Homework III, Theory of Computation 2022

Submission: The deadline for Homework 3 is 23:59 on Friday 27 May. Please submit your solutions on Moodle. Typing your solutions using a typesetting system such as LaTeX is strongly encouraged! If you must handwrite your solutions, write cleanly and with a pen. Messy and unreadable homeworks will not be graded. No late homeworks will be accepted.

Writing: Please be precise, concise and (reasonably) formal. Keep in mind that many of the problems ask you to provide a proof of a statement (as opposed to, say, just to provide an example). Therefore, make sure that your reasoning is correct and there are no holes in it. A solution that is hard/impossible to decipher/follow might not get full credit (even if it is in principle correct). You do not need to reprove anything that was shown in the class—just state clearly what was proved and where.

Collaboration: These problem sets are meant to be worked on in groups of 3–5 students. Please submit only one writeup per team—it should contain the names of all the students. You are strongly encouraged to solve these problems by yourself. If you must, you may use books or online resources to help solve homework problems, but you must credit all such sources in your writeup and you must never copy material verbatim. Even though only one writeup is submitted, it is expected that each one of the team members is able to fully explain the solutions if requested to do so.

Grading: Each of the two problems will be graded on a scale from 0 to 5.

Warning: Your attention is drawn to the EPFL policy on academic dishonesty. In particular, you should be aware that copying solutions, in whole or in part, from other students in the class or any other source without acknowledgement constitutes cheating. Any student found to be cheating risks automatically failing the class and being referred to the appropriate office.

Homework 3

1 Let $G = (V, E)$ be an undirected graph. A set of vertices $D \subseteq V$ is *dense* if every vertex $v \in V \setminus D$ has a neighbour in D (that is, $\{v, u\} \in E$ for some $u \in D$). Define

$$\text{DENSESET} = \{\langle G, k\rangle : G \text{ has a dense set of size at most } k\}.$$

Prove that DENSESET is **NP**-complete. In your proof, you may assume the **NP**-completeness of any of the problems discussed in class (SAT, Independent Set, Clique, Subset Sum, Vertex Cover, Set Cover, etc.).

Solution: We need to show two things: That DENSESET is in **NP** and that DENSESET is **NP**-hard.

DENSESET is in NP: The polynomial time verifier will take $(G = (V, E), k)$ as the problem instance and a set D as the certificate. The algorithm will check if:

1. $D \subseteq V$
2. $|D| \leq k$
3. For every vertex $v \in V \backslash D$, v has at least one edge to D.

The running time is $O(|V||E|)$, which is polynomial in the input.

DENSESET is NP-hard: We reduce the known **NP**-complete problem VERTEXCOVER to DENSESET.

Reduction: Given a graph $G = (V, E)$ and a number k, we define the instance $G' = (V', E')$ of DENSESET as follows: Remove any isolated vertices. For each edge $(u, v) \in E$, create a new vertex w_{uv} and add the edges (w_{uv}, u) and (w_{uv}, v). Let $V' = V \cup \{w_{uv} : (u, v) \in E\} \backslash \{\text{isolated vertices in } V\}$ and let $E' = E \cup \{(w_{uv}, u), (w_{uv}, v) : (u, v) \in E\}$. This is clearly a polynomial-time reduction.

Claim: G has a vertex cover of size k if and only if G' has a dense set of size k.

Proof of claim:

[$\Longrightarrow$]: Suppose that S is a vertex cover of size k for G. Then $D = S$ is a dense set in G'. Indeed, given a vertex $v' \in V'$, if v' is one of the "old" vertices from V, then v' must have at least one edge $(u, v') \in E$. Since $S = D$ is a vertex cover, we have that either $v' \in D$, or $u \in D$, as required. If instead $v' = w_{uv}$ is a vertex of the new type, then w_{uv} has edges to both u and v, and either $u \in D$ or $v \in D$ (because D is a vertex cover in G), so w_{uv} has an edge to an element of D.

