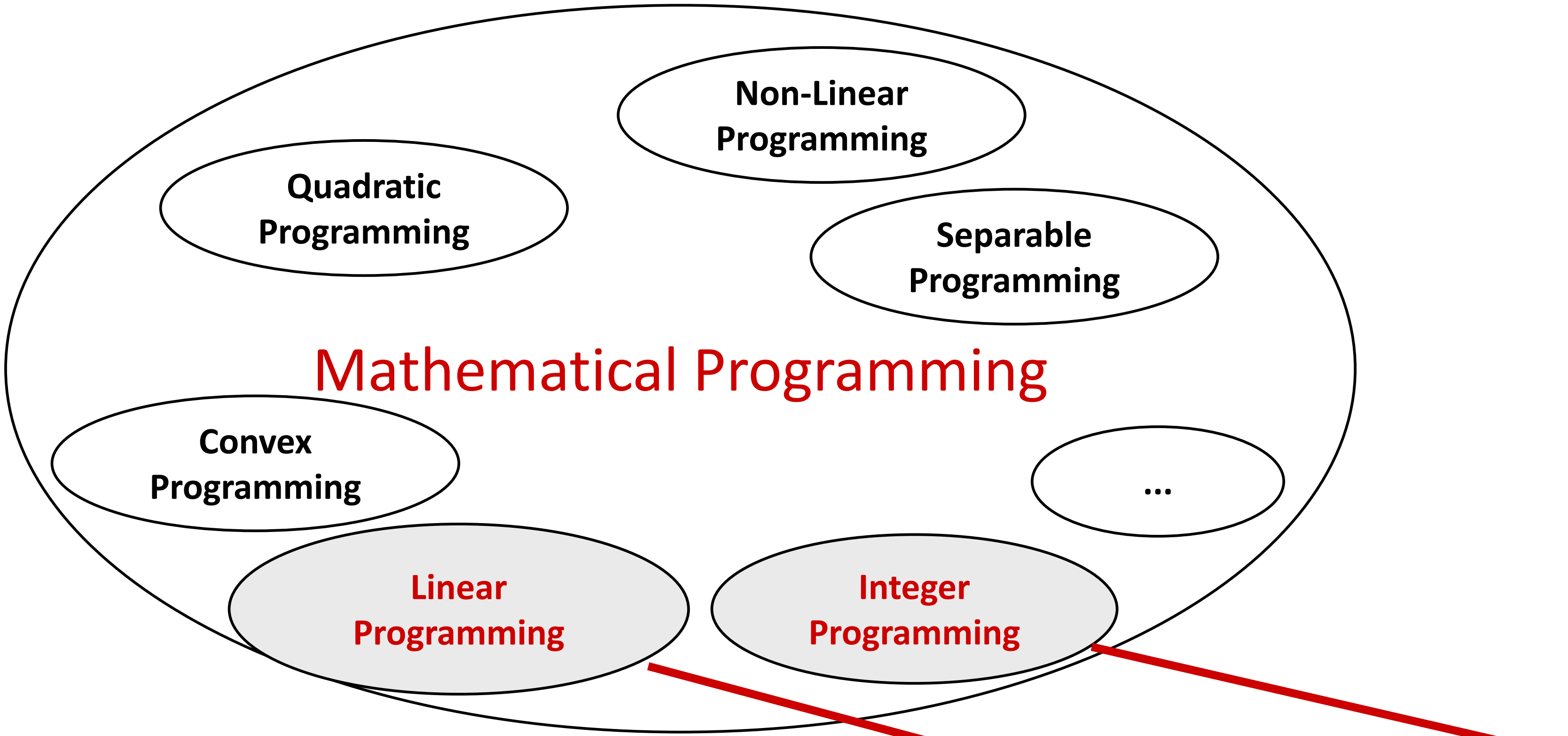
LINEAR PROGRAMMING

Some models we are going to learn in this course

This Linear Programming Problem belongs to a group of problems known as Mathematical Programming Problems, which are characterized by having a single objective and are subject to a set of constraints (which features are different for each class of problems).



In this course we will only address Linear Programming and Integer Programming

Linear Programming

- First stated in this form by **George B. Dantzig**, it is an amazing fact that literally thousands of decision (programming) problems from business, industry, government and the military can be stated (or approximated) as linear programming problems.
- Although there were some precursor attempts at stating such problems in mathematical terms, notably by the Russian mathematician **Leonid V. Kantorovich** in 1939, Dantzig's general formulation, combined with his method of solution, the Simplex Method, revolutionized decision making.
- The name "linear programming" was suggested to Dantzig by the economist **Tjalling C. Koopmans**.

Both Kantorovich and Koopmans were awarded the 1975 Nobel prize in economics for their contributions to the theory of optimum allocation of resources.

The untold story

- Most people familiar with the origins and development of linear programming were amazed and disappointed that **Dantzig did not receive the Nobel prize** along with Koopmans and Kantorovich (a Nobel prize can be shared by up to three recipients).
- Shortly after the award, Koopmans talked about his displeasure with the Nobel selection and told he had earlier written to Kantorovich suggesting that they both refuse the prize, certainly a most difficult decision for both, but especially so for Kantorovich who was not recognized in URSS….

Kantorovich said :

"In the spring of 1939 I gave some more reports – at the Polytechnic Institute and the House of Scientists, but several times met with the objection that the work used mathematical methods, and in the West the mathematical school in economics was an anti-Marxist school and mathematics in economics was a means for apologists of capitalism."

Linear Programming and the Simplex method were explained by George Dantzig in 1948 at a meeting held at the University of Wisconsin.

In the discussion after his lecture, someone from the audience said:

"Yes, but... we all know the world is nonlinear..."

John von Neumann, who was also there, stood up and said:

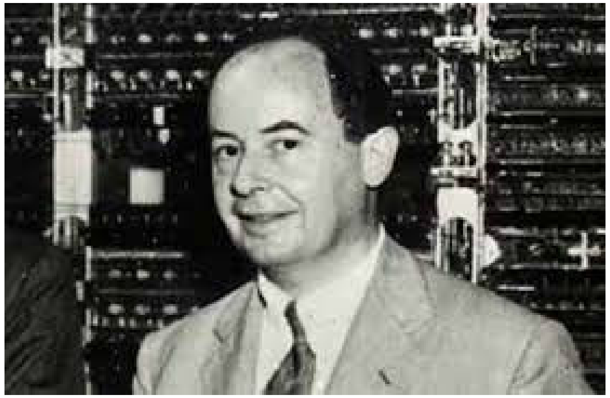
"Mr. Chairman, Mr. Chairman,

if the speaker does not mind, I would like to reply for him.

The speaker titled his talk 'linear programming' and carefully stated his axioms.

If you have an application that satisfies the axioms, well use it.

If it does not, then don't."



John von Neumann (1903-1957) was a Hungarian-American mathematician, physicist, inventor, computer scientist. He was a pioneer of quantum mechanics and of concepts of cellular automata, the universal constructor and the digital computer.

After this episode, Dantzig's colleagues decided to hang this cartoon outside his office…

Top Ten Algorithms of the XXth Century

Computing in Science& Engineering, a joint publication of the American Institute of Physics and the IEEE Computer Society January/February 2000

1946 - Metropolis Algorithm for Monte Carlo

1947 - Simplex Method for Linear Programming

1950 - Krylov Subspace Iteration Methods

1951 - The Decompositional Approach to Matrix Computations

1957 - The Fortran Optimizing Compiler

1959 - QR Algorithm for Computing Eigenvalues

1962 - Quicksort Algorithm for Sorting

1965 - Fast Fourier Transform

1977 - Integer Relation Detection

1987 - Fast Multipole Method

Cereals, Ltd

Cereals, Ltd is a company specialized in preparing and packing wheat and corn for several retailing stores.There are no limits concerning the supply of these cereals and demand can be considered unlimited (in other words, the whole production is sold). The production is processed in three phases: Pre-Processing (I), Processing (II) and Packing (III).

	I Pre-Processing	II Processing	III Packing	
Weekly production capacity (h)	120	100	150	
Time (h) needed to prepare 1 ton				Profit(€/ton)
Wheat	6 h	1 h	5 h	4
Corn	2 h	4 h	5 h	3

Currently, the company produces 18 tons of wheat and 6 tons of corn per week.
Is this the best option?

