Theory of Computation
Homework 3

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Exercice 1

Let $G = (V, E)$ be an undirected graph. A set of vertices $D \subseteq V$ is *dense* if every vertex $v \in V \setminus D$ has a neighbour in D (that is, $(u,\ v) \in E$ for some $u \in D$). We define :

$$\textsc{DenseSet} \ = \ \{\, \langle G,\ k\rangle \ :\ G \text{ has a dense set of size at most } k \,\}.$$

Prove that DenseSet is **NP**-complete.

Solution To prove DenseSet is **NP**-complete, we will first show that DenseSet is in **NP**. After that, we will show it is **NP**-hard by reducing the VertexCover problem (known to be **NP**-complete) to DenseSet.

[NP-membership] We give a poly-time verifier for DenseSet. It will take (G, k) as the problem instance and a set C as the certificate.
The algorithm will check if:

1. $|C| \leq k$
2. C is a subset of V
3. For every vertex v in V, v is either in C or is adjacent to a vertex in C.

This can be done in polynomial time, that is $O(V{+}E)$.

[NP-hardness] We will carry out a polynomial-time mapping reduction from VertexCover to DenseSet, written as VertexCover $\leq_p$ DenseSet.

Reduction: Given a graph $G = (V, E)$ and a number $k \in \mathbb{N}^\star$, we design a poly-time computable function f such that $\langle G, k\rangle \in$ VertexCover iff $f(\langle G, k\rangle) \in$ DenseSet. Let $f(\langle G, k\rangle) = \langle G', k'\rangle$, where $G' = (V', E')$, and V', E', and k' are constructed as follows:

- V' : add all the vertices of V.

- E' : add all the edges of E, and for every edge $\{u, v\} \in E$, create and add a new vertex to V', that is connected to u and v.
- $k' = k+n$, where n denotes the number of isolated vertices (that is incident to no edges) in G. Because if a vertex is isolated, the only way it gets dominated is by including it in the dense set.

Note that every step can be performed in polynomial time.

Claim: G has a vertex cover of size k iff G' has a dense set of size k'.

($\Rightarrow$) If $\langle G, k\rangle \in$ VertexCover, then for every edge $\{u, v\} \in E$, either u or v or both are in the vertex cover. Indeed, these vertices are connected to some of the elements in the vertex cover. Same goes for the newly created vertices, since the vertex is adjacent to u and v. Finally, if there were any isolated vertices in G, those are added (if not already) inside the vertex cover. This way, all the edges are covered by the vertex cover, and its set of vertices of size $k' = k + n$ forms a dense set in graph G'. Therefore, if G has a vertex cover of size k, then G' has a dense set of size k'.

($\Leftarrow$) If $\langle G', k'\rangle \in$ DenseSet, we first remove the n isolated vertices in the dense set. Then for every newly created vertex in the dense set, we replace it by one of the two regular ones, and we still dominate the same vertices. We argue that the resulting set V, after this transformation, is a vertex cover of size k. Indeed, it holds, since otherwise it would have meant that an edge is not covered, contradicting the initial hypothesis that we have a dense set.

This completes our reduction and

$$\langle G, k\rangle \in \textsc{VertexCover} \;\Leftrightarrow\; \langle G', k'\rangle \in \textsc{DenseSet}.$$

As DenseSet is in **NP** and it is **NP**-hard, DenseSet is **NP**-complete.

Exercice 2

We denote by $\mathbb{Z}$ the set of integers and $\mathbb{N} = \{1, 2, 3, \dots\}$ the set of positives integers. We consider the following two variants of SubsetSum:

$$\begin{aligned}
\textsc{SubsetSum}^+ &= \{\, \langle X,\, s\rangle \,:\, X \subseteq \mathbb{N} \text{ is a multiset and some subset of } X \text{ sums to } s \,\},\\
\textsc{SubsetSum}^\pm &= \{\, \langle X,\, s\rangle \,:\, X \subseteq \mathbb{Z} \text{ is a multiset and some subset of } X \text{ sums to } s \,\}.
\end{aligned}$$

We describe a *direct* reduction $\textsc{SubsetSum}^\pm \leq_p \textsc{SubsetSum}^+$.

