Problem 1

Problem 1. $r_t \sim N(0, \sigma^2)$

— Classical brownian motion

$P_t = P_{t-1} + r_t$

$E[P_t] = E[P_{t-1}] + E(r_t)$

$E[P_t] = E[P_{t-1}] + 0 = E[P_{t-1}] = P_{t-1}$

$SD(P_t) = \sqrt{(SD(P_{t-1}))^2 + (SD(r_t))^2} = \sqrt{\sigma^2} = \sigma$

— Arithmetic Return System

$P_t = P_{t-1}(1+r_t) = P_{t-1} + P_{t-1} \cdot r_t$

$E[P_t] = E[P_{t-1}] + E[P_{t-1} \cdot r_t]$

$= P_{t-1} + E[P_{t-1}] \cdot E[r_t]$

$E[P_t] = P_{t-1}$

$Var[P_t] = E[P_t^2] - E[P_t]^2$

$E[P_t^2] = E[P_{t-1}^2 \cdot (1+r_t)^2] = E[P_{t-1}^2] \cdot E[(1+r_t)^2]$

$= E[P_{t-1}^2] \cdot E[1 + 2r_t + r_t^2]$

$\because E[r_t^2] = Var(r_t) + E[r_t]^2 = \sigma^2$

$\therefore E[P_t^2] = P_{t-1}^2 \cdot (1+\sigma^2)$

$\therefore Var(P_t) = P_{t-1}^2 \cdot (1+\sigma^2) - P_{t-1}^2 = P_{t-1}^2 \cdot \sigma^2$

$\therefore SD(P_t) = \sqrt{P_{t-1}^2 \cdot \sigma^2} = P_{t-1} \cdot \sigma$

— Log Return

$P_t = P_{t-1} \cdot e^{r_t}$

$E[P_t] = E[P_{t-1}] \cdot E[e^{r_t}]$

$E[P_t] = P_{t-1} \cdot e^{\frac{1}{2}\sigma^2}$

since e^{r_t} is a log normal distribution,
the expected value of a log normal distribution is $e^{u+\frac{1}{2}\sigma^2}$
and in this case, $u=0$.

$Var[P_t] = E[P_t^2] - E[P_t]^2$

$E[P_t^2] = E[P_{t-1}^2 \cdot e^{2r_t}]$

$= E[P_{t-1}^2] \cdot E[e^{2r_t}]$

$= P_{t-1}^2 \cdot e^{2\sigma^2}$

since e^{r_t} is a log normal distribution,
the expected value of a log normal distribution is $e^{u+\frac{1}{2}\sigma}$
and in this case, $u=0$.

$\therefore Var[P_t] = P_{t-1}^2 \cdot e^{2\sigma^2} - (P_{t-1} \cdot e^{\frac{1}{2}\sigma^2})^2$

$= P_{t-1}^2 \cdot (e^{2\sigma^2} - e^{\sigma^2})$

$SD(P_t) = P_{t-1} \cdot \sqrt{e^{2\sigma^2} - e^{\sigma^2}}$

In the code, I set P0 = 100 and theta = 0.2

In Classical Brownian Motion, the result is below:
Sample mean of simulated prices: 100.00019637357447
Sample standard deviation of simulated prices: 0.20013157425283348
The result corresponds with the expected value and standard deviation, which are 100 and 0.2

In Arithmetic Return System, the result is below:
Sample mean of simulated prices: 100.01963735744897
Sample standard deviation of simulated prices: 20.013157425283353
The result corresponds with the expected value and standard deviation, which are 100 and 20

In Log Return, the result is below:
Sample mean of simulated prices: 102.04224179655556
Sample standard deviation of simulated prices: 20.61372162463448
The result corresponds with the expected value and standard deviation, which are 102.02013 and 20.60977

$$100e^{0.5 \cdot 0.04} = 102.02013\ldots$$

$$100\sqrt{e^{2 \cdot 0.04} - e^{0.04}} = 20.60977\ldots$$

In conclusion, in three cases (Classical Brownian Motion, Arithmetic Return System, Log Return), the result of the mean and standard deviation based on the simulation match my expectations.

Problem 2

Calculate the arithmetic returns for all prices.

	Date	SPY	AAPL	MSFT	AMZN	TSLA	GOOGL
1	2/15/2022 0:00	0.016127	0.023152	0.018542	0.008658	0.053291	0.007987
2	2/16/2022 0:00	0.001121	-0.001389	-0.001167	0.010159	0.001041	0.008268
3	2/17/2022 0:00	-0.021361	-0.021269	-0.029282	-0.021809	-0.050943	-0.037746
4	2/18/2022 0:00	-0.006475	-0.009356	-0.009631	-0.013262	-0.022103	-0.016116
5	2/22/2022 0:00	-0.010732	-0.017812	-0.000729	-0.015753	-0.041366	-0.004521

Remove the mean from the series so that the mean(META)=0
Centered META mean: 9.227253e-19

	META
1	0.017826
2	-0.017513
3	-0.038110
4	-0.004795
5	-0.017123
..	...
184	0.067983
185	0.000083
186	0.054497
187	0.105161
188	0.012948

VaR:
Using a normal distribution: (5% confidence level): 0.06534605
Using a normal distribution with an Exponentially Weighted variance (λ = 0. 94): 0.0967
Using a MLE fitted T distribution: 0.059
Using a fitted AR(1) model: 0.0655
Using a Historic Simulation: 0.0568

Using normal distribution with an Exponentially Weighted variance has the highest VaR which is 0.0655, and Historic Simulation has the lowest VaR which is 0.0568

Problem 3

Using an exponentially weighted covariance with lambda = 0.94, calculate the VaR of each portfolio as well as your total VaR (VaR of the total holdings). Express VaR as a $.

arithmetic method VaR:
Portfolio A: -0.029139752917664838
Portfolio B: -0.023451930005335624
Portfolio C: -0.020245358398332185
Portfolio Total: -0.024184779812501O3
Express VaR as a $.
Portfolio A: 8514.201940137194
Portfolio B: 7194.656898739166
Portfolio C: 6012.2212601721985
Portfolio Total: 21582.070704863407

log model VaR:
Portfolio A: -0.029921111060540063
Portfolio B: -0.023828369773231644
Portfolio C: -0.020610134566705195
Portfolio Total: -0.02462695687473913

In conclusion, PortfolioA has the highest VaR and thus most risky, while PortfolioC has the lowest VaR and thus the least risky.

My method:
I choose arithmetic method and then use log method. The log model will increase the VaR results into the following. In each of the portfolio, the VaR increases.

Explanation:
The choice of return calculation method can have a significant impact on the VaR estimate, particularly for large changes in price. The log method is generally preferred for VaR calculations because it better captures the compounding effect of returns, particularly in the case of large price changes. In contrast, the arithmetic method can underestimate the risk of large price changes because it assumes that returns are additive. This can result in a higher probability of large losses and a lower VaR estimate when using the arithmetic method compared to the log method.