Cereals, Ltd - Formulation

Decision variables

x = tons of wheat to produce weekly

y = tons of corn to produce weekly

Constraints

$$6x + 2y \leq 120$$

$$x + 4y \leq 100$$

$$5x + 5y \leq 150$$

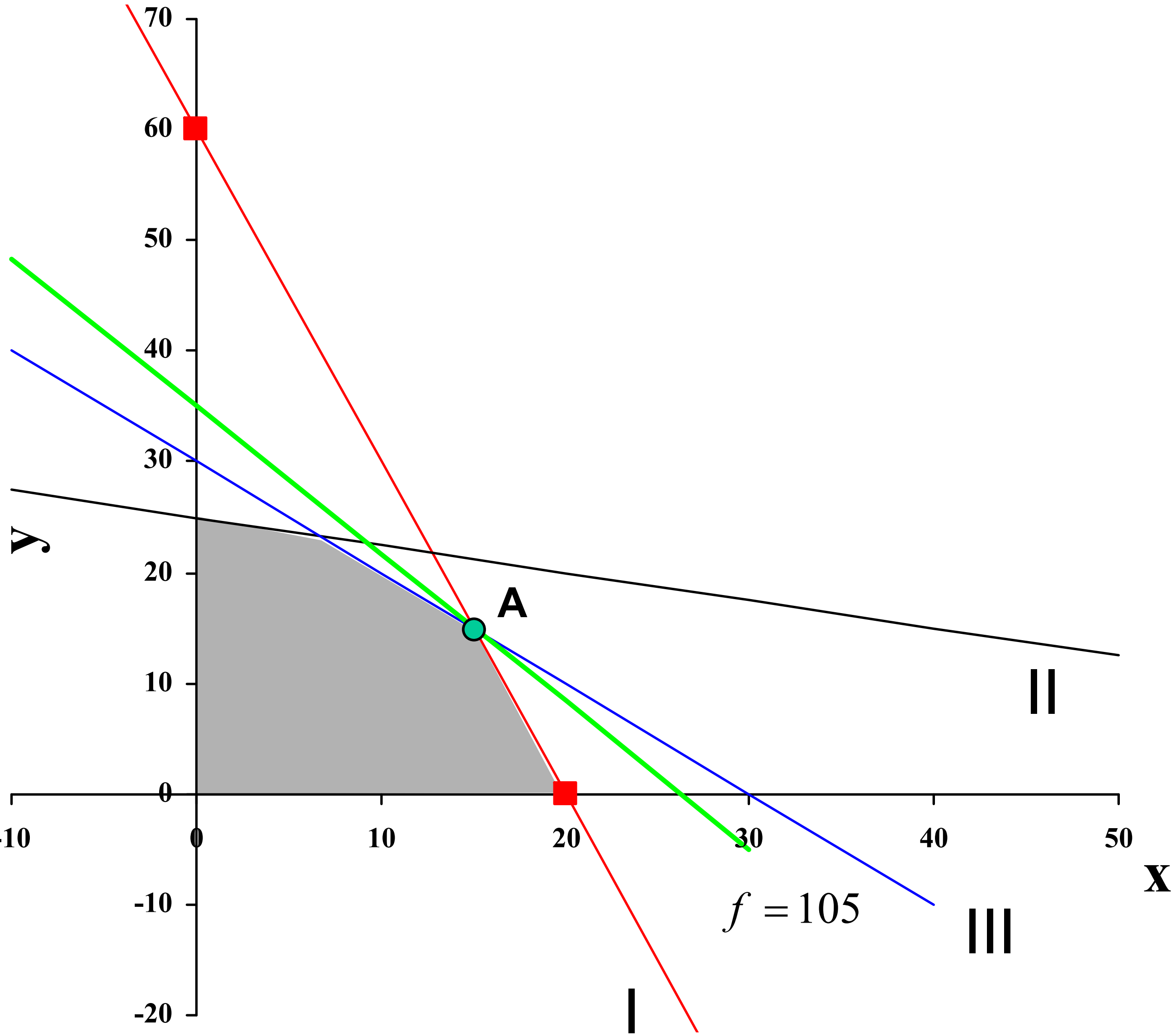
$$x \geq 0, y \geq 0$$

Objective function: to maximize the profit $\max 4x + 3y$

$$\max\ 4x+3y$$

$$\text{s.a}\quad 6x+2y\le 120$$
$$x+4y\le 100$$
$$5x+5y\le 150$$
$$x\ge 0,\ y\ge 0$$

Consider again the example of Cereals, Ltd



Optimal solution A = (15,15)

Profit: $4x+3y=105$

What happens if it would be possible to increase the profit of each ton of wheat to 4,35 €?

And what if the production capacity in section III (packing) is reduced to 126h per week?

Once the optimal solution of a linear problem is obtained, what should we do if changes in the parameters occur?

Sensitivity Analysis

Analyses the effect of (small) changes on the parameter values in the optimal solution.

Case 1: changes in the coefficients of the objective function (c_j)

Example: what is the possible variation for unitary profit of wheat (x) and corn (y) without changing the optimal solution (x=15 and y=15)?

Caso 2: changes in the right side of constraints alterações (b_i)

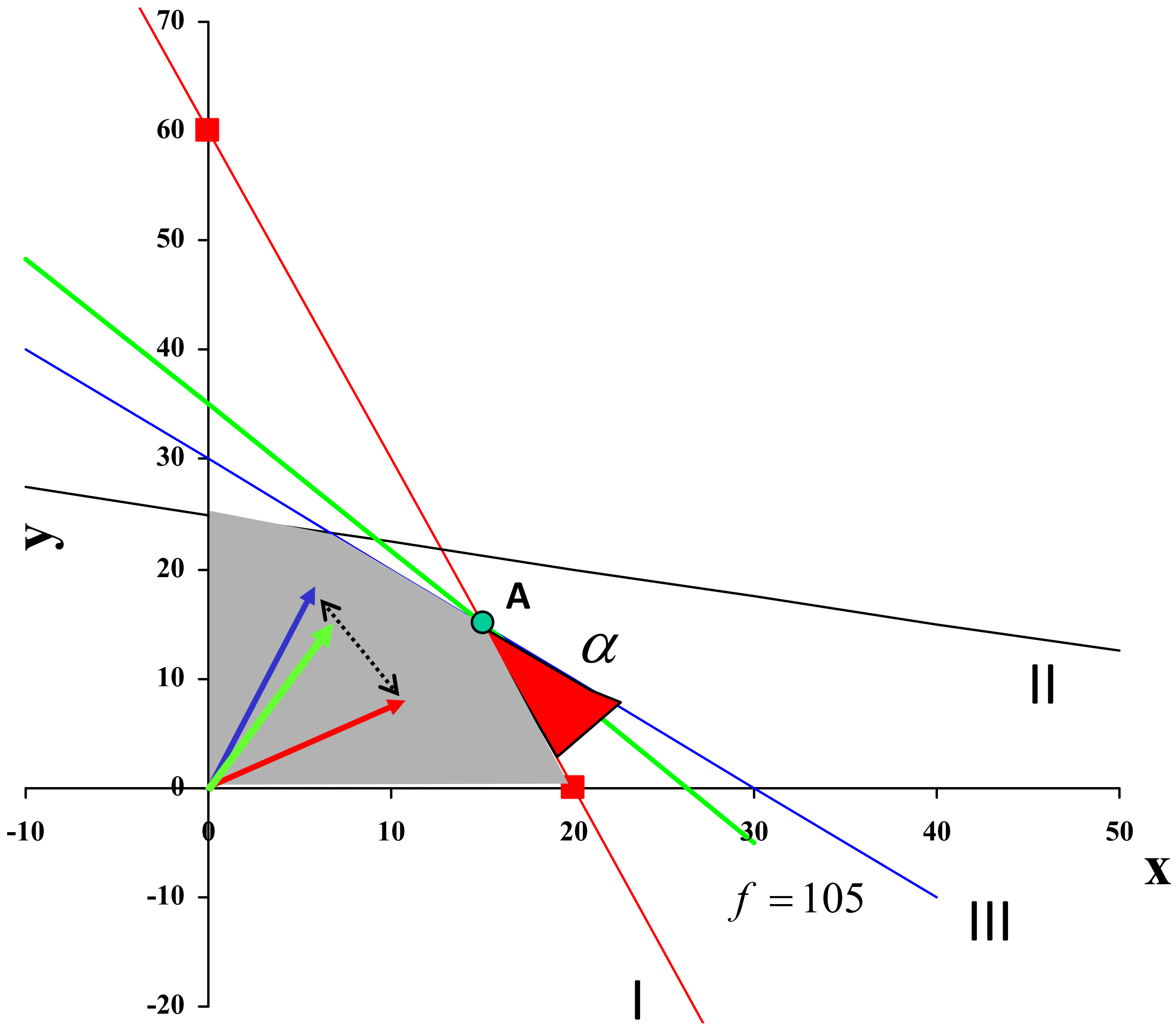
Example: what is the effect of changing the production capacity in each section (I, II and III)?

Case 1 - changes in the coefficients of the objective function (c_j)

Lines **I** and **III** make an angle **a** at point **A**.

If **f** rotates inside angle **a**, the optimal solution is maintained.

Rotating **f** inside angle **a** means that the slope of f varies between the slopes of **I** and **III**.



Case 1 - changes in the coefficients of the objective function (c_j)

Slope of I = **- 3**

$$6x+2y=120 \Leftrightarrow y=-3x+60$$

Slope of III = **-1**

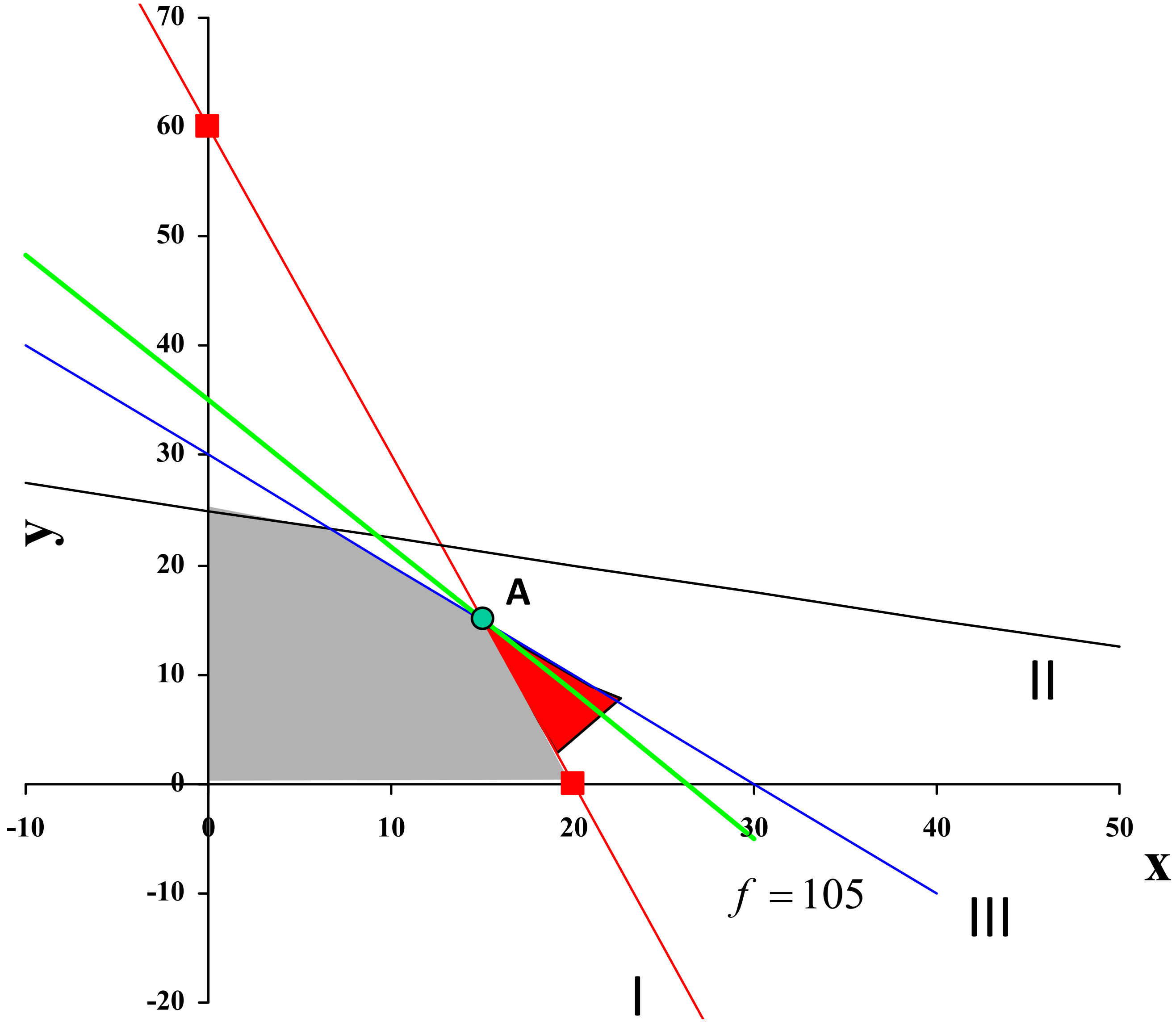
$$5x+5y=150 \Leftrightarrow y=-x+30$$

Slope of f = **- 4 / 3**

$$4x+3y=k \Leftrightarrow y=-\frac{4}{3}x+\frac{k}{3}$$

In fact,

$$-3 \leq -\frac{4}{3} \leq -1$$



Case 1 - changes in the coefficients of the objective function (c_j)

