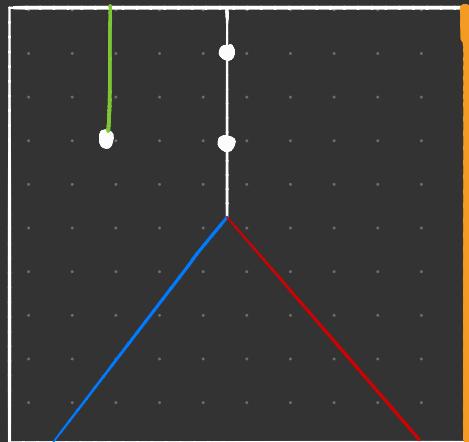


Defects in topological phases.

- 1907. 06692 w/ D. Barter
- 1901 . 08069 "
- 1810 . 09469 w/ D. Barter & C. Jones .
- 1806 . 01279 " "



String net models

- Playing with pictures.

$$|\text{ground state}\rangle \sim | \rangle + | \textcircled{0} \rangle + | 0_0 \rangle + | \textcircled{0}_0 \rangle + \dots$$

$\{ \begin{array}{c} / \\ a \\ \backslash \end{array} \}$

1 :



: V_{ab}^c



=



?



Example : $\text{Vec}(\mathbb{Z}_2)$ (Toric code)

String types : { ,  }

Fusion rules { , , ,  }

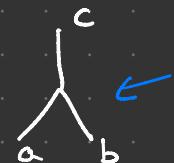
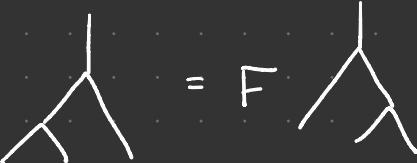
$$|\Psi_1\rangle = \begin{array}{c} \text{square frame} \\ + \\ \text{square frame with blue circle} \\ + \\ \text{square frame with two blue circles} \\ + \\ \text{square frame with one blue circle} \\ + \dots \end{array}$$

$$|\Psi_2\rangle = \begin{array}{c} \text{square frame with blue wavy line} \\ + \\ \text{square frame with blue circle and blue wavy line} \\ + \\ \text{square frame with blue circle and blue wavy line} \\ + \\ \text{square frame with blue wavy line} \\ + \dots \end{array}$$

$$\begin{array}{ccc} \text{string type 1} & = & \text{string type 2} \\ \text{string type 2} & = & \text{string type 1} \end{array}$$

(Skeletal) Fusion category

- Finite set of simple objects : $\text{obj} = \{1, a, b, c, \dots\}$

- Fusion rules : $a \otimes b = \sum N_{ab}^c c$ \equiv  $\leftarrow V_{ab}^c$
- Local relations : 

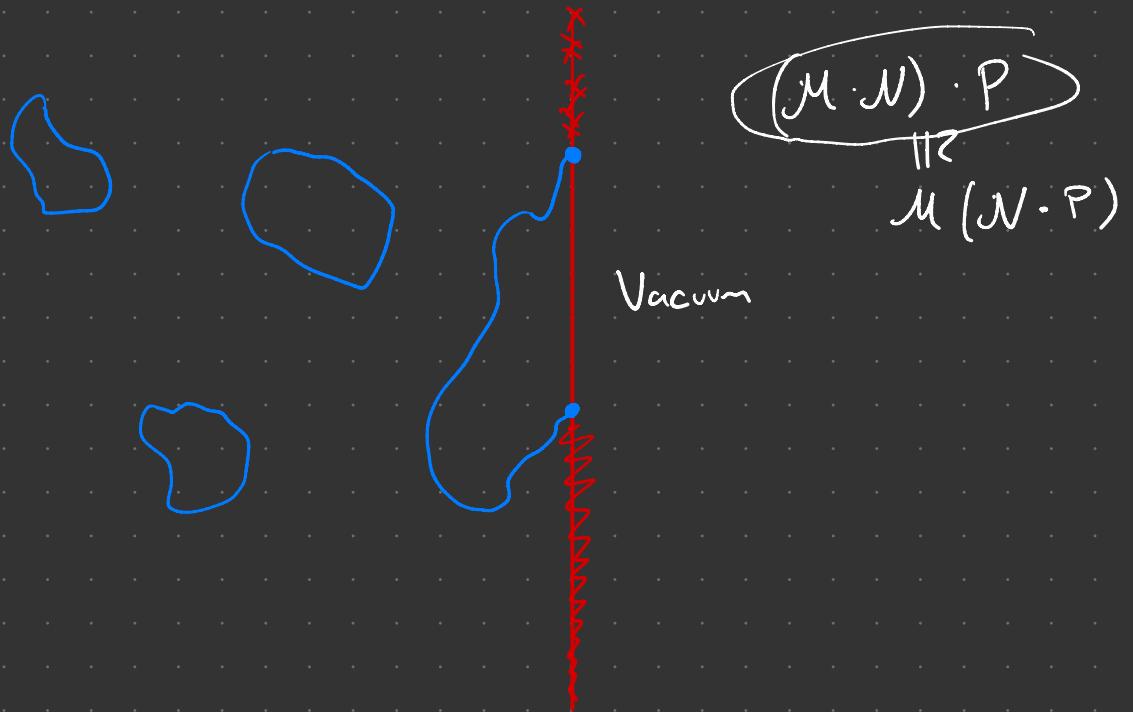
$a \otimes b \underset{\sim}{\approx} b \otimes a$



