Theory of Computation
Homework 1

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1 Exercice 1

We consider the following automaton, denoted $\mathcal{M}_1$:

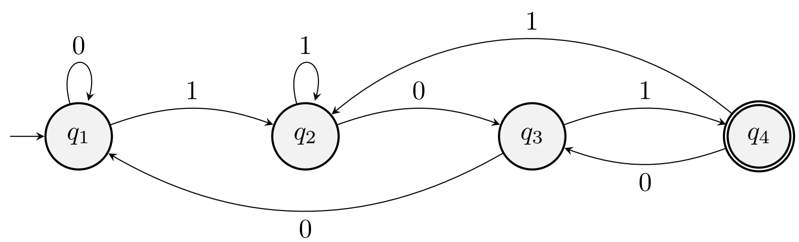

Figure 1: Caption of the DFA $\mathcal{M}_1$

We can describe $\mathcal{M}_1$ formally by writing $\mathcal{M}_1 = (Q, \Sigma, \delta, q_1, F)$, where :

$Q = \{q_1, q_2, q_3, q_4\}$, the set of states
$\Sigma = \{0, 1\}$, the binary alphabet
$F = \{q_4\}$, the set of accepting states
q_1 is the starting state
$\delta : Q \times \Sigma \to Q$, the transition function described as follows :

	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_1	q_4
q_4	q_3	q_2

Table 1: $\mathcal{M}_1$ Transition function

After a few trials, the language $\mathcal{L}(\mathcal{M}_1)$ recognized by the DFA seems to be :

$$\mathcal{L}(\mathcal{M}_1) \;=\; \{w \in \Sigma^\star \,|\, w \text{ ends on "101"}\}.$$

We denote an input string $x \in \Sigma^\star$, and l its length. We prove by *induction* on l the following claim.

Claim If the input string x does not contain any "1" digits the $\mathcal{M}_1$ finishes in q_1, if x ends on "00" and contains at least a "1" the $\mathcal{M}_1$ finishes in q_1, if x ends on a "1" but does not contain "101" as a substring the $\mathcal{M}_1$ finishes in q_2, if x ends on "10" the $\mathcal{M}_1$ finishes in q_3, if x ends on a "1" after the last "101" substring the $\mathcal{M}_1$ finishes in q_2, and finally if x ends on "101" the $\mathcal{M}_1$ finishes in q_4. Note that these 6 cases are mutually exclusive and every input falls into one of the cases.

Base case If $l = 0$ then x is the empty string so it does not contain any "1" digits and indeed the $\mathcal{M}_1$ finishes in q_1 (the starting state) and the claim holds.

Induction hypothesis Suppose that the claims is true for all $l < n$, where n is an integer such that $n > 0$.

Induction step Let $l = n$ and let x' be the first $n-1$ digits of x. Since the length of x' is less than n, the induction hypothesis applies. We have 6 cases :

1. Suppose x' does not contain any "1" digits (by the induction hypothesis we are at q_1) and consider the last input digit σ. If $\sigma =$ "0", x does not contain any "1" digits either and indeed the $\mathcal{M}_1$ stays at q_1 and finishes. However if $\sigma =$ "1", then x ends on a "1" but does not contain "101" as a substring and indeed the $\mathcal{M}_1$ transitions to q_2 and finishes.

2. Suppose x' ends on "00" and contains at least a "1" (so by the induction hypothesis we are at q_1) and consider the last input digit σ. If $\sigma =$ "0", x still ends on "00" and contains at least a "1" and indeed the $\mathcal{M}_1$ stays at q_1 and finishes. However if $\sigma =$ "1", then x ends on a "1" and x can either contain or not "101" as a substring, in both cases the $\mathcal{M}_1$ indeed transitions to q_2 and finishes.

3. Suppose x' ends on a "1" but does not contain "101" as a substring (so by the induction hypothesis we are at q_2) and consider the last input digit σ. If $\sigma =$ "0", x ends on "10" and indeed the $\mathcal{M}_1$ transitions to q_3 and finishes. However if $\sigma =$ "1", then x still ends on a "1" and does not contain "101" as a substring and indeed the $\mathcal{M}_1$ stays at q_2 and finishes.