[$\Longleftarrow$]: Suppose that D is a dense set of size k in G'. Then we can create a vertex cover S for G as follows: For each vertex $w_{uv} \in D$ of the new type, replace w_{uv} by u or v. Let D' be the resulting set. Note that D' is still a dense set in G', and the size has not increased. Moreover, we have that $D' \subseteq V$ by construction. Let $S = D'$. We claim that S is a vertex cover in G. Indeed, suppose for contradiction that there exists an edge $(u, v) \in E$ such that $u \notin S$ and $v \notin S$. Then w_{uv} does not have any neighbours in D', and $w_{uv} \notin D'$ (by construction of D'), which contradicts that D' is a dense set in G'.

2 Denote by $\mathbb{Z}$ the set of integers and by $\mathbb{N} = \{1, 2, 3, \ldots\}$ the set of positive integers. Consider the following two variants of SubsetSum (the first one of which is the version defined in class):

$$\begin{aligned}\textsc{SubsetSum}^+ &= \{\langle X, s\rangle : X \subseteq \mathbb{N} \text{ is a multiset and some subset of } X \text{ sums to } s\},\\ \textsc{SubsetSum}^\pm &= \{\langle X, s\rangle : X \subseteq \mathbb{Z} \text{ is a multiset and some subset of } X \text{ sums to } s\}.\end{aligned}$$

Describe a *direct* reduction $\textsc{SubsetSum}^\pm \leq_p \textsc{SubsetSum}^+$. That is, your reduction must directly transform an input of $\textsc{SubsetSum}^\pm$ to an input of $\textsc{SubsetSum}^+$. You are not allowed to use the Cook–Levin theorem or the knowledge that $\textsc{SubsetSum}^+$ is **NP**-complete.

Solution: The idea is to increase all members of X by the same large number A to make the set positive—some care must however be taken to offset s accordingly and to do so we introduce enough zeroes to get a hold on the *size* of the subset. We now describe the reduction more formally. The reduction first removes all zeroes from X (this doesn't change the problem) and ensure that $s \geq 0$ (if not, everything is multiplied by -1). Then, the reduction computes the sum $P = \sum_{x \in X : x \geq 0} x$. If $P < s$, there is no solution and the reduction outputs anything not contained in $\textsc{SubsetSum}^+$ e.g. $\langle\{1\}, 2\rangle$. If $P = s$, there is a solution and the reduction outputs something in $\textsc{SubsetSum}^+$, e.g. $\langle\{1\}, 1\rangle$. If $P > s$, the reduction computes $A = \sum_{x \in X} |x|$ and proceeds in four steps:

1. Add 0 with multiplicity $|X|$ to obtain the multiset $X' \subseteq \mathbb{Z}$.
2. Add A to each element of X' to obtain the multiset $X'' \subseteq \mathbb{N}$.
3. Set $s'' := s + A \cdot |X|$.
4. Return the $\textsc{SubsetSum}^+$-instance $\langle X'', s''\rangle$.

This reduction can be performed in polynomial and is indeed direct. We now show that it is correct and focus on the case $P > s$ as the other two hold directly.

$\Longrightarrow$**:** Suppose $\langle X, s\rangle \in \textsc{SubsetSum}^\pm$ and let $S \subseteq X$ be a subset summing to s. Let S' be the completion of S with zeroes so that $|S'| = |X|$ and observe that by construction $S' \subseteq X'$ and S' still sums to s. Now, offsetting each element in S' by A increases the sum from s to $s + A \cdot |X|$ as $|S'| = |X|$. This shows that $\langle X'', s''\rangle \in \textsc{SubsetSum}^+$.

$\Longleftarrow$**:** Suppose $\langle X'', s''\rangle \in \textsc{SubsetSum}^+$ and let $S'' \subseteq X''$ be a subset summing to s''. Let $S' = \{x - A : x \in S''\}$ and note that $S' \subseteq X'$. We now show that $|S'| = |X|$. To begin with, if $|S'| \leq |X| - 1$, then we reach the following contradiction:

$$\sum_{x \in S''} x = A \cdot |S''| + \sum_{x \in S'} x \overset{\text{(a)}}{<} A \cdot |S''| + A \overset{\text{(b)}}{\leq} s + A \cdot |X| = s''$$