$$F^2 = F^3$$

Boundaries

Toric code example ($Vee(\mathbb{Z}_2)$)



Left $\overset{FC \text{ bulk.}}{\leftarrow} \mathcal{C}$ -module category : $\mathcal{C} \curvearrowright \mathcal{M}$

- Set of objects $\{m, n, \dots\}$

- Left \mathcal{C} action $a \triangleright m = \sum \tilde{N}_{am}^n m$



- = L

$$L^2 = F L F$$

Solutions

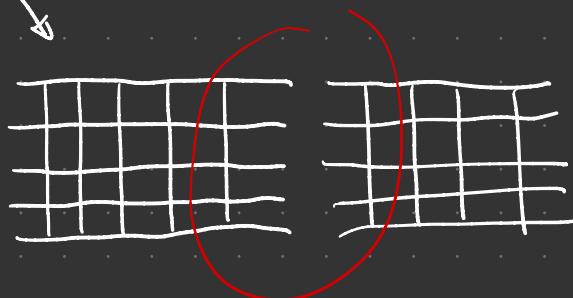
- $\underline{\mathcal{M}} = \underline{\mathbb{C}}$ forgetting the fusion structure.

$$\text{Toric code } (\text{Vec } \mathbb{Z}_2) = \{1, g \mid g \otimes g = 1\}.$$