4. Suppose x' ends on "10" (so by the induction hypothesis we are at q_3) and consider the last input digit σ. If $\sigma =$ "0", x ends on "00" and contains at least a "1" and indeed the $\mathcal{M}_1$ transitions to q_1 and finishes. However if $\sigma =$ "1", then x ends on "101" and indeed the $\mathcal{M}_1$ transitions to q_4 and finishes.

5. Suppose x' ends on a "1" after the last "101" substring (so by the induction hypothesis we are at q_2) and consider the last input digit σ. If $\sigma =$ "0", x ends on "10" and indeed the $\mathcal{M}_1$ transitions to q_3 and finishes. However if $\sigma =$ "1", then x still ends on a "1" after the last "101" substring and indeed the $\mathcal{M}_1$ stays at q_2 and finishes.

6. Suppose x' ends on "101" (so by the induction hypothesis we are at q_4) and consider the last input digit σ. If $\sigma =$ "0", x ends on "10" and indeed the $\mathcal{M}_1$ transitions to q_3 and finishes. However if $\sigma =$ "1", then x ends on a "1" after the last "101" substring and indeed the $\mathcal{M}_1$ transitions to q_2 and finishes.

The hypothesis holds for $l = n$ and this completes the proof.

2 Exercice 2

2.1 Probleme 2a

For a language $\mathcal{L} \subseteq \Sigma^\star$, we define its *triple* by :

$$\mathcal{L}^3 \;\coloneqq\; \{www : \, w \in \mathcal{L}\}$$

Let us show that regular languages are *not* closed under tripling. For that matter, we consider the following DFA, denoted $\mathcal{M}_2$:

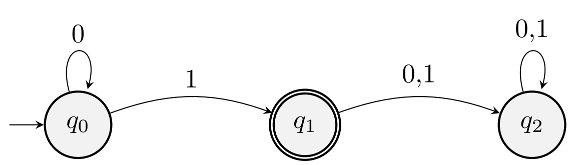

Figure 2: Caption of the DFA $\mathcal{M}_2$

We can describe this automaton formally by writing $\mathcal{M}_2 = (Q, \Sigma, \delta, q_0, F)$, where :

$$Q = \{q_0,\, q_1,\, q_2\},\ \text{the set of states}$$
$$\Sigma = \{0,\, 1\},\ \text{the alphabet}$$
$$F = \{q_1\},\ \text{the set of accepting states}$$
$$q_0\ \text{is the starting state}$$
$$\delta\, :\, Q\, \times\, \Sigma \to Q,\ \text{the transition function described as follows :}$$

	0	1
q_0	q_0	q_1
q_1	q_2	q_2
q_2	q_2	q_2

Table 2: $\mathcal{M}_2$ Transition function

Clearly, $\mathcal{M}_2$ recognizes the language :

$$\mathcal{L}(\mathcal{M}_2) = \{0^n 1 \,|\, n \geq 0\},$$

and $\mathcal{L}(\mathcal{M}_2)$ is regular.

Next we define the *triple* $\mathcal{L}^3$:

$$\mathcal{L}^3 = \{www : w \in \mathcal{L}(\mathcal{M}_2)\} = \{0^n 1 0^n 1 0^n 1 \,|\, n \geq 0\}.$$

Suppose, for the sake of contradiction, that $\mathcal{L}^3$ is regular. Then we know that there must exist a positive integer p satisfying the premises of the pumping lemma.

We pick $s := 0^p 1 0^p 1 0^p 1 \in \mathcal{L}^3$. According to the pumping lemma, there exists a split $s = xyz$, $|xy| \leq p$, $|y| \geq 1$, such that for all $i \geq 0$, $xy^iz \in \mathcal{L}^3$. Hence, we define $y := 0^k$, $1 \leq k \leq p$. From the standpoint of the lemma, $\tilde{s} := xy^2z \in \mathcal{L}^3$, for $i = 2$. However, the string $\tilde{s} = 0^{p+k}10^p10^p1$ and for any $k \in [\![1..p]\!]$, $\tilde{s}$ is not the 3-time concatenation of the same string anymore.

