Mathematics used by the <u>FourBarLinkage</u> package

Referring to fig. 1 the four-bar linkage can be described by four vectors namely

$\vec{a}$	Left rotating bar
$\vec{b}$	Floating bar
$\vec{c}$	Right rotating bar
$\vec{d}$	Ground fixed bar

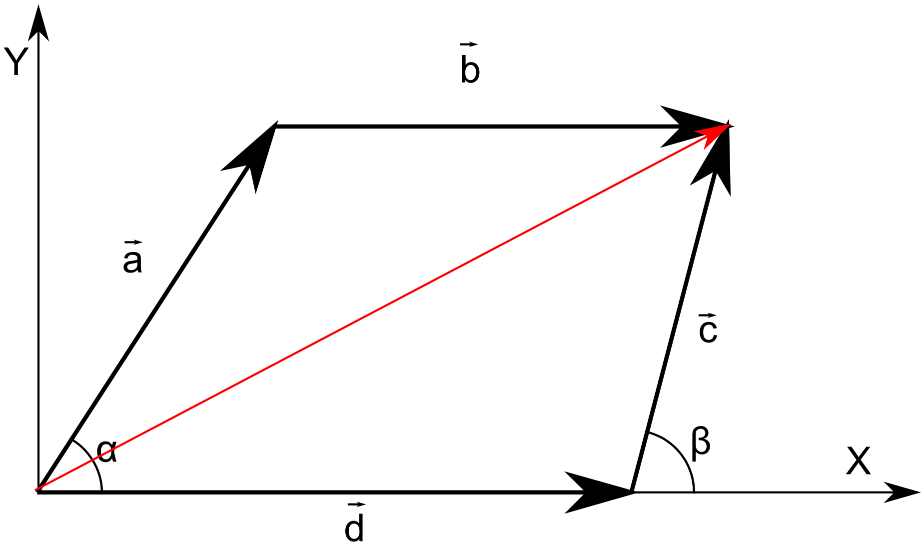

fig. 1: Four-bar linkage geometry

Its configuration is completely defined by the angles α and β formed by the rotating bars with the positive direction of the X axis.
The linkage, though, has only one degree of freedom, therefore α and β are not independent and the first aim of this document is to find a mapping $\alpha \rightarrow \beta$.
The aforementioned vectors must satisfy the vector relation $\vec{a}+\vec{b}=\vec{d}+\vec{c}$ that can be solved with respect to the floating bar $\vec{b}=\vec{d}+\vec{c}-\vec{a}$.
Considering that

$$\vec{a}=\begin{bmatrix} a\cos\alpha \\ a\sin\alpha \end{bmatrix};\vec{c}=\begin{bmatrix} c\cos\beta \\ c\sin\beta \end{bmatrix};\vec{d}=\begin{bmatrix} d \\ 0 \end{bmatrix}$$

this equation can be written as

$$\vec{b}=\begin{bmatrix} d \\ 0 \end{bmatrix}+\begin{bmatrix} c\cos\beta \\ c\sin\beta \end{bmatrix}-\begin{bmatrix} a\cos\alpha \\ a\sin\alpha \end{bmatrix}=\begin{bmatrix} d+c\cos\beta-a\cos\alpha \\ c\sin\beta-a\sin\alpha \end{bmatrix}$$

where $a=|\vec{a}|, b=|\vec{b}|, c=|\vec{c}|, d=|\vec{d}|$.
The modulus[1] of $\vec{b}$ is $\sqrt{(d+c\cos\beta-a\cos\alpha)^2+(c\sin\beta-a\sin\alpha)^2}$ and squaring both sides of the equation yields $b^2=(d+\cos\beta-a\cos\alpha)^2+(c\sin\beta-a\sin\alpha)^2$.
Some elementary algebraic manipulations yield

$$a^2-b^2+c^2+d^2-2ad\cos\beta+2cd\cos\beta-2ac\cos(\alpha-\beta)=0$$

that cannot easily be solved with respect to angle α therefore a numerical procedure is used by means of the **fsolve()** function in the octave[2] language.
Actually **fsolve()** is a minimizer that numerically approximates to zero the function

1 By definition the modulus of a vector is positive, therefore the square root is taken with the positive determination only and squaring both sides of the equation is possible.

