

From Quaternions to Rotation Matrices

In Gen6D, the model represents the ground truth and estimated poses using the format $\mathbf{P} = (\mathbf{R}, \mathbf{t})$. Here, \mathbf{R} denotes the rotation matrix and \mathbf{t} is the translation vector. The thing is, in the SPACECRAFT dataset, the poses have the format $\mathbf{P} = (\mathbf{q}, \mathbf{t})$, where \mathbf{q} is a *quaternion*. We use quaternions for three-dimensional rotation calculations because they offer several benefits over rotation matrices. Notably, quaternions are more compact, requiring only four elements to be stored compared to nine for a matrix. Additionally, they are more efficient when composing rotations thanks to their algebraic properties.

Despite the benefits of quaternions mentioned earlier, other datasets frequently represent poses using a combination of rotation matrices and translation vectors. Therefore, to align with Gen6D's pose format, this section will focus on converting quaternions into rotation matrices.

Quaternions were first introduced by the Irish mathematician W. R. Hamilton in 1843 as an extension of the complex numbers. In the first place, we provide the definition of a *quaternion* : it is the sum of a scalar q_0 and a vector $\mathbf{q} = (q_1, q_2, q_3)$, that is,

$$\underline{\mathbf{q}} \stackrel{\text{def.}}{=} q_0 + \mathbf{q} = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}.$$

In the above, \mathbf{i} , \mathbf{j} and \mathbf{k} denote the three unit vectors of the canonical basis for the set of all ordered triples of real numbers \mathbb{R}^3 . The set of quaternions is denoted by the 4-space \mathbb{H} .

The quaternion addition is component-wise. Regarding the product of two quaternions, it is essential to first outline the foundational rule established by Hamilton :

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\mathbf{j}\mathbf{k} = -1.$$

We derive the following multiplication table :

\times	1	\mathbf{i}	\mathbf{j}	\mathbf{k}
1	1	\mathbf{i}	\mathbf{j}	\mathbf{k}
\mathbf{i}	\mathbf{i}	-1	\mathbf{k}	$-\mathbf{j}$
\mathbf{j}	\mathbf{j}	$-\mathbf{k}$	-1	\mathbf{i}
\mathbf{k}	\mathbf{k}	\mathbf{j}	$-\mathbf{i}$	-1

Let $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{H}^2$, we are now able to present the multiplication of $\underline{\mathbf{p}}$ and $\underline{\mathbf{q}}$:

$$\underline{\mathbf{p}}\underline{\mathbf{q}} = \underbrace{p_0q_0 - \mathbf{p} \cdot \mathbf{q}}_{\text{scalar part}} + \underbrace{p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q}}_{\text{vector part}}.$$

In the preceding, we recall that the function $\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R} : (\mathbf{p}, \mathbf{q}) \mapsto \mathbf{p} \cdot \mathbf{q} \stackrel{\text{def.}}{=} \mathbf{p}^\top \mathbf{q}$ is the *dot product*, and the function

$$\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 : (\mathbf{p}, \mathbf{q}) \mapsto \mathbf{p} \times \mathbf{q} \stackrel{\text{def.}}{=} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} = [\mathbf{p}]_{\times} \mathbf{q}$$

is the *cross product*, and $[\mathbf{p}]_{\times}$ denotes the transformation matrix that when multiplied from the right with a vector \mathbf{q} gives $\mathbf{p} \times \mathbf{q}$.

Several additional definitions are essential before we can begin the conversion of quaternions to rotation matrices.

Let $\underline{\mathbf{q}} = q_0 + \mathbf{q}$ be a quaternion. The *complex conjugate* of $\underline{\mathbf{q}}$, denoted by $\underline{\mathbf{q}}^*$, is given by the map

$$\mathbb{H} \rightarrow \mathbb{H} : \underline{\mathbf{q}} \mapsto \underline{\mathbf{q}}^* \stackrel{\text{def.}}{=} q_0 - \mathbf{q}.$$

The *norm* of a quaternion $\underline{\mathbf{q}}$, denoted $|\underline{\mathbf{q}}|$, is the distance obtained from the map

$$\mathbb{H} \rightarrow \mathbb{R}^+ : \underline{\mathbf{q}} \mapsto |\underline{\mathbf{q}}| \stackrel{\text{def.}}{=} \sqrt{\underline{\mathbf{q}}^* \underline{\mathbf{q}}}.$$

Note that a quaternion whose norm is 1 is referred to as a *unit quaternion*. The *reciprocal* of a quaternion is defined as the map

$$\mathbb{H}^* \rightarrow \mathbb{H}^* : \underline{\mathbf{q}} \mapsto \underline{\mathbf{q}}^{-1} \stackrel{\text{def.}}{=} \frac{\underline{\mathbf{q}}^*}{|\underline{\mathbf{q}}|^2}.$$

We observe that if $\underline{\mathbf{q}}$ is a unit quaternion, we simply have $\underline{\mathbf{q}}^{-1} = \underline{\mathbf{q}}^*$. Furthermore, the subsequent proposition is accepted : if $\underline{\mathbf{q}}$ is a unit quaternion, there exists a unique $\theta \in [0, 2\pi]$ such that

$$\underline{\mathbf{q}} = q_0 + \mathbf{q} = \cos \frac{\theta}{2} + \mathbf{u} \sin \frac{\theta}{2},$$

where the unit vector \mathbf{u} is defined as $\mathbf{u} \stackrel{\text{def.}}{=} \frac{\mathbf{q}}{|\mathbf{q}|}$.