Let a, b be the coefficients of the objective function $f(x,y) = ax + by = k \Leftrightarrow y = -\frac{a}{b}x + \frac{k}{b}$

The optimal solution (i.e, the values of x and y) remains unchanged if $-3 \leq -\frac{a}{b} \leq -1$

although the **f** value (in this example, the profit) may vary.

- In particular, if we modify the value of **a**, keeping **b = 3**:

$$-3 \leq -\frac{a}{3} \leq -1 \Leftrightarrow -9 \leq -a \leq -3 \Leftrightarrow 3 \leq a \leq 9$$

- If we change the value of **b** instead, keeping **a = 4**:

$$-3 \leq -\frac{4}{b} \leq -1 \Leftrightarrow 1 \leq -\frac{4}{b} \leq 3 \Leftrightarrow \frac{4}{3} \leq b \leq 4$$

Case 1 - changes in the coefficients of the objective function (c_j)

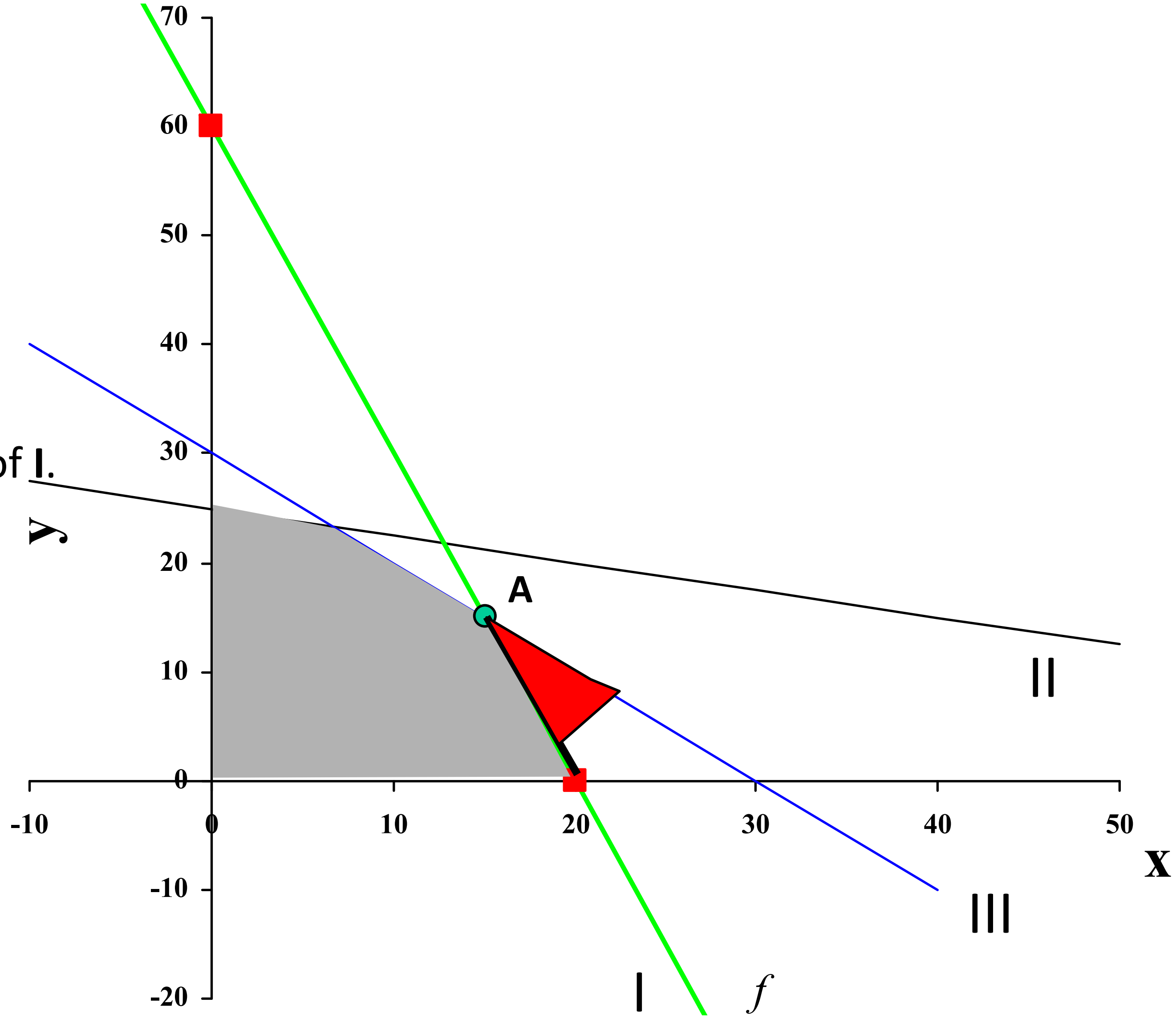
- If $-3 < -\frac{a}{b} < -1$, the optimal solution is unique (point A).

- If $-\frac{a}{b} = -3$

(*e.g, a=9, b=3*),

the slope of **f** is equal to the slope of **I**.

In this case, we will have an infinite number of optimal solutions.



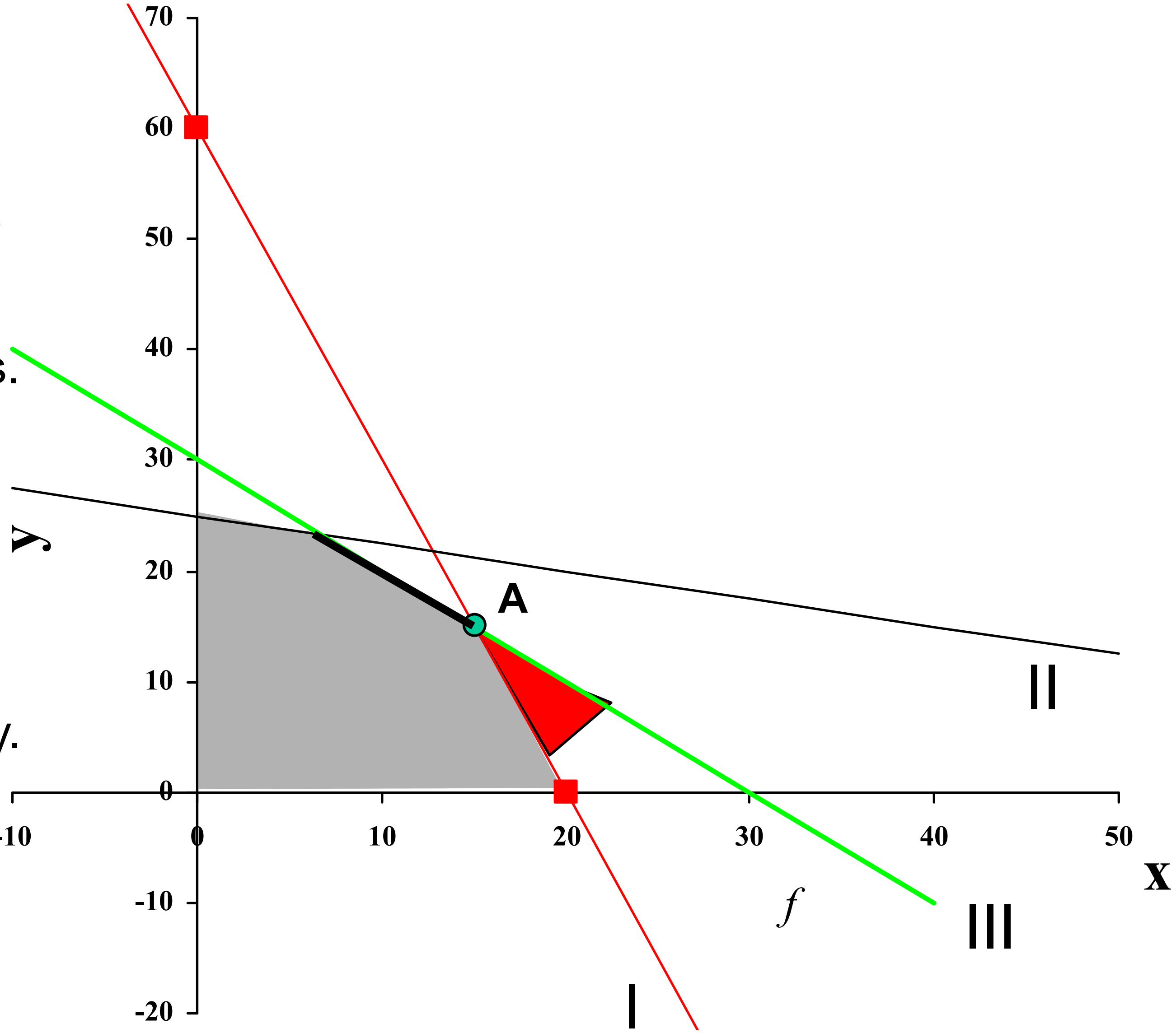
Case 1 - changes in the coefficients of the objective function (c_j)

- If $-\frac{a}{b} = -1$

(e.g, a = 4, b=4 or a=3, b=3. …),

The slope of **f** is equal to the slope of **III** and we will also have an infinite number of optimal solutions.