2 Octave is an open source language targeted at numerical computation which is designed to be compatible with Matlab.

$y = a^2 - b^2 + c^2 + d^2 - 2\,a\,d\cos\beta + 2\,c\,d\cos\beta - 2\,a\,c\cos(\alpha - \beta)$ therefore the code is composed by the definition of the function to be minimized and the call to the minizer.

```
function y = f(beta)
    global a b c d alpha;
    y = a^2-b^2 +c^2 + d^2 - 2*a*d*cos(alpha) + 2*c*d*cos(beta) - 2*a*c*cos(alpha-beta);
endfunction
...
    [beta, fval, info] = fsolve(@f, initialApprox);
...
```

The solver is called in its simplest version, i.e. without the Jacobian as the numerical convergence in such kind of trigonometric functions is not a problem.
So the algorithm solves numerically the equation for all the angles covered by the left rotating bar. The initial approximation of β is fixed at $\pi/2$ for the first value of α and as the α values are incremented, the last β is used as initial approximation for the following iteration.

Classification of the leverages

The rotating bars can rotate completely or oscillate in an angle range. The literature calls crank a bar that can rotate completely, rocker a bar that oscillates. The angle range of a rocker is always symmetrical in respect to the fixed ground bar (X axis in fig. 1).
The extreme conditions for both angles are summarized in fig. 2.

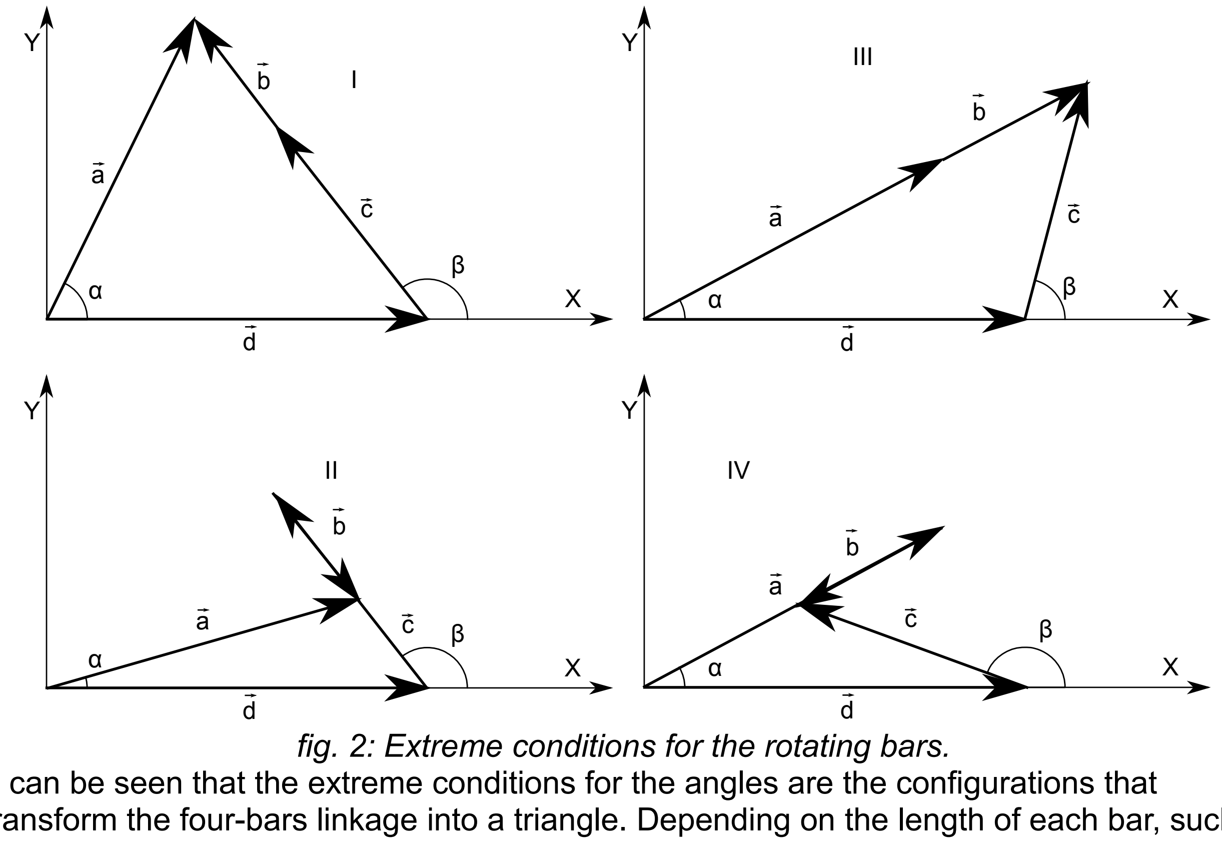

fig. 2: Extreme conditions for the rotating bars.