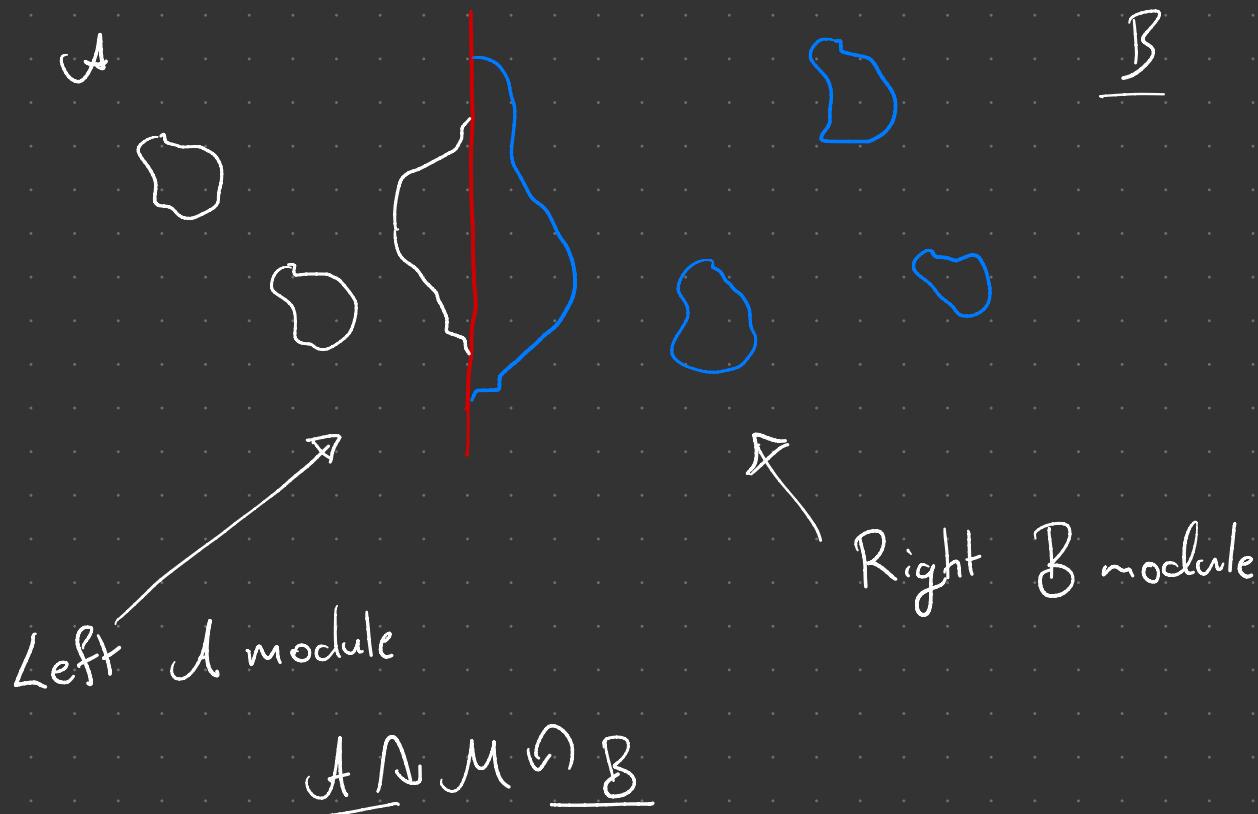
$$\mathcal{M}_{\text{smooth}} = \{\tilde{1}, \tilde{g}\} \quad g \triangleright \tilde{g} = \tilde{1}$$

$$\mathcal{M}_{\text{Routh}} = \{*\}$$

Lattice
model :



Interfaces



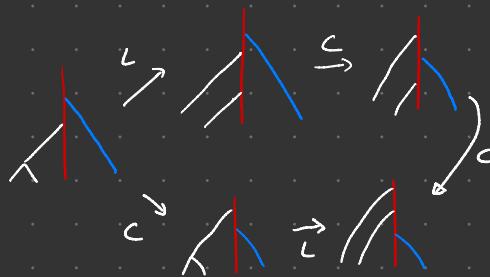
Bimodule category

ANMOS

$$= \begin{matrix} & & \\ & C & \\ & & \end{matrix}$$

$$C^L L = LC$$

$$C^T R = R C$$

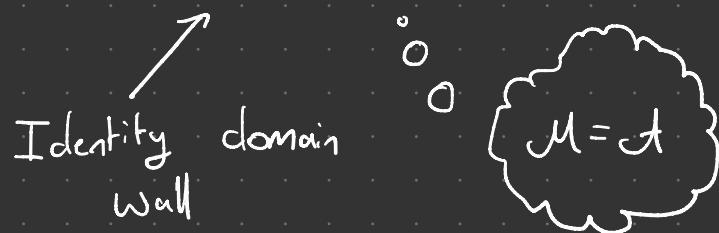


Solution $\rightarrow A \times B^{\text{op}}$

Always exist : $T : \{(a, b) \mid a \in A, b \in B\}$.



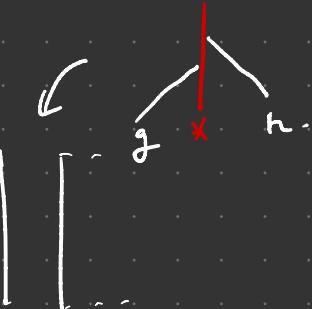
if $A = B$ $X_1 : \{a \mid a \in A\}$.