Note that, for the same production plan (x=15, y=15), the profit is different if we have 4x+4y or 3x+3y.

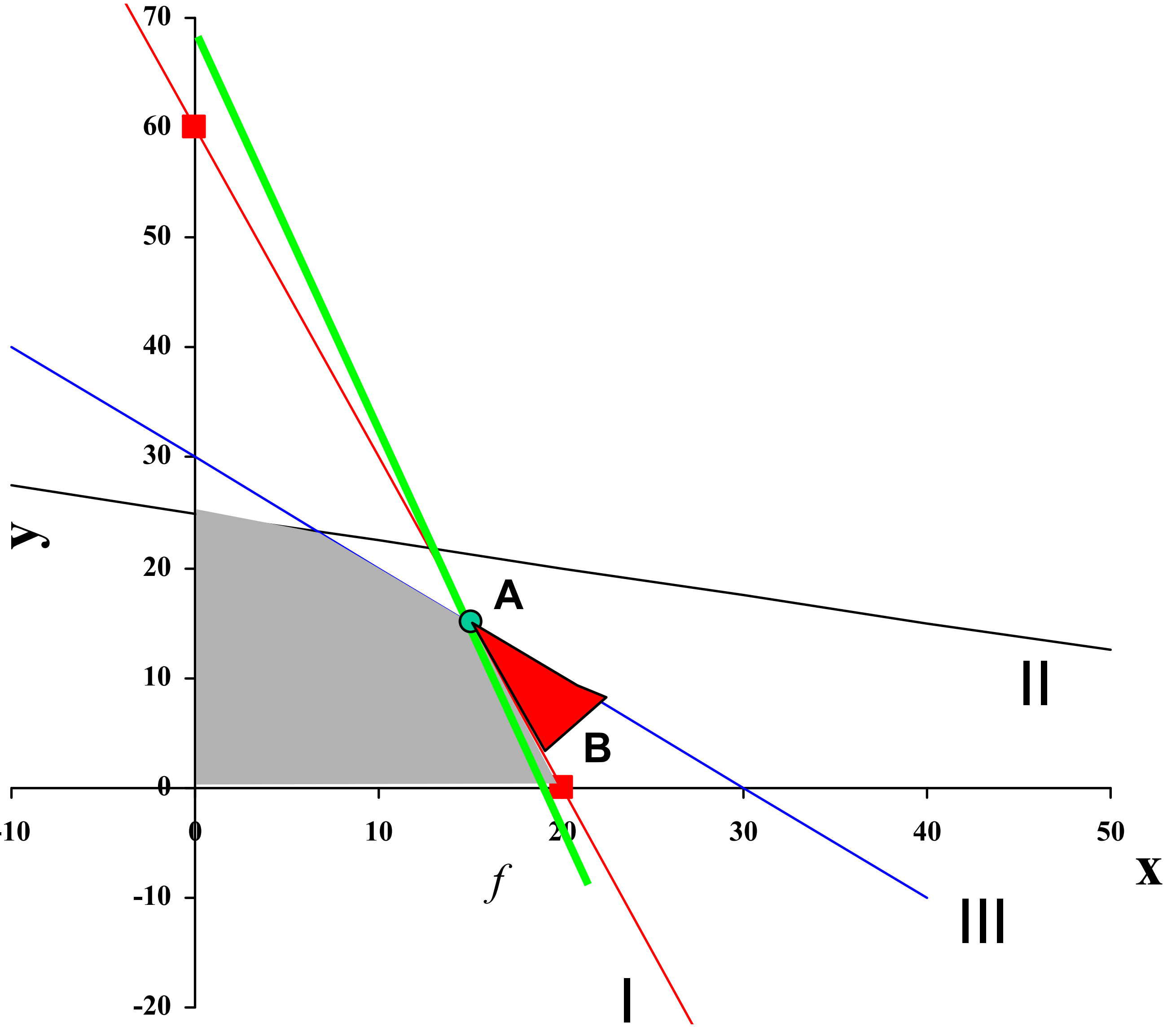


Case 1 - changes in the coefficients of the objective function (c_j)

What happens if **f** rotates beyond angle α?

- If $-\frac{a}{b} < -3$, we can see graphically that the new optimal solution is **B**.

- But, in the general case, we can only say that the optimal solution will change and it is necessary to solve the new problem.



Case 2: changes in the right side of constraints (b_i)

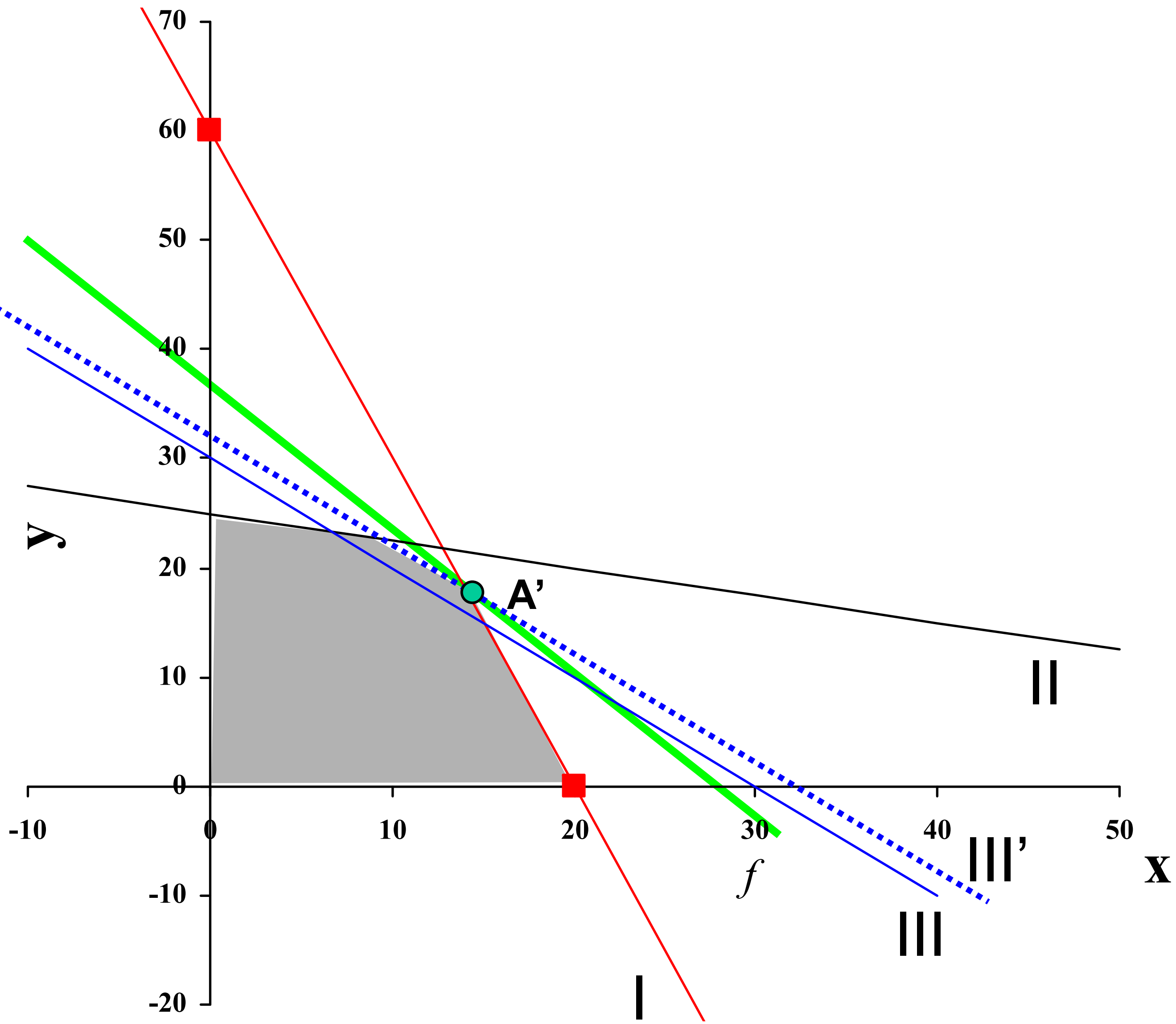
Consider now constraint **III**:

$$5x + 5y \leq 150$$

What happens if we increase the production capacity (k) in section **III**?

Let $5x + 5y \leq k$

As **k** increases, we will have lines (like **III'**) parallel to **III**, and the optimal solution is in the intersection point of **I** and **III'** (A').



