

## HOMEWORK ASSIGNMENT 3

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(1) Problem A

See code.

(2) Problem B

$$r = 3, p = 0.4$$

$$EN = r \cdot \frac{1}{p} = 7.5$$

$$Var(N) = r \cdot \frac{1-p}{p^2} = 11.25$$

$$\text{Standard Deviation of } B = \sqrt{Var(N)} = 3.35$$

$$pN(k) = \binom{k-1}{r-1} \cdot (1-p)^{k-r} \cdot p^r$$

$$skewness = E[(N - 7.5)^3 / 3.35^3] = E[(N - 7.5)^3 / 37.73]$$

$$= \sum_{N=3}^{\infty} [((N - 7.5)^3) / 37.73] \cdot \binom{N-1}{2} \cdot (0.4^3) \cdot (0.6^{N-3}) = 1.1925$$

(3) Problem C

(a) long run average number of rolls between wins

Let's call the number of rolls between wins  $M$ . We are to find  $E(M)$ ; if there are  $M$  rolls between consecutive wins, then in long run, every  $(M + 1)$  rolls contain a win;

let  $Avg(R)$  denotes the average of all the rolls we got between one win and another;

$$E(M + 1) = E\left(\frac{8}{Avg(R)} + 1\right) = \frac{E(8)}{E(Avg(R))} + 1$$

the expected value of  $Avg(R)$  is just  $ER$ , which is 3.5 according to textbook;

$$\text{so } E(M) = E(M + 1) - 1 = \frac{E(8)}{ER} - 1 = 8/3.5 - 1 = 2.2857 - 1 = 1.2857$$

(b) long run value of total winnings per turn.

this should be equal to  $\sum_{i=0}^5 P(\text{land on } i \text{ and win})(i + 1)$

$$P(\text{land on } i \text{ and win}) = P(\text{land on } i) \cdot P(\text{win} \mid \text{land on } i)$$

$P(\text{land on } i)$  for all  $i$  should be equal, which will be  $1/8$ .

$$P(\text{land on } 0 \text{ and win}) = (1/8) \cdot (1)$$

$$P(\text{land on } 1 \text{ and win}) = (1/8) \cdot (5/6)$$

$$\begin{aligned}
P(\text{land on 2 and win}) &= (1/8) \cdot (4/6) \\
P(\text{land on 3 and win}) &= (1/8) \cdot (3/6) \\
P(\text{land on 4 and win}) &= (1/8) \cdot (2/6) \\
P(\text{land on 5 and win}) &= (1/8) \cdot (1/6) \\
\sum_{i=0}^5 P(\text{land on } i \text{ and win})(i+1) \\
&= 1 \cdot (1/8) \cdot 1 + 2 \cdot (1/8) \cdot (5/6) + 3 \cdot (1/8) \cdot (4/6) + 4 \cdot (1/8) \cdot (3/6) + 5 \cdot \\
&\quad (1/8) \cdot (2/6) + 6 \cdot (1/8) \cdot (1/6) \\
&= 1.666667 \\
&\text{so the value should be } 1.666667
\end{aligned}$$

(c)  $ET_j$ :

if we start at square  $j$ , we need to get a total number of  $(8-j)$  to win;  
let  $S_i$  denotes the total squares we've advanced given  $i$  rolls; for example, if we rolled a 1, then a 2, then  $S_2 = 1+2 = 3$ ;  
 $P(T_j = N) = P[(S_n - 1 < (8-j)) \text{ and } (S_n \geq (8-j))]$   
for  $ET_7$ : we definitely will win in 1 roll, so  $ET_7 = 1$ ;  
for  $ET_6$ : we have  $5/6$  chance to win in 1 roll, and  $1/6$  chance that we roll a 1 first, then roll anything to win  
so  $ET_6 = 1 \cdot 5/6 + 2 \cdot 1/6 = 1.16666$   
for  $ET_5$ : we have  $4/6$  chance to win in 1 roll, and  $1/6$  chance that we enter a situation which is exactly the same as  $ET_6$ , and  $1/6$  chance we enter a situation that's exactly like  $ET_7$ ;  
so  $ET_5 = 1 \cdot 4/6 + (1 + ET_6) \cdot 1/6 + (1 + ET_7) \cdot 1/6 = 1.36111$

so we see the pattern here: if we don't make it to win from our original state, we are entering one of the other states: for example, if we start out at position 5 and didn't win, we are either entering position 6 or 7, and thus we can use the ET of those state, except that we take one more steps to arrive, so we can do  $(1 + ET_i) \cdot P(j \rightarrow i)$  where  $j$  is our original state and  $i$  is the expected new state.

consequently:

$$\begin{aligned}
ET_4 &= 1 \cdot 3/6 + (1 + ET_5) \cdot 1/6 + (1 + ET_6) \cdot 1/6 + (1 + ET_7) \cdot 1/6 = 1.58796 \\
ET_3 &= 1 \cdot 2/6 + (1 + ET_4) \cdot 1/6 + (1 + ET_5) \cdot 1/6 + (1 + ET_6) \cdot 1/6 + \\
&\quad (1 + ET_7) \cdot 1/6 = 1.85262 \\
ET_2 &= 1 \cdot 1/6 + (1 + ET_3) \cdot 1/6 + (1 + ET_4) \cdot 1/6 + (1 + ET_5) \cdot 1/6 + \\
&\quad (1 + ET_6) \cdot 1/6 + (1 + ET_7) \cdot 1/6 = 2.16139 \\
ET_1 &= (1 + ET_2) \cdot 1/6 + (1 + ET_3) \cdot 1/6 + (1 + ET_4) \cdot 1/6 + (1 + ET_5) \cdot \\
&\quad 1/6 + (1 + ET_6) \cdot 1/6 + (1 + ET_7) \cdot 1/6 = 2.52162 \\
ET_0 &= (1 + ET_1) \cdot 1/6 + (1 + ET_2) \cdot 1/6 + (1 + ET_3) \cdot 1/6 + (1 + ET_4) \cdot \\
&\quad 1/6 + (1 + ET_5) \cdot 1/6 + (1 + ET_6) \cdot 1/6 = 2.77523
\end{aligned}$$

(4) Problem D

$$EV = P(U = 1)E(V|U = 1) + P(U = 0)E(V|U = 0)$$

$$EV = p \cdot 1 + (1 - p) \cdot (1 + EV) \text{ (plus one failed time)}$$

$$EV = p + 1 - p + EV - p \cdot EV$$

$$0 = 1 - p \cdot EV$$

$$p \cdot EV = 1$$

$$EV = \frac{1}{p}$$