Homework II, Theory of Computation 2022

Submission: **The deadline for Homework 2 is 23:59 on 28 April.** Please submit your solutions on Moodle. Typing your solutions using a typesetting system such as LaTeX is strongly encouraged! If you must handwrite your solutions, write cleanly and with a pen. Messy and unreadable homeworks will not be graded. No late homeworks will be accepted.

Writing: Please be precise, concise and (reasonably) formal. Keep in mind that many of the problems ask you to provide a proof of a statement (as opposed to, say, just to provide an example). Therefore, make sure that your reasoning is correct and there are no holes in it. A solution that is hard/impossible to decipher/follow might not get full credit (even if it is in principle correct). You do not need to reprove anything that was shown in the class—just state clearly what was proved and where.

Collaboration: These problem sets are meant to be worked on in groups of 3–5 students. Please submit only one writeup per team—it should contain the names of all the students. You are strongly encouraged to solve these problems by yourself. If you must, you may use books or online resources to help solve homework problems, but you must credit all such sources in your writeup and you must never copy material verbatim. Even though only one writeup is submitted, it is expected that each one of the team members is able to fully explain the solutions if requested to do so.

Grading: Each of the two problems will be graded on a scale from 0 to 5.

Warning: Your attention is drawn to the EPFL policy on academic dishonesty. In particular, you should be aware that copying solutions, in whole or in part, from other students in the class or any other source without acknowledgement constitutes cheating. Any student found to be cheating risks automatically failing the class and being referred to the appropriate office.

Homework 2

1 Consider the following two languages

$$\begin{aligned} L_1 &= \{ \langle D, D' \rangle : D \text{ and } D' \text{ are DFAs and } L(D) \subseteq L(D') \}, \\ L_2 &= \{ \langle M \rangle : M \text{ is a TM and } L(M) \text{ is infinite} \}. \end{aligned}$$

Classify each of these languages as one of: decidable, undecidable but recognisable, unrecognisable. Justify your answer with proofs.

Solution: L_1 is decidable. Note that for any two sets A and B we have $A \subseteq B \iff A \cap \overline{B} = \emptyset$. We can then reformulate L_1 as

$$\begin{aligned} L_1 &= \{ \langle D, D' \rangle : D \text{ and } D' \text{ are DFAs and } L(D) \cap \overline{L(D')} = \emptyset \} \\ &= \{ \langle D, D' \rangle : D \text{ and } D' \text{ are DFAs and } L(D \cap \overline{D'}) = \emptyset \}. \end{aligned}$$

From Lecture 2, we know that regular languages are closed under complementation and intersection. From Lecture 5, we know that the language

$$E_{DFA} = \{ \langle D'' \rangle : D'' \text{ is a DFA and } L(D'') = \emptyset \}$$

is decidable. Hence, to decide L_1 we can build $D'' = D \cap \overline{D'}$ and check if its language is empty by running the algorithm that decides E_{DFA}.

L_2 is unrecognizable. To prove it, we show that $\overline{A_{TM}} \leq_m L_2$. Suppose towards contradiction that L_2 is recognizable and let T_{L_2} be a Turing machine recognizing L_2. We now want to construct a reduction F that takes as input a pair $\langle T, w \rangle$ (where T is a Turing machine and w is a string) and returns a description of a Turing machine $F(\langle T, w \rangle) = \langle M \rangle$ such that:

(1) if $\langle T, w \rangle \in \overline{A_{TM}} \implies \langle M \rangle \in L_2$,

(2) if $\langle T, w \rangle \in A_{TM} \implies \langle M \rangle \in \overline{L_2}$.

The following construction for M satisfies the two properties above:

– input: x,

– simulate T on w for $|x|$ steps,

– if T accepts, reject; otherwise accept.

Note that (1) is satisfied since since if T rejects or loops on w, then M accepts all strings; (2) holds because if T accepts w, then M only accepts strings s such that $|s|$ is less than the running time of T on w. Now, if we use F together with T_{L_2} we can then recognize $\overline{A_{TM}}$, which is a contradiction. Hence, L_2 is unrecognizable.

2 Let A and B be two disjoint languages (that is, $A \cap B = \emptyset$). Say that a language C *separates* A and B iff $A \subseteq C$ and $B \subseteq \overline{C}$.

2a Suppose that A and B are co-recognisable (that is, $\overline{A}$ and $\overline{B}$ are recognisable). Show that there is a decidable language C that separates A and B.

2b Define two disjoint languages by

$$
\begin{aligned}
A &= \{\langle M, w\rangle : M \text{ is a TM and } M \text{ accepts } w\},\\
B &= \{\langle M, w\rangle : M \text{ is a TM and } M \text{ rejects } w\}.
\end{aligned}
$$

Prove that there does not exist any decidable language C that separates A and B.

Solution:

2a) Since A, B are co-recognisable, $\overline{A} = L(M_{\overline{A}}), \overline{B} = L(M_{\overline{B}})$ for some Turing machines $M_{\overline{A}}, M_{\overline{B}}$. Consider the Turing machine M that on input $w \in \Sigma^*$ runs $M_{\overline{A}}$ and $M_{\overline{B}}$ in parallel.

During the parallel execution:

- if $M_{\overline{A}}$ accepts then M rejects w.
- if $M_{\overline{B}}$ accepts then M accepts w.

We show that M is a decider for a language C separating A and B:

- M always halts:
  - if $w \in A$ then $M_{\overline{A}}$ rejects or loops and $M_{\overline{B}}$ accepts. Thus, M accepts.
  - if $w \in B$ then $M_{\overline{A}}$ accepts and $M_{\overline{B}}$ rejects or loops. Thus, M rejects.
  - if $w \in \overline{A} \cap \overline{B}$ then $M_{\overline{A}}$ and $M_{\overline{B}}$ accept. Thus, M halts (may accept or reject).
- $L(M)$ separates A and B:
  - $w \in A \implies w \in \overline{B} \implies M_{\overline{B}}$ accepts $w \implies M$ accepts $w \implies w \in C$
  - $w \in B \implies w \in \overline{A} \implies M_{\overline{A}}$ accepts $w \implies M$ rejects $w \implies w \in \overline{C}$

  where $A \subseteq \overline{B}, B \subseteq \overline{A}$ follows from $A \cap B = \emptyset$.

2b) Suppose towards a contradiction that A and B are separated by a decidable language C. Then we claim that the following two languages are also separated by a decidable language

$$
\begin{aligned}
A' &= \{\langle M\rangle : M \text{ is a TM and } M \text{ accepts } \langle M\rangle\},\\
B' &= \{\langle M\rangle : M \text{ is a TM and } M \text{ rejects } \langle M\rangle\}.
\end{aligned}
$$

Indeed, $C' = \{\langle M\rangle : \langle M, \langle M\rangle\rangle \in C\}$ separates A' and B', and C' is decidable since C is. Let M be a TM that decides $\overline{C'}$. Let us see what happens when M is run on its own encoding $\langle M\rangle$:

$$
\begin{aligned}
\langle M\rangle \in \overline{C'} &\iff M \text{ accepts } \langle M\rangle && \text{(Definition of } M)\\
&\iff \langle M\rangle \in A' && \text{(Definition of } A')\\
&\iff \langle M\rangle \in C' && \text{(Definition of } C' \text{ and } M \text{ never loops)}
\end{aligned}
$$

This is a contradiction.