where (a) holds because S' does not contain all of X and $0 \notin X$ and (b) because $s \geq 0$. Now, if $|S'| \geq |X| + 1$, then we reach the following contradiction:

$$\sum_{x \in S''} x = A \cdot |S''| + \sum_{x \in S'} x \overset{\text{(a)}}{\geq} A \cdot |X| + \sum_{x \in X} |x| + \sum_{x \in S'} x \geq A \cdot |X| + \sum_{\substack{x \in X \\ x \geq 0}} x \overset{\text{(b)}}{>} s''$$

where (a) is by definition of A and (b) because $P > s$. Now, note that if S is a copy of S' with all zeroes removed, $S \subseteq X$ and S sums to s:

$$\sum_{x \in S} x = \sum_{x \in S'} x = -A \cdot |S'| + \sum_{x \in S''} x = -A \cdot |X| + s'' = s$$

so that $\langle X, s \rangle \in \textsc{SubsetSum}^{\pm}$, as desired.

An alternative solution: The reduction goes as follows. Given a multiset $X = \{x_1, \cdots, x_m\} \subseteq$ and an integer s, let $I^+ := \{i : x_i > 0\}$ be the set of indices of all positive integers in X, and $I^- := \{i : x_i < 0\}$ be the set of indices of all negative integers in X. Let $X' = \{|x_1|, |x_2|, \cdots, |x_m|\}, s' = s - \sum_{i \in I^-} x_i$.

It is clear that the above reduction runs in polynomial time. It suffices to prove $\langle X, s \rangle \in \textsc{SubsetSum}^{\pm} \Leftrightarrow \langle X', s' \rangle \in \textsc{SubsetSum}^{+}$.

$\Longrightarrow$: Suppose $\langle X, s \rangle \in \textsc{SubsetSum}^{\pm}$. Let $I_s \subseteq [m]$ be a set of indices such that $\sum_{i \in I_s} x_i = s$. Let $I_s^+ = I_s \cap I^+$ be the set of indices in I_s whose corresponding elements are positive, $I_s^- = I_s \cap I^-$ be the set of indices in I_s whose corresponding elements are negative. Let $I'_{s'} = I_s^+ \cup (I^- \setminus I_s^-)$. Then we have

$$\begin{aligned}
\sum_{i \in I'_{s'}} |x_i| &= \sum_{i \in I_s^+} x_i - \sum_{i \in I^- \setminus I_s^-} x_i \\
&= \left(\sum_{i \in I_s^+} x_i + \sum_{i \in I_s^-} x_i \right) - \sum_{i \in I^-} x_i \\
&= \sum_{i \in I_s} x_i + \left(s' - s\right) \\
&= s',
\end{aligned}$$

which implies that $\langle X, s \rangle \in \textsc{SubsetSum}^{+}$.

$\Longleftarrow$: Suppose $\langle X', s' \rangle \in \textsc{SubsetSum}^{+}$. Let $I'_{s'} \subseteq [m]$ be a set of indices such that $\sum_{i \in I'_{s'}} |x_i| = s'$. Let $I'^{+}_{s'} = I'_{s'} \cap I^+$ be the set of indices in $I'_{s'}$ whose corresponding elements are positive, $I_{s'}^- = I'_{s'} \cap I^-$ be the set of indices in $I'_{s'}$ whose corresponding elements are negative. Let $I_s = I'^{+}_{s'} \cup \left(I^- \setminus I'^{-}_{s'}\right)$. Then we have

$$\begin{aligned}
\sum_{i \in I_s} x_i &= \sum_{i \in I'^{+}_{s'}} x_i + \sum_{i \in I^- \setminus I'^{-}_{s'}} x_i \\
&= \left(\sum_{i \in I'^{+}_{s'}} x_i - \sum_{i \in I'^{-}_{s'}} x_i \right) + \sum_{i \in I^-} x_i \\
&= \sum_{i \in I'_{s'}} |x_i| + \left(s - s'\right) \\
&= s,
\end{aligned}$$

which implies that $\langle X, s \rangle \in \textsc{SubsetSum}^{\pm}$.