gamma} + \tau^{\alpha\beta} R^{\alpha\gamma}$$

R is the rotation tensor, where rotation and deformation are defined as:

$$\epsilon^{\alpha\beta} = \frac{1}{2} \left(\frac{\partial v^\alpha}{\partial x^\beta} + \frac{\partial v^\beta}{\partial x^\alpha} \right)$$

$$R^{\alpha\beta} = \frac{1}{2} \left(\frac{\partial v^\alpha}{\partial x^\beta} - \frac{\partial v^\beta}{\partial x^\alpha} \right)$$

State equation

One needs to calculate pressure from the other state variables, here the Mie-Gruenisen equation is used:

$$P(e, \rho) = \left(1 - \frac{1}{2} \Gamma \eta \right) P_H(\rho) + \Gamma \rho e$$

where sub-index H refers to Hugoniot curve, Γ is the Gruenisen parameter and η is the density change rate $\eta = \frac{\rho}{\rho_0} - 1$

Material Model

Luna employs the Grady & Kipp fracture model. This model determines a statistical failure distribution inside the material. In SPH we need to explicitly determine those failures inside the material.[2]

Chapter 3

High Strain Lagrangian hydrodynamics: a three-dimensional SPH code for dynamic material response

SPH is a pure Lagrangian particle method. The absence of mesh means that large deformations can be computed in a pure Lagrangian frame.[3]

The Method

The foundation of SPH is the interpolation theory. The conservation laws of continuum fluid dynamics, in the form of partial differential equations, are transformed into integral equations through the use of an interpolation function that gives the *kernel estimate* of the field variables at a point. Computationally, information is known at discrete points, so that the integrals are evaluated as sums over neighboring points. These interactions result in a net force which will accelerate the particle.[3]

The interesting point in the SPH formulation is that the spatial derivative of physical properties is evaluated through the sum of the function discrete values and derivative of the kernel operator.

SPH equations

The equations of SPH are the equations of continuum mechanics, where the dependent variables are:

- Density
- Specific internal energy
- velocity
- Stress tensor

On the other hand, the independent variables are the spatial coordinates and the time. The total time derivatives are taken in the moving Lagrangian frame.

Artificial Viscosity & wall heating ***

Constitutive Relations

The stress tensor is defined in terms of an isotropic part which is the Pressure P and the traceless symmetric deviatoric stress S :

$$\sigma^{\alpha\beta} = P\delta^{\alpha\beta} - S^{\alpha\beta}$$

The pressure is normally computed using an equation of state, such as the Mie-Grüneisen equation for solids:

Mie-Grüneisen $P = P_H(1 - \Gamma'\eta) + \Gamma'\rho E$

$$P_H = \begin{cases} a_0\eta + b_0\eta^2 + c_0\eta^3 & \eta > 0 \\ a_0\eta & \eta < 0 \end{cases}$$

The subscript H refers to the Hugoniot curve, while $\eta = \rho/\rho_0 - 1$ is used to represent the compression and $\Gamma' = \Gamma\rho/\rho_0$.

For the anisotropic part, we write a prognostic equation for the deviatoric stress assuming small displacements

$$\dot{S}^{\alpha\beta} = \mu\epsilon^{\alpha\beta} = \mu\left(\epsilon^{\alpha\beta} - \frac{1}{3}\delta^{\alpha\beta}\epsilon^{\gamma\gamma}\right)$$

where μ is the shear modulus and ϵ is the traceless rate of strain. However, for finite displacements this equation is not material frame indifferent, that is, the material response will depend in an unphysical way on rotations of the material and of the observer describing it. The Jaumann rate is the most widely used in codes and we adopt it also. With Jaumann rate, our constitutive equation is:

$$\dot{S}^{\alpha\beta} - S^{\alpha\beta}R^{\beta\gamma} - S^{\gamma\beta}R^{\alpha\gamma} = \mu\bar{\epsilon}^{\alpha\beta}$$

*** It looks like those equations are a little bit different from the equations of Luna. *I think that Luna just wrote them wrong in his document* The strain rate and rotation rate tensors that have been used are defined as follows: —

Time integration

The discrete SPH equations can be integrated with standard methods such as Runge-Kutta.

The explicit time integration schemes are subjected to the CFL condition (Courant-Friederichs-Levy) for stability. The CFL condition states that the computational domain of dependence in a numerical simulation should include the physical domain of dependence, or the maximum speed of numerical propagation must exceed the maximum speed of physical propagation. This CFL condition requires the time step to be proportional to the smallest spatial particle resolution, which in SPH applications is represented by the smallest smoothing length

$$\Delta = \min \left(\frac{h_i}{c} \right)$$

Chapter 4

Simulations of brittle solids using smooth particle hydrodynamics

The hydrodinamical quantities in SPH are known at a finite number of points, called particles, which move with the flow. given any spatial distribution of these points, the SPH technique allows the computation of saptial derivatives without an underlying grid. Once the spatial derivative have been computed and forces determined, time integration proceeds in the usual way.[4] SPH is able to simulate solids by including the non-diagonal terms of the stress tensor (Using Hoo'ks law) modified by a von Mises yielding relation for stress beyond the elastic limit. At the lower stress associated with brittle fracture, we developed a new fracture model based on the nucleation of incipient flaws whose number density is given by a Weibull distribution.

Equations for an Elastic-perfectly plastic strength model

The stress tensor is given by $\sigma^{\alpha\beta} = -P\delta^{\alpha\beta} + S^{\alpha\beta}$ where P is the pressure, $S^{\alpha\beta}$ the traceless deviatoric tensor and $\delta^{\alpha\beta}$ the kronecker symbol.

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