Example : $\text{Vec } \mathbb{Z}_2$

$$T = \left[\begin{array}{c|c} & \\ \hline & \end{array} \right], \quad L = \left[\begin{array}{c|c} & \\ \hline & \end{array} \right]$$

$$R = \left[\begin{array}{c|c} & \\ \hline & \end{array} \right], \quad F_0 = \left[\begin{array}{c|c} & \\ \hline & \end{array} \right]$$



$$X_1 : \left[\begin{array}{c|c} & \\ \hline & \end{array} \right]$$

$$F_1 : \left[\begin{array}{c|c} * & \\ \hline * & \end{array} \right] = (-1) \quad \left[\begin{array}{c|c} * & \\ \hline * & \end{array} \right]$$

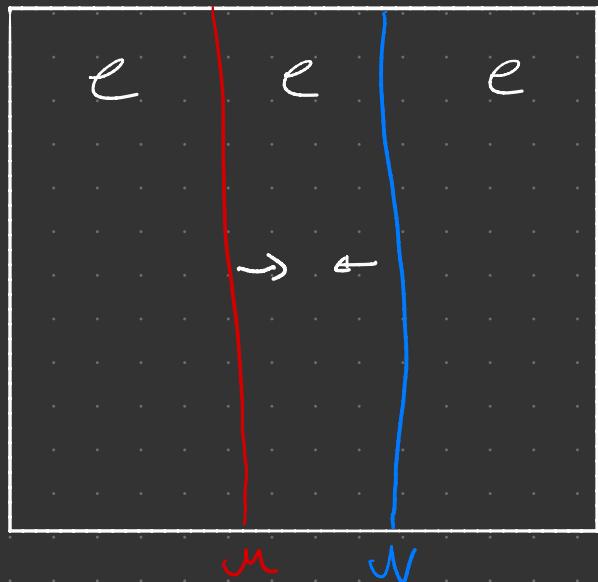
"Fusing" Domain walls



Brauer Picard Ring

ANALOGY

$A^{\text{PGL}} \otimes_B N^{\text{G}} e$



$e^{\text{PGL}} \otimes_e N^{\text{G}} e$

Composite
domain wall

$$M \otimes_e N = \bigoplus_P N_{\mu\nu}^P e^P$$

Aside : Representations of a category.

$\text{ob} : \mathbb{C}^d$

$F : \mathcal{C} \longrightarrow \text{Vec}$

$\text{Hom}(v, w)$

$L : V \rightarrow W$.

$F_a : V_a$

$F(f : a \rightarrow b) : L_f : V_a \rightarrow V_b$

$F(f) \circ F(g) = F(f \circ g)$

$$\underline{G} : \text{ob } \underline{G} = \{ * \}$$

$$\text{Hom}(*, *) = \mathbb{C}\underline{G}$$

$$\begin{array}{ccc} * & \xrightarrow{g} & * \\ h \circ g = hg & \searrow & \downarrow \text{id} \\ & * & \end{array}$$

$$F : \underline{G} \rightarrow \text{Vec}$$

$$F_* = \mathbb{C}^d$$

$$F(g) \circ F(h) = F(g \circ h) = F(gh)$$

||

$$M_g M_h$$

||

$$M_{gh}$$

$$\text{Category of rep's} : [e, \text{Vec}]$$

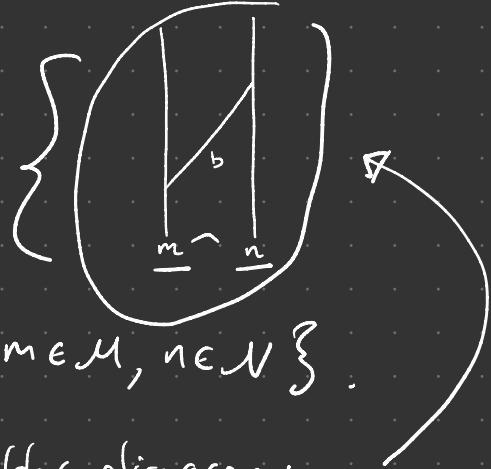
obj $M \otimes_{\mathcal{B}} N$: $\left\{ \begin{matrix} (m, n, \eta : (m \triangleleft b, n) \cong (m, b \triangleright n)) \\ \uparrow \quad \uparrow \end{matrix} \right\}$.

$$M \otimes_{\mathcal{B}} N \simeq \underline{\text{Fun}_{\mathcal{B}}(M^{\text{op}}, N)}$$

Ladder category :

objects $\{(m, n) | m \in M, n \in N\}$.

morphisms : Ladder diagrams



Thm: $M \otimes_{\mathcal{B}} N \simeq [\text{Lad}(M, N)^*, \text{Vec}] \simeq \text{Kar}(\text{Lad}(M, N))$

Proof : See 1806.01279

Computing representations: Karoubi envelope

Objects: $(A\text{el}, i: A \rightarrow A)$.