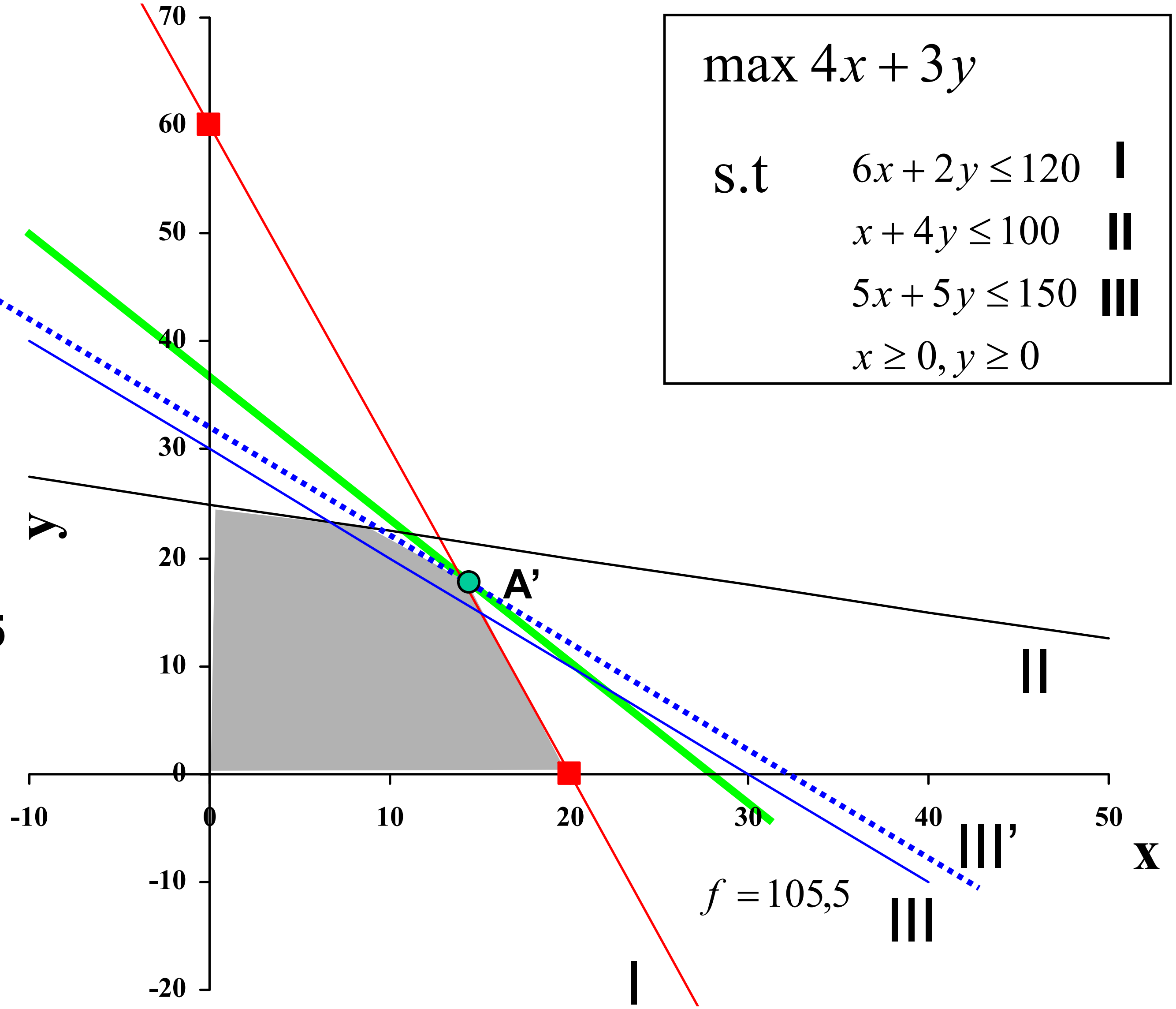
Case 2: changes in the right side of constraints (b_i)

- If we increase one unit to resource **III**: **k = 151**.
- The new optimal solution is the intersection point of:

$$\begin{cases} 5x+5y=151 \\ 6x+2y=120 \end{cases} \Leftrightarrow \begin{cases} x=14{,}9 \\ y=15{,}3 \end{cases}$$

The profit changed from **f* =105** to:

$$4x+3y = 4\times 14{,}9 + 3\times 15{,}3$$
$$= 105{,}5$$



$$\max\ 4x+3y$$

s.t

$6x+2y \le 120$ I

$x+4y \le 100$ II

$5x+5y \le 150$ III

$x \ge 0, y \ge 0$

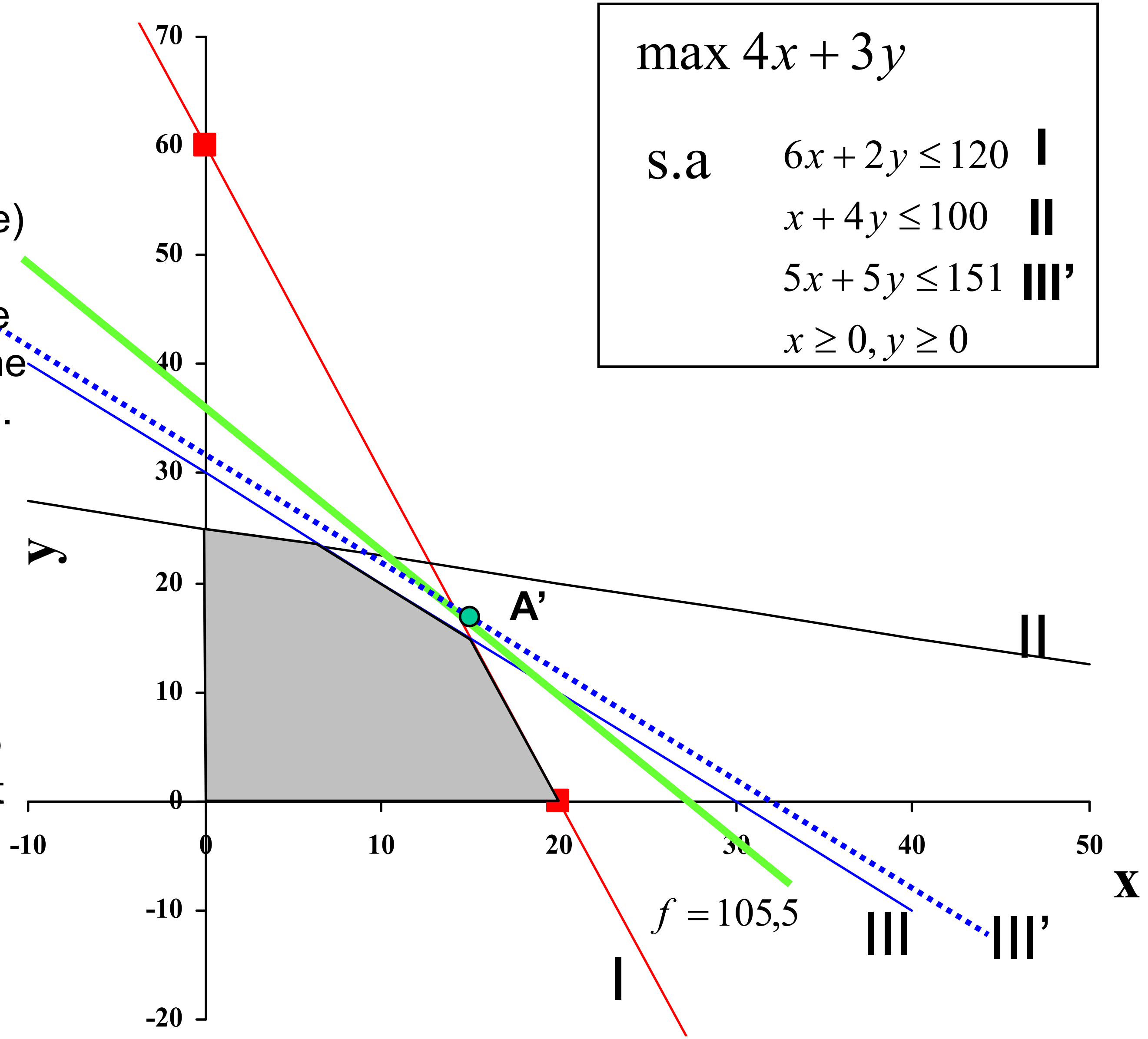
Shadow price

- The amount added to profit (in this case) as a result of the additional unit of resources is seen as the marginal value of the resources and is referred to as the **opportunity cost** or the **shadow price**.

- In this example, the shadow price of resource **III** is the marginal profit obtained when we have an additional hour in the packing section.
- Since the profit has increased from **105** to **105,5**, the shadow price of constraint **III** is **0,5 €**.

$$\max 4x+3y$$

$$\text{s.a} \quad \begin{aligned} 6x+2y &\le 120 && \textbf{I} \\ x+4y &\le 100 && \textbf{II} \\ 5x+5y &\le 151 && \textbf{III'} \\ x\ge 0,\ y &\ge 0 \end{aligned}$$



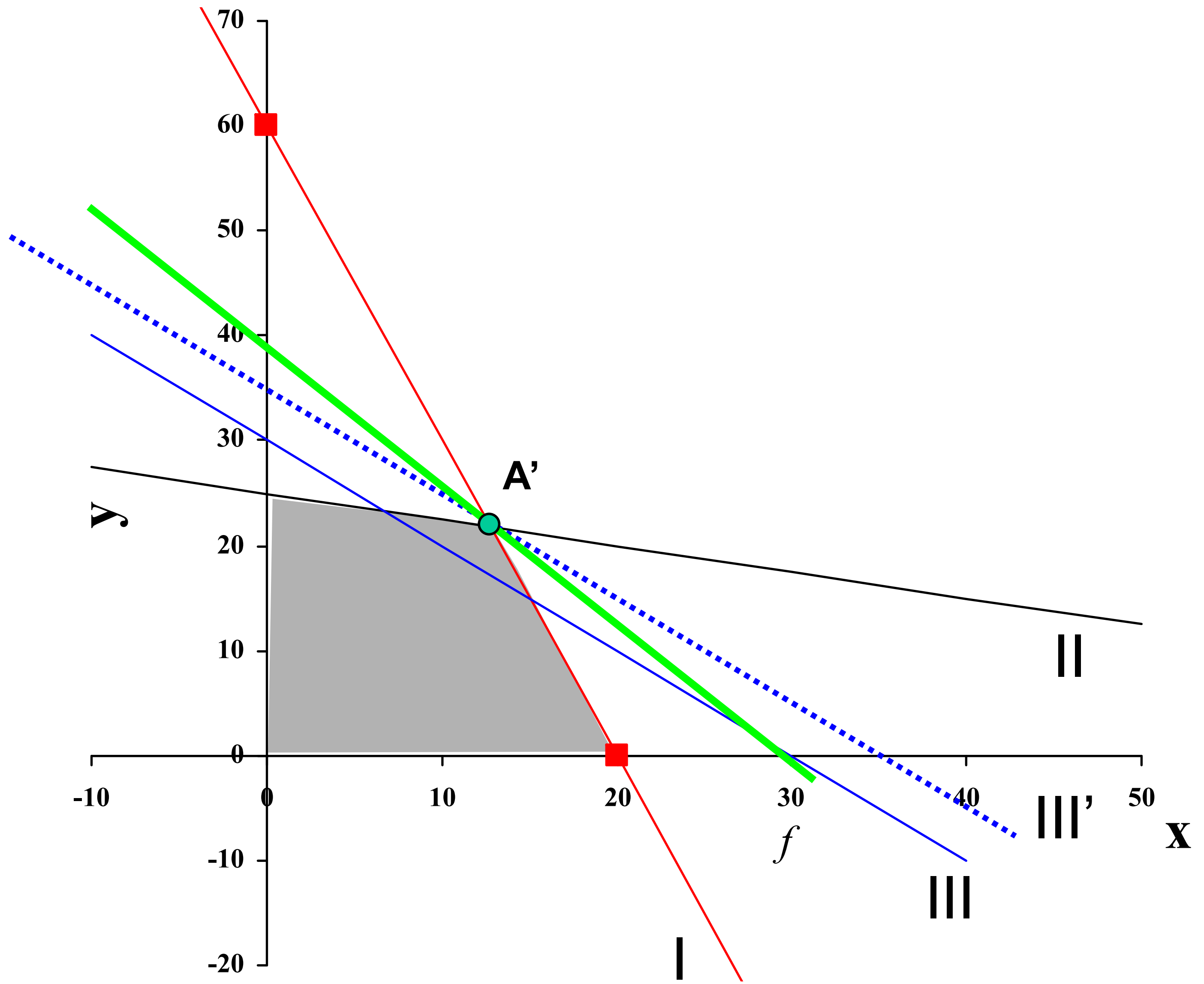
Case 2: changes in the right side of constraints (b_i)

- When **k = 172,727,** the optimal solution lies in the intersection point of **I**, **II** and **III'**.