Quaternion Rotation Operator Let $\underline{\mathbf{q}} \in \mathbb{H}$ be a unit quaternion, and let $\mathbf{v} \in \mathbb{R}^3$ be a vector. The action of the $L_{\underline{\mathbf{q}}}$ function

$$\mathbb{R}^3 \rightarrow \mathbb{R}^3 : \mathbf{v} \mapsto L_{\underline{\mathbf{q}}}(\mathbf{v}) \stackrel{\text{def.}}{=} \underline{\mathbf{q}} \mathbf{v} \underline{\mathbf{q}}^*$$

on \mathbf{v} is equivalent to a rotation of the vector through an angle θ about the axis of rotation \mathbf{u} .

At last, we can proceed with our conversion task : our aim is to find a 3×3 rotation matrix \mathbf{R} , such that

$$\begin{cases} L_{\mathbf{R}}(\mathbf{v}) \stackrel{\text{def.}}{=} \mathbf{R} \mathbf{v} \\ L_{\mathbf{R}}(\mathbf{v}) = L_{\underline{\mathbf{q}}}(\mathbf{v}), \end{cases}$$

which means we wish to obtain an expression for \mathbf{R} by manipulating $L_{\underline{\mathbf{q}}}$, utilizing principles of linear algebra and vector calculus. We consider the vector \mathbf{v} as a quaternion $\underline{\mathbf{v}}$ with a zero scalar component :

$$\begin{aligned}
L_{\underline{\mathbf{q}}}(\mathbf{v}) &= \underline{\mathbf{q}}\mathbf{v}\underline{\mathbf{q}}^* \\
&= (q_0 + \mathbf{q})(0 + \mathbf{v})(q_0 - \mathbf{q}) \\
&= (\underbrace{-\mathbf{q} \cdot \mathbf{v}}_{\text{scalar part}} + \underbrace{q_0\mathbf{v} + \mathbf{q} \times \mathbf{v}}_{\text{vector part}})(q_0 - \mathbf{q}) \\
&= q_0(-\mathbf{q} \cdot \mathbf{v}) - (q_0\mathbf{v} + \mathbf{q} \times \mathbf{v}) \cdot (-\mathbf{q}) \\
&\quad + (-\mathbf{q} \cdot \mathbf{v})(-\mathbf{q}) + q_0(q_0\mathbf{v} + \mathbf{q} \times \mathbf{v}) \\
&\quad + (q_0\mathbf{v} + \mathbf{q} \times \mathbf{v}) \times (-\mathbf{q}) \\
&= \underbrace{-q_0(\mathbf{q} \cdot \mathbf{v}) + q_0(\mathbf{q} \cdot \mathbf{v})}_{\text{scalar part}} \\
&\quad + \underbrace{\mathbf{q}(\mathbf{q} \cdot \mathbf{v}) + q_0^2\mathbf{v} + q_0(\mathbf{q} \times \mathbf{v}) + \mathbf{q} \times (q_0\mathbf{v} + \mathbf{q} \times \mathbf{v})}_{\text{vector part}} \\
&= \mathbf{q}(\mathbf{q}^T \mathbf{v}) + q_0^2\mathbf{v} + q_0(\mathbf{q} \times \mathbf{v}) + \mathbf{q} \times (q_0\mathbf{v} + \mathbf{q} \times \mathbf{v}) \\
&= (\mathbf{q} \otimes \mathbf{q} + q_0^2 \mathbf{I}_{3 \times 3} + 2 q_0 [\mathbf{q}]_{\times} + [\mathbf{q}]_{\times}^2) \mathbf{v}
\end{aligned}$$

In the above, \otimes stands for the *outer product* and $\mathbf{I}_{3 \times 3}$ is the *identity matrix*. Since $L_{\mathbf{R}}(\mathbf{v}) = L_{\underline{\mathbf{q}}}(\mathbf{v})$, we can identify \mathbf{R} as $(\mathbf{q} \otimes \mathbf{q} + q_0^2 \mathbf{I}_{3 \times 3} + 2 q_0 [\mathbf{q}]_{\times} + [\mathbf{q}]_{\times}^2)$. We expand \mathbf{R} to find its expression :

$$\begin{aligned}
\mathbf{R} &= \mathbf{q} \otimes \mathbf{q} + q_0^2 \mathbf{I}_{3 \times 3} + 2 q_0 [\mathbf{q}]_{\times} + [\mathbf{q}]_{\times}^2 \\
&= \begin{bmatrix} q_1^2 & q_1 q_2 & q_1 q_3 \\ q_2 q_1 & q_2^2 & q_2 q_3 \\ q_3 q_1 & q_3 q_2 & q_3^2 \end{bmatrix} + q_0^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2 q_0 \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \\
&= 2 \begin{bmatrix} (q_0^2 + q_1^2) - \frac{1}{2} & q_1 q_2 - q_0 q_3 & q_1 q_3 + q_0 q_2 \\ q_1 q_2 + q_0 q_3 & (q_0^2 + q_2^2) - \frac{1}{2} & q_2 q_3 - q_0 q_1 \\ q_1 q_3 - q_0 q_2 & q_2 q_3 + q_0 q_1 & (q_0^2 + q_3^2) - \frac{1}{2} \end{bmatrix}
\end{aligned}$$

This marks the completion of our derivation, we found the expression of \mathbf{R} . The Python code can be found in the Listing ??.