Kas(\mathcal{C}):

SII

$[\mathcal{C}^{\text{op}}, \text{Vec}]$

$$i \circ i = i$$

$$\text{f} \in \mathcal{E} \quad i \circ f = f = f \circ i'$$

$$f: (A, i) \rightarrow (B, i')$$

In summary, we can compute the "fusion" of domain walls
by finding idempotent ladders.

$$\text{Example : } \text{Vee TL}_2. \quad F_1 \otimes F_1 \cong X_1 \xrightarrow{\tilde{e}} \begin{array}{c} | \\ \tilde{1} \\ | \end{array} \quad \begin{array}{c} | \\ \tilde{g} \\ | \end{array}$$

$$\begin{array}{c} * \\ \diagup \\ \diagdown \\ * \end{array} = (-1) \begin{array}{c} | \\ \diagup \\ \diagdown \end{array}$$

$$I_{\pm} = \frac{1}{2} \left(\begin{array}{c|c} | & | \\ \hline \pm & \mp \end{array} \right).$$

$$M_1 \\ M_g : M_g^2 = M_1$$

$$\begin{array}{c} * \\ | \\ * \end{array}$$

$$\begin{array}{c} | \\ \diagup \\ | \\ | \end{array} + \begin{array}{c} | \\ \diagup \\ | \\ | \end{array}$$

$$\begin{array}{c} | \\ \diagup \\ * \\ | \\ * \end{array}$$

$$: I_- \rightarrow I_+$$

$$g \triangleright I_- = I_+$$

	T	L	R	F_0	X_1	F_1
T	$2T$	T	$2R$	R	T	R
L	$2L$	L	$2F_0$	F_0	L	F_0
R	T	$2T$	R	$2R$	R	T
F_0	L	$2L$	F_0	$2F_0$	F_0	L
X_1	T	L	R	F_0	X_1	F_1
F_1	L	T	F_0	R	F_1	X_1

X_1 F_1
 F_1 X_1



Brauer-Picard Group

Point Defects

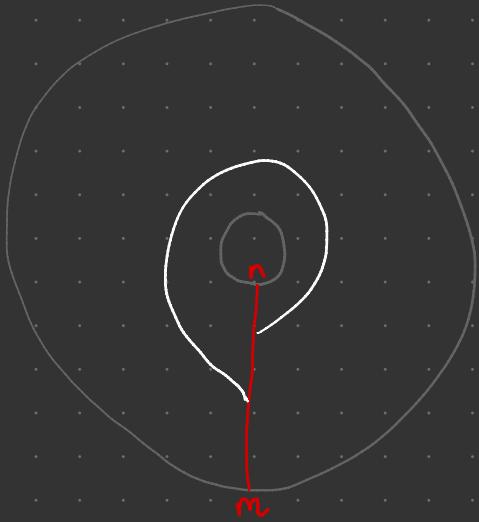
So far, we modified the rules at a line. What about at a point?

- Argons
- "twists"
- :



Tube category

$\text{Hom}(m, n)$:



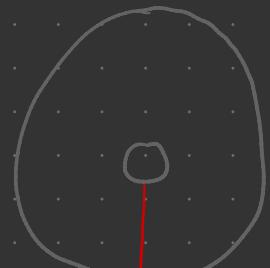
composition



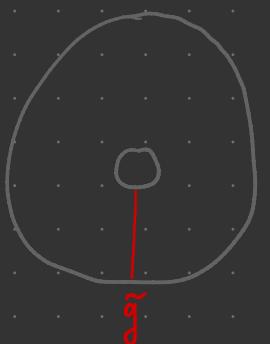
$L' =$



Example. $\text{Vec}(\mathbb{Z}_2)$



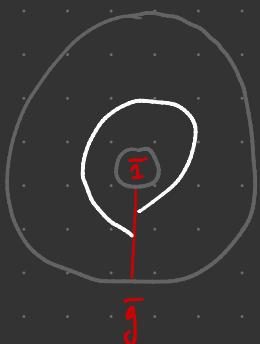
α



\tilde{g}



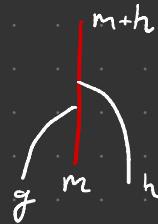
\tilde{i}



\tilde{g}

$L:$

$\{\tilde{i}, \tilde{g}\}$



$\Rightarrow 1$ type of point excitation

Toric code

Physically:



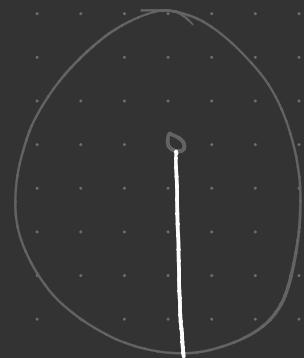
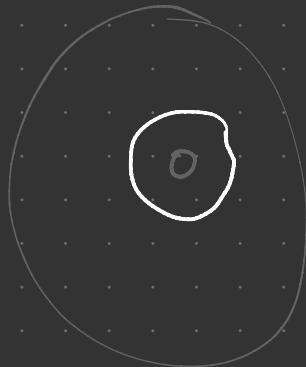
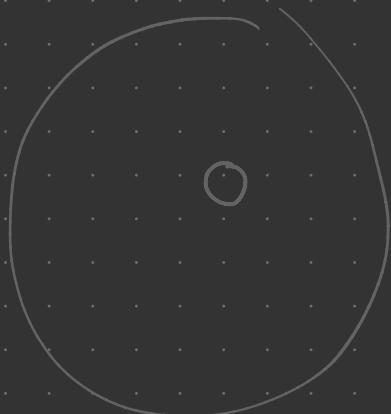
Kar (Tub) : ob (m, n)

Mor



m

$\text{Vec}(\mathbb{Z}_2)$



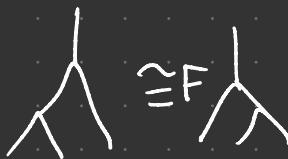
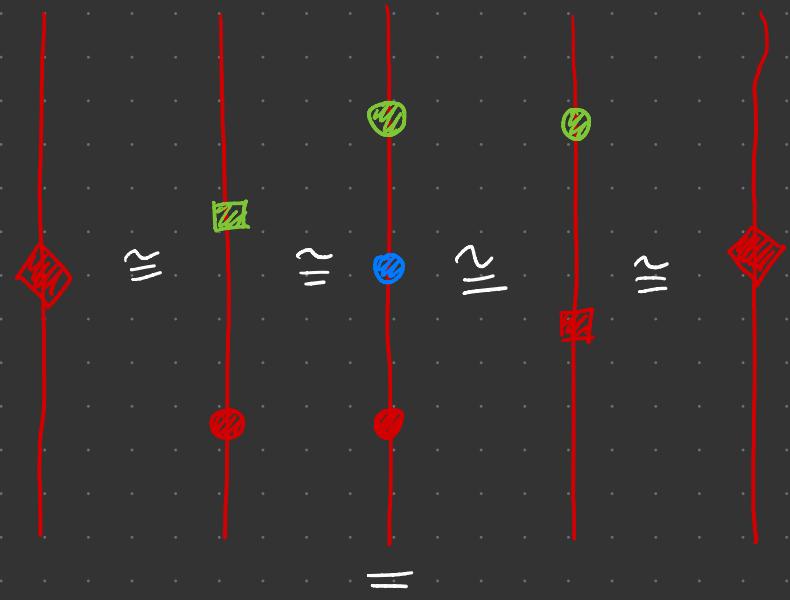
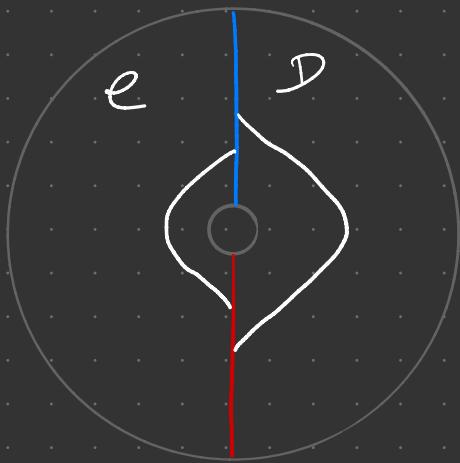
$$1 = \frac{1}{2} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right)$$