Solution Let f be a computable function in polynomial time that procedes as follows:
On input $\langle X,\, s\rangle$:

1. Compute $A = 1 + \sum_{i=1}^{n} |a_i|$, where n is the cardinal of X.
2. Let $X' = \bigcup_{i=1}^{n}\{a_i + A\} \,\cup\, \bigcup_{i=1}^{n}\{A\}$
3. Let $s' = s + nA$
4. From there, return $\langle X',\, s'\rangle$

Claim: The set of integers X has a subset that sums precisely to s iff the set of positive integers X' has a subset that sums precisely to s'.

($\Rightarrow$) Suppose $\langle X,\, s\rangle \in \textsc{SubsetSum}^\pm$.
Hence s can be written as: $s = \sum_{i=1}^{m} x_i$, for some $x_i \in X$, $m \leq n$.
This implies that: $\sum_{i=1}^{m}(x_i + A) = x + mA$.
Note that $x_i + A > 0$ as $A = 1 + \sum_{i=1}^{n} |a_i| \geq |x_i| + 1 > |x_i|$ for all $x_i \in X$.
Then we have that:
$\sum_{i=1}^{m}(x_i + A) + \sum_{i=1}^{n-m} A = s + mA + (n-m)A = x + nA = s'$
Here note that $m \leq n \Rightarrow n - m \geq 0$.
We have that $x_i + A \in X'$ and we also have the number A shows n times in X', which makes it possible to choose $(n-m)$ times the number A in X' (as $n \leq m$).

Therefore, if $\langle X,\, s\rangle$ is in $\textsc{SubsetSum}^\pm$, then $\langle X',\, s'\rangle$ is in $\textsc{SubsetSum}^+$.

($\Leftarrow$) Suppose $f(\langle X,\, s\rangle) = \langle X',\, s'\rangle \in \textsc{SubsetSum}^+$.
Let $n = |X|$ and $s' = s + nA$.
We also have $X' = \{\, a_1 + A,\, \dots,\, a_n + A,\, A,\, \dots,\, A \,\}$ with A appearing n times.

Let $\{\, r_1,\, \dots,\, r_m \,\} \subset \{\, 1,\, \dots,\, 2n \,\}$ be a list of indices of X' such that:
$\sum_{i=1}^{m} X'_{r_i} = s + An$

$\Rightarrow \sum_{i=1}^{m}(b_{r_i} + A) = s + An$, where

$$b_{r_i} = \begin{cases} a_{r_i} & \text{if } 1 \leq r_i \leq n \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow \sum_{i=1}^{m} b_{r_i} + mA = s + An \qquad (1)$

Note that the sum, denoted z, of any subset of S is such that $-(A-1) \leq z \leq A-1 \Rightarrow -A+1 \leq \sum_{i=1}^{m} b_{r_i} \leq A-1$. We also have that $-A+1 \leq s \leq A-1$.

And as $\langle X', s' \rangle$ is in SubsetSum$^+$, in equation (1), n must be equal to m. If it was not the case, then it wouldn't be possible to have b_{r_i} and s such that the equation (1) holds.

$\Rightarrow n = m \Rightarrow \sum_{i=1}^{m} b_{r_i} + mA = \sum_{i=1}^{n} b_{r_i} + nA$

$\Rightarrow \sum_{i=1}^{n} b_{r_i} + nA = s + An$

$\Rightarrow \sum_{i=1}^{n} b_{r_i} = s$

We can just remove from the sum the values of b_{r_i} that are 0.
Thus we get that $\sum_{i=1,\, 1 \leq r_i \leq n}^{n} a_{r_i} = s$.

Therefore, if $f(\langle X, s \rangle)$ is in SubsetSum$^+$, then $\langle X, s \rangle$ is in SubsetSum$^\pm$.

This completes our reduction and SubsetSum$^\pm \leq_p$ SubsetSum$^+$.