- What happens if **k > 172,727** ?

  For example, **k=175**

$$5x + 5y \leq 175$$



Case 2: changes in the right side of constraints (b_i)

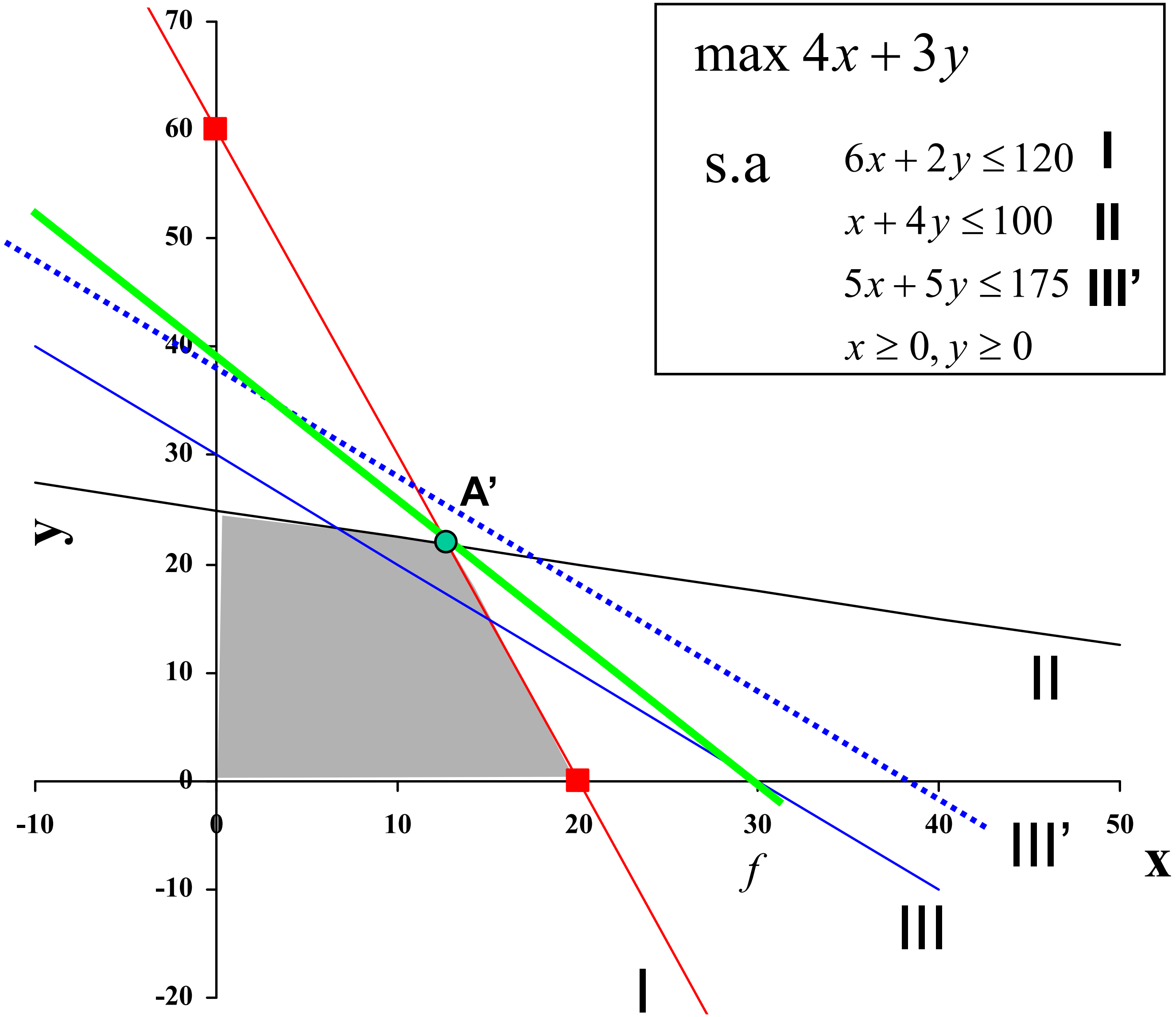
- If $5x + 5y \leq 175$ (**III'**) the optimal solution **A'** lies in the intersection point of **I** and **II** and constraint **III'** becomes redundant.

  In this case the shadow price meaning no longer applies, since the increase of a unit in resource III does no longer corresponds to a variation in the objective function

  We will have to solve the new problem.

$$\max 4x + 3y$$

$$\text{s.a} \quad \begin{array}{ll} 6x + 2y \leq 120 & \textbf{I} \\ x + 4y \leq 100 & \textbf{II} \\ 5x + 5y \leq 175 & \textbf{III'} \\ x \geq 0, y \geq 0 & \end{array}$$



Case 2: changes in the right side of constraints (b_i)

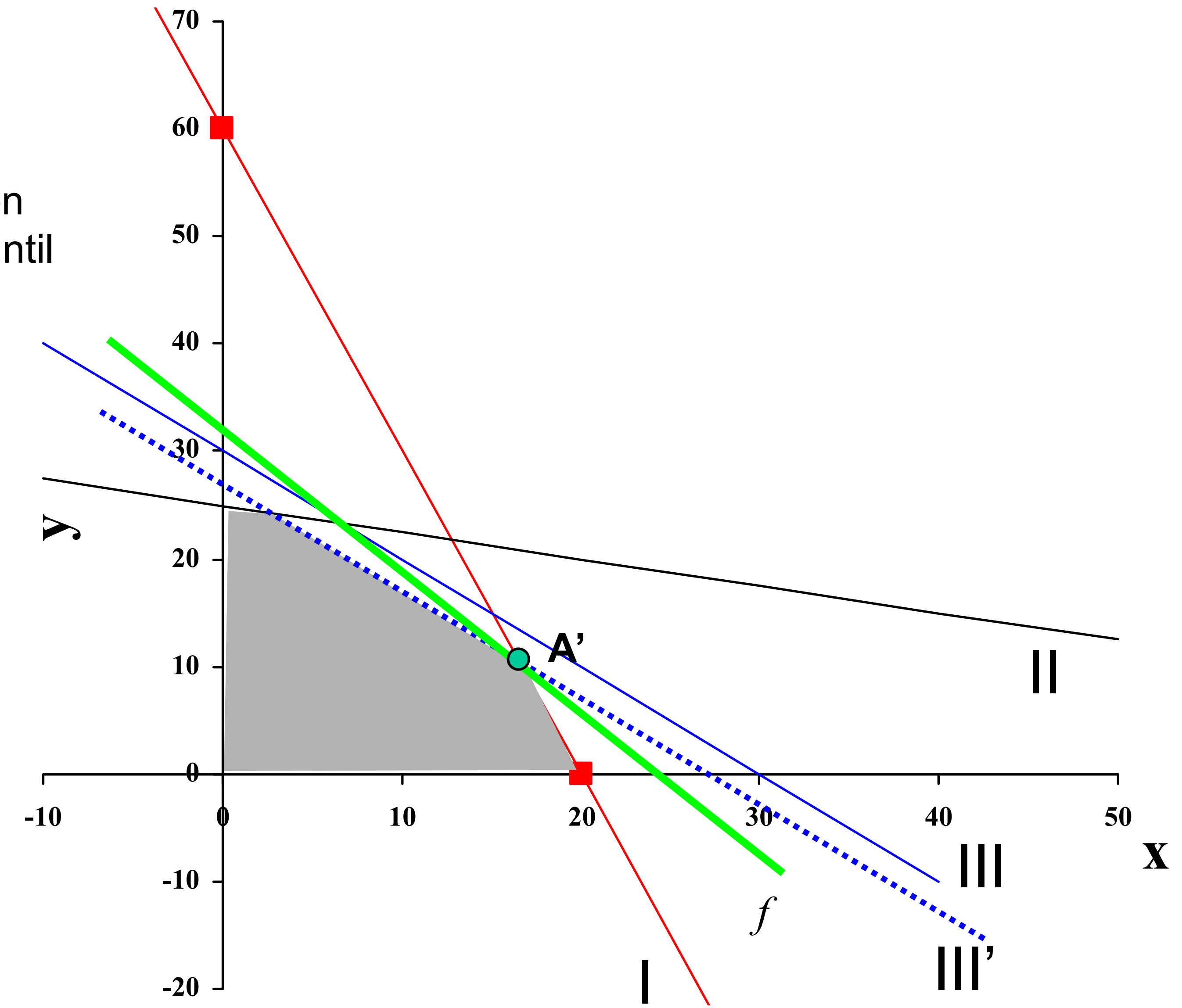
- What happens when **k < 150**?

  As k decreases, the optimal solution lies in the intersection of **I** and **III'** until **k=100** (why?).

- And when **k < 100** ?