It can be seen that the extreme conditions for the angles are the configurations that transform the four-bars linkage into a triangle. Depending on the length of each bar, such triangles are possible or not.

The situation is the following.

		Triangle sides		
α_{min}	I	a	c-b	d
α_{max}	II	a	c+b	d
β_{min}	III	a+b	c	d
β_{max}	IV	a-b	c	d

For a triangle to exist the sum of the length of two sides must be higher than the other side. Mathematically it can also be verified using the Heron's[3] formula to compute the area of a triangle: if la, lb, lc is the length of each side and $p=(la+lb+lc)/2$ is the semi-perimeter, the area can be computed with $A=\sqrt{p(p-la)(p-lb)(p-lc)}$.
For a triangle to exist, the radicand of this equation must be positive.
The area is also used to determine the height of the triangle with respect to the ground side needed to compute the angle itself using the **tan2()** function.

Trajectory of the a point integral to the floating bar.

The trajectory of points integral to the rotating bars are circumferences. The floating bar, on the contrary, has a complex displacement and the points integral to it move following interesting trajectories. In order to compute them, the displacement of the floating bar must be known.

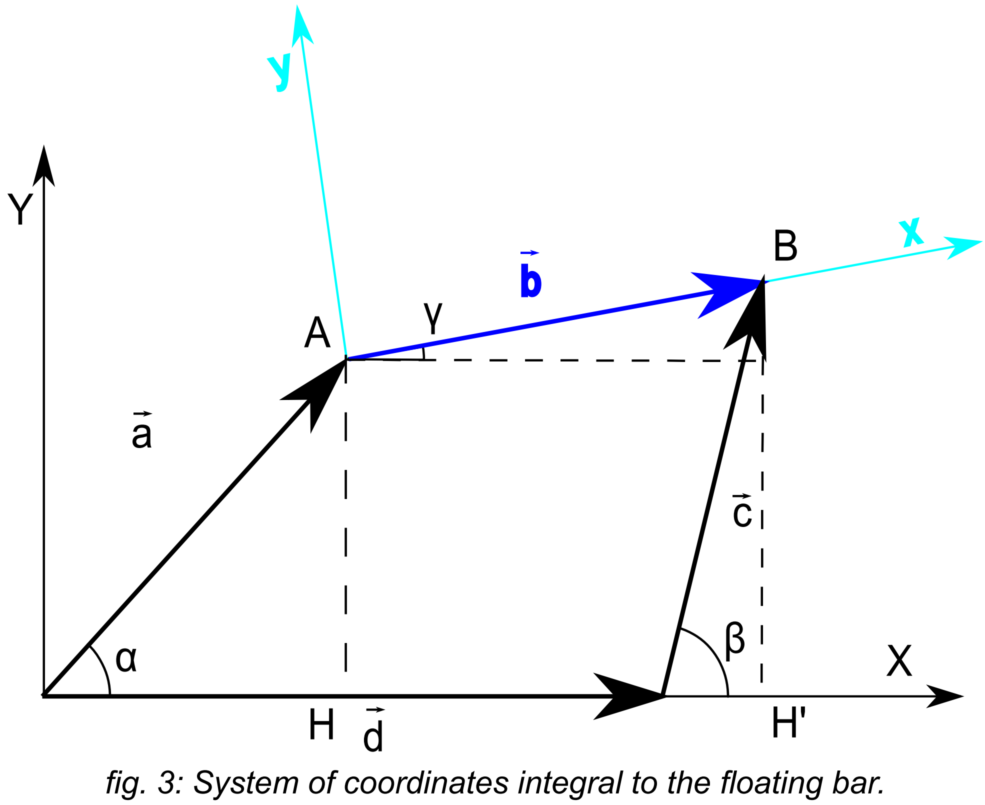

fig. 3: System of coordinates integral to the floating bar.

The angle γ of the floating bar can be computed with the following equations (ref. fig. 3)

$$\overline{AH}=a\sin\alpha$$
$$\overline{BH'}=c\sin\beta$$
$$\overline{HH'}=d+c\cos\beta-a\cos\alpha$$

therefore $\tan\gamma=\frac{\overline{BH'}-\overline{AH}}{\overline{HH'}}=\frac{c\sin\beta-a\sin\alpha}{d+c\cos\beta-a\cos\alpha}$ and evenutally

3 https://en.wikipedia.org/wiki/Heron%27s_formula

$$\gamma = atan2(c\sin\beta - a\sin\alpha, d + c\cos\beta - a\cos\alpha)$$

The (x,y) coordinate of the point on the moving reference is transformed in the absolute coordinates (X,Y) of the ground reference using basic matrix algebra.