Thus a contradiction is unavoidable if we make the assumption that $\mathcal{L}^3$ is regular, so $\mathcal{L}^3$ is *not* regular. Quod Erat Demonstrandum.

2.2 Probleme 2b

Our purpose is to show that for a regular language $\mathcal{L} \subseteq \Sigma^\star$ over a *unary alphabet*, *i.e.* $|\Sigma| = 1$, the *triple* $\mathcal{L}^3$ as previously defined is regular.

Let $\mathcal{D}_1$ be a DFA such that :

$$\mathcal{D}_1 = (Q_1,\, \Sigma,\, \delta_1,\, q_1,\, F_1)$$

accepting the language $\mathcal{L}$ with $|\Sigma| = 1$. In particular, we have : $F_1 = \cup_{i=1}^{|F_1|} F_{1,i}$

Let $\mathcal{D}_{2,i}$ and $\mathcal{D}_{3,i}$ be copies of the DFA $\mathcal{D}_1$ such that :

$$\begin{aligned}
\mathcal{D}_{2,i} &= (Q_{2,i},\, \Sigma,\, \delta_{2,i},\, q_{2,i},\, F_{2,i}) \quad \text{and} \\
\mathcal{D}_{3,i} &= (Q_{3,i},\, \Sigma,\, \delta_{3,i},\, q_{3,i},\, F_{3,i}) \quad \text{with} \quad i \in [\![1\,..\,|F_1|]\!]
\end{aligned}$$

We modify every $\mathcal{D}_{2,i}$ and $\mathcal{D}_{3,i}$ such that the sets of accepting sets $F_{2,i}$ and $F_{3,i}$ contain only one accepting state $f_{2,i}$ and $f_{3,i}$ respectively, which corresponds to the ith final state.

Using the DFAs previously established, we create an NFA, denoted $\mathcal{N}$, such that :

$$\mathcal{N} = (Q',\, \Sigma',\, \delta',\, q',\, F').$$

We describe each of its components :

$$\begin{aligned}
Q' &= Q_1 \cup \{\cup_{i=1}^{|F_1|} Q_{2,i}\} \cup \{\cup_{i=1}^{|F_1|} Q_{3,i}\} \quad \text{the states} \\
\Sigma' &= \Sigma \quad \text{the } \textit{unary alphabet} \\
q' &= q_1 \quad \text{the starting state} \\
F' &= \bigcup_{i=1}^{|F_1|} F_{3,i} \quad \text{the set of accepting sets}
\end{aligned}$$

$$\delta'(q,a) = \begin{cases}
\delta_1(q,a), & \text{if } q \in Q_1 \text{ and } q \notin F_1 \\
\delta_1(q,a), & \text{if } q \in F_1 \text{ and } a \neq \epsilon \\
\delta_1(q,a) \cup \{q_{2,i}\}, & \text{if } q = f_{1,i} \text{ and } a = \epsilon \\
\delta_{2,i}(q,a), & \text{if } q \in Q_{2,i} \text{ and } q \notin F_{2,i} \\
\delta_{2,i}(q,a), & \text{if } q \in F_{2,i} \text{ and } a \neq \epsilon \\
\delta_{2,i}(q,a) \cup \{q_{3,i}\}, & \text{if } q = f_{2,i} \text{ and } a = \epsilon \\
\delta_{3,i}(q,a), & \text{if } q \in Q_{3,i}
\end{cases} \qquad \text{with} \quad i \in [\![1\,..\,|F_1|]\!]$$

This completes our construction, and $\mathcal{L}^3$ is regular over a *unary alphabet*. However, it is important to point out that all of this does not hold for any alphabet Σ, *cf.* probleme 2a.