  For example, if k=90

$$5x + 5y \le 90$$

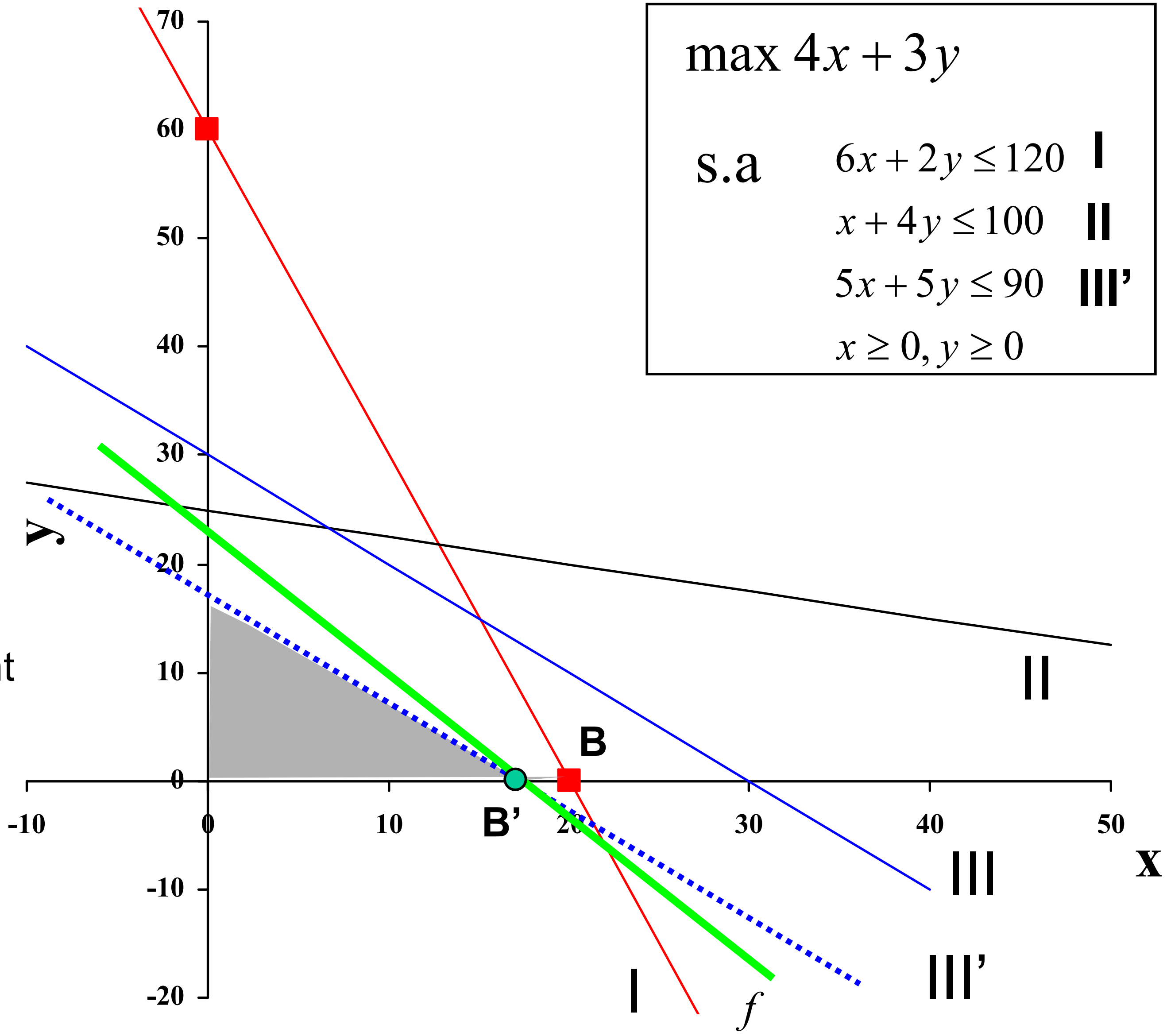


Case 2: changes in the right side of constraints (b_i)

- If $5x+5y \le 90$ the optimal solution no longer lies in the intersection of **I** with **III'**. Now it lies in the intersection of **III'** with **y=0** (we say that the *basis has changed*).

- Now we have a slack in constraint I. The value and the meaning of the shadow price associated to constraint **III** no longer apply.

- We will have to solve the new problem.



$$\max\ 4x+3y$$

s.a
$$6x+2y \le 120 \quad \text{I}$$
$$x+4y \le 100 \quad \text{II}$$
$$5x+5y \le 90 \quad \text{III'}$$
$$x \ge 0,\ y \ge 0$$

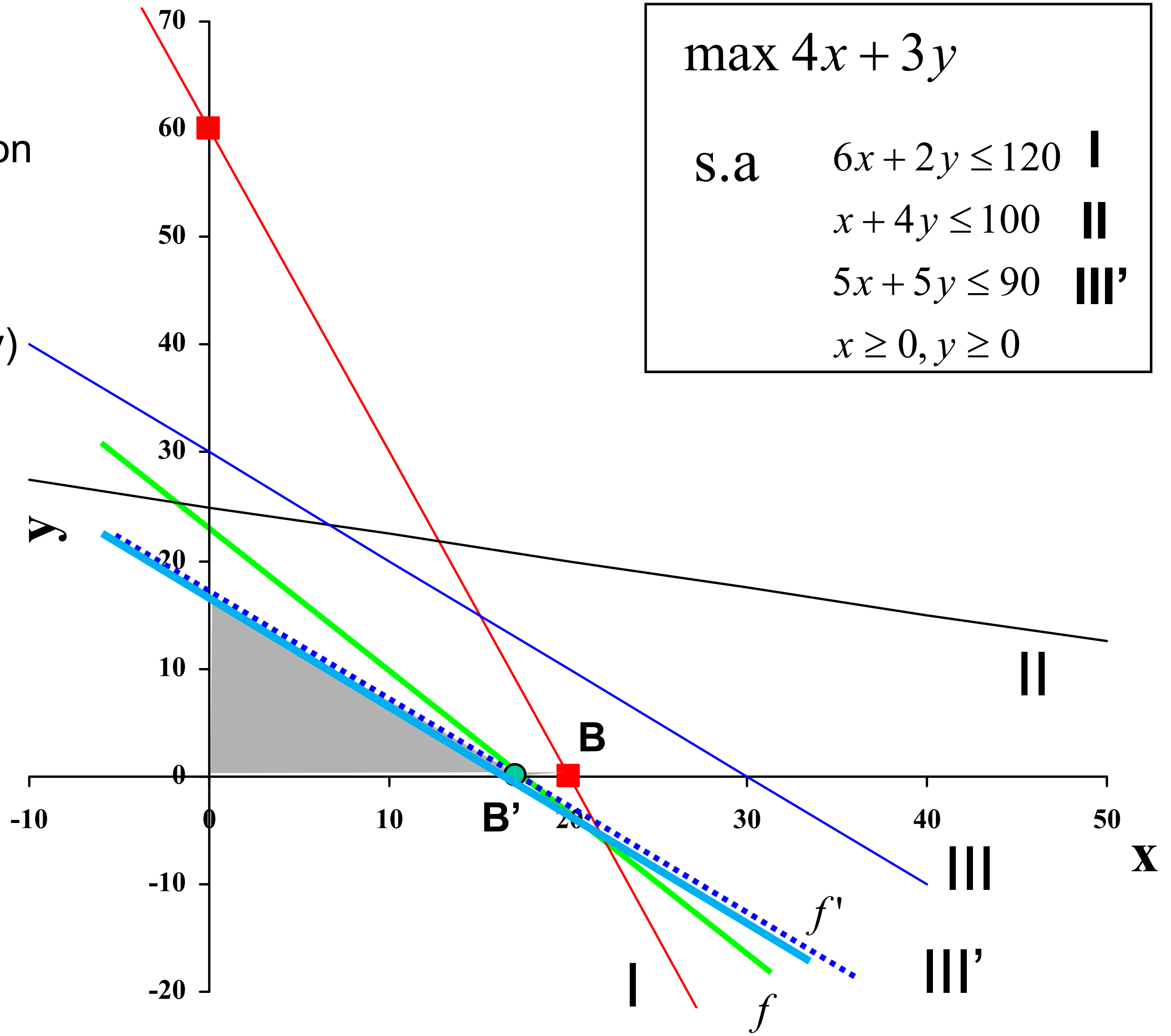
Reduced cost

- In this case, the optimal solution indicates that the optimal production plan should not include the production of corn (**y=0**).

- The **reduced cost** of this product shows the increment that the corn (y) unitary profit should have in order to include it in the optimal production plan.

- In this example, if the corn unitary profit is 4 €/ton (f’), it may be considered in the production plan.

Hence, since the increment is of 1€/ton, the **reduced cost of y is 1**.



$$\max 4x + 3y$$

s.a

$$6x + 2y \le 120 \quad \textbf{I}$$
$$x + 4y \le 100 \quad \textbf{II}$$
$$5x + 5y \le 90 \quad \textbf{III'}$$
$$x \ge 0, y \ge 0$$

Case 2: changes in the right side of constraints (b_i)

Consider now constraint **II**

$$x + 4y \le 100$$

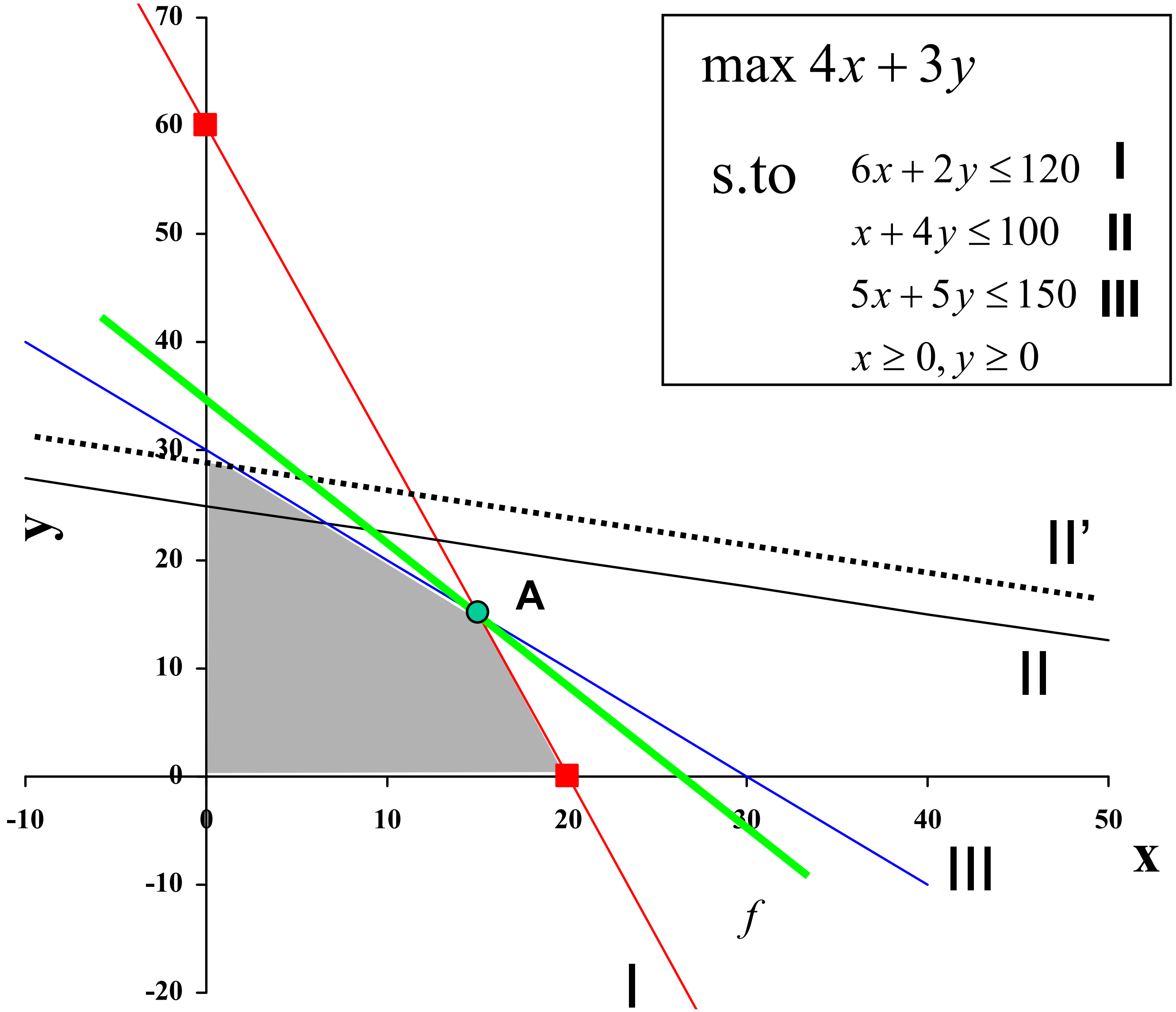
What happens if we increase the production capacity (k) in section **II**?

$$x + 4y \le k$$

- For k > 100, the optimal solution does not change.

- And when k < 100?
  (for example, if k = 75?)

- And if k < 75?



$$\max\ 4x + 3y$$

s.to

$$6x + 2y \le 120 \quad \text{I}$$
$$x + 4y \le 100 \quad \text{II}$$
$$5x + 5y \le 150 \quad \text{III}$$
$$x \ge 0,\ y \ge 0$$

In short, for a maximization problem

Case 1 - changes in the coefficients of the objective function (c_j)

For a particular optimal solution, and for each coefficient of the objective function (cj), it is possible to determine an interval of variation that will keep the optimal solution unchanged (note that the value of the objective function may change).

If, in the optimal solution, the value of a decision variable is zero (xi = 0), its reduced cost is the increment that the corresponding coefficient in the objective function should have in order to include that variable in the optimal solution (xi > 0).

Case 2: changes in the right side of constraints (b_i)

(i) If a constraint is active (there is no slack or surplus) then increasing or decreasing the amount of the resource associated to that constraint could lead to a change in the value of the objective function in the optimal solution.
The shadow price of a resource is the increment in the objective function generated by an additional unit of that resource.

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(ii) If there is a slack in a constraint (the constraint is inactive), the value of the objective function in the optimal solution does not alter if we increase the amount of available resource. However, if we decrease the amount of available resource, the value of the objective function in the optimal solution may change.

Solution of Cereals, Ltd obtained by Lindo software (http://www.lindo.com)

LP OPTIMUM FOUND AT STEP 1

OBJECTIVE FUNCTION VALUE

1) 105.0000

VARIABLE	VALUE	REDUCED COST
X	15.000000	0.000000
Y	15.000000	0.000000

ROW	SLACK OR SURPLUS	**DUAL PRICES**
2)	0.000000	0.250000
3)	25.000000	0.000000
4)	0.000000	0.500000

SENSITIVITY ANALYSIS

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VAR	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X	4.000000	5.000000	1.000000
Y	3.000000	1.000000	1.666667

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	120.000000	60.000000	33.333332
3	100.000000	INFINITY	25.000000
4	150.000000	22.727272	49.999996

Exercise
Sensitivity Analysis using the optimal solution obtained by Lindo

Consider again the example of Cereals, Ltd.

Due to new market challenges, it was decided to produce a new cereal, barley (‘cevada’, in Portuguese).
The marginal profits now are of 4 € per ton of wheat, 1 € per ton of corn and 3 € per ton of barley. In each section (I, II and III) , the production times for each ton of barley are 3.5, 6 and 4 hours, respectively. Also, the company is committed to produce at least 12 tons of wheat and 10 tons of barley each week.

Problem formulation:

$$
\begin{aligned}
&\max\ 4x + y + 3z \\
&\text{s.to}\quad 6x + 2y + 3.5z \le 120 \\
&\qquad\quad x + 4y + 6z \le 100 \\
&\qquad\quad 5x + 5y + 4z \le 150 \\
&\qquad\quad x \ge 12 \\
&\qquad\quad z \ge 10 \\
&\qquad\quad x, y, z \ge 0
\end{aligned}
$$

The new problem has been solved using Lindo, and we obtained the following tableaux for the optimal solution

OBJECTIVE FUNCTION VALUE

1) 89.14286

VARIABLE	VALUE	REDUCED COST
X	12.000000	0.000000
Y	0.000000	0.714286
Z	13.714286	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	0.857143
3)	5.714286	0.000000
4)	35.142857	0.000000
5)	0.000000	-1.142857
6)	3.714286	0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X	4.000000	1.142857	INFINITY
Y	1.000000	0.714286	INFINITY
Z	3.000000	INFINITY	0.666667

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	120.000000	3.333333	12.999999
3	100.000000	INFINITY	5.714286
4	150.000000	INFINITY	35.142857
5	12.000000	2.166667	0.615385
6	10.000000	3.714286	INFINITY

Questions:

1. What is the profit in the optimal solution? What is the optimal production plan?
2. Suppose that the wheat (x) profit has an increment of 0.75 €/ton. Which is the impact of this change on the optimal production plan and on the profit? And if the increment is of 1.5 €/ton? And if the wheat profit decreases 0.05 €/ton? And if it decreases 2 €/ton?
3. What should be done to make corn production (y) profitable?
4. If the company was forced to produce some corn, which would be the impact of that decision on the company profit?
5. Suppose that there is an additional hour available in section I. What is the impact on profit? And if there are 2 additional hours?
4. Suppose that the number of available hours in section I diminishes to 119 hours. What is the impact on profit? And if we only dispose of 105 hours in this section?

6) Comment the importance of hiring multifunctional employees that can work in different sections.
7) What happens if the minimum amount of wheat to produce increases of 1.5 tons? And if it decreases of 0,29 tons?
8) What happens if the minimum amount of corn to produce increases of 0.5 tons? And if it decreases of 7 tons?

Linear Programming

Simplex Method

Simplex Method

Motivation:

The graphical method cannot be applied to problems with more than 2 variables.

Basic Idea:

The Simplex method is based in the fact that any LP optimal solution lies on a vertex of the feasible region.

Basic Method:

- Start by calculating the objective function value for any vertex of the domain.
- Jump to an adjacent vertex corresponding to a better objective function value.
- Continue with this process until it is no possible to improve the objective function.

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Weekly production capacity (h)	120	100	150	
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Wheat	6 h	1 h	5 h	4
Corn	2 h	4 h	5 h	3

Currently, the company produces 18 tons of wheat and 6 tons of corn per week.
Is this the best option?

Cereals, Ltd - Formulation

Decision variables

x = tons of wheat to produce weekly

y = tons of corn to produce weekly

Constraints

$$6x + 2y \le 120$$

$$x + 4y \le 100$$

$$5x + 5y \le 150$$

$$x \ge 0, y \ge 0$$

Objective function: to maximize the profit

$$\max 4x + 3y$$

Simplex Method

Step 1: Writing the problem in the canonical form

Consider that all the inequalities (constraints) are of <= type with positive values in the right side. The LP is in canonical form when the inequalities are changed to equalities by adding a **slack variable** to each constraint.

Maximizar (ou Minimizar) $f = c_1 \cdot x_1 + c_2 \cdot x_2 + ... + c_n \cdot x_n$

sujeito a

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + ... + a_{1n} \cdot x_n = b_1$$
$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + ... + a_{2n} \cdot x_n = b_2$$
$$\cdots\cdots\cdots$$
$$a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + ... + a_{mn} \cdot x_n = b_m$$

Com: $m < n$ restrições

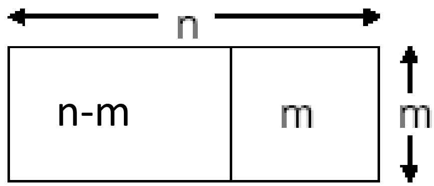
$$b_1, b_2, ..., b_m \geq 0$$
$$x_1, x_2, ..., x_n \geq 0$$

Exemplo:

$$\max f = 4x_1 + 3x_2$$
$$6x_1 + 2x_2 + s_1 = 120$$
$$x_1 + 4x_2 + s_2 = 100$$
$$5x_1 + 5x_2 + s_3 = 150$$
$$x_1 \geq 0, x_2 \geq 0$$

Simplex Method (Step 2)

Step 2: Find an initial feasible basic solution



Consider a LP in the canonical form with

- n variables
- m constraints (equalities), with m<n

A **basic solution** is obtained by assigning n-m variables to zero and solving the constraints (equations) for the remaining variables (m).

The n