$$e = \frac{1}{2} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} - \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right)$$

$$m = \frac{1}{2} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right)$$

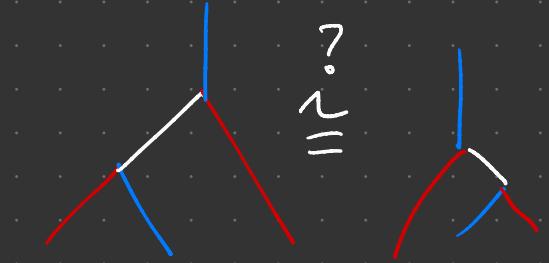
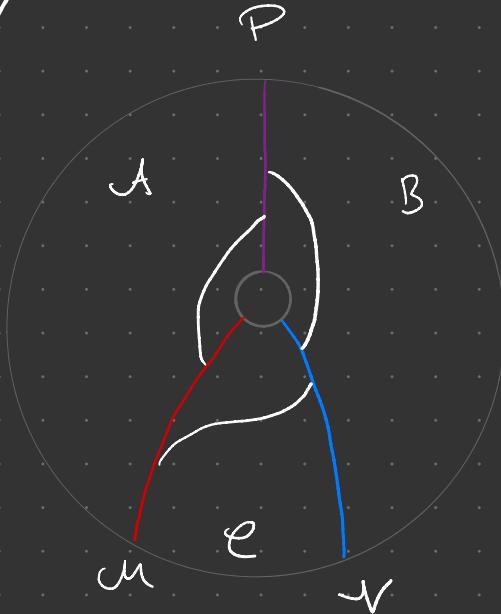
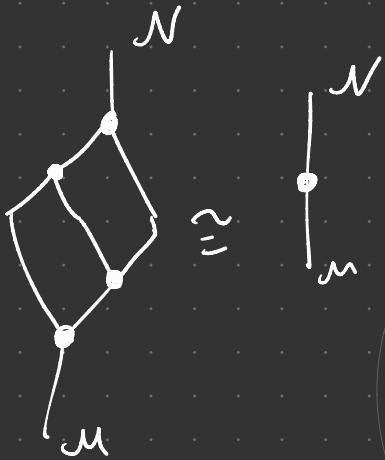
$$em = \frac{1}{2} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} - \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right)$$

More tube categories.



End(\mathcal{U})

More tube categories.



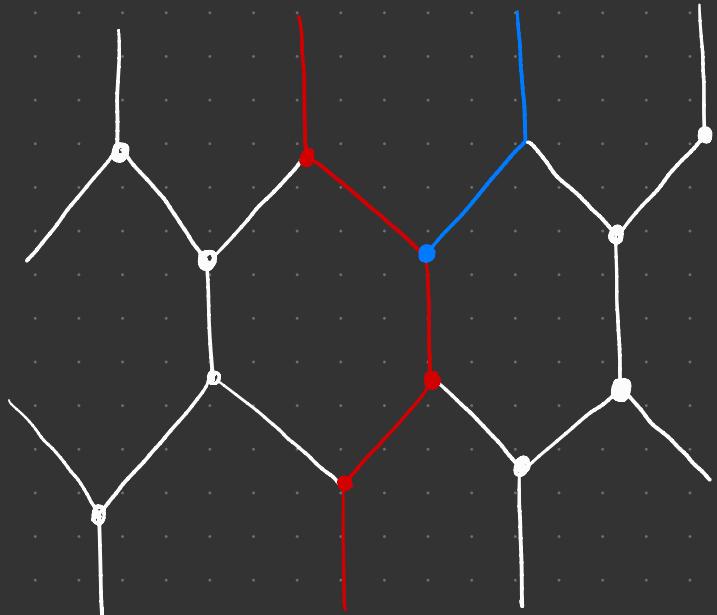
$$\rightsquigarrow F: \mathcal{M}_e @ N \rightarrow P$$

{

Gauging domain walls

String net models.

Draw network of bimodules



Hamiltonian designed
to project to
a given point defect

Summary

- Generalised tube algebras \leadsto Point defects.
- Representations of categories, via Karoubi gives way to compute many properties.

1907.06692

1901.08069

1810.09469

1806.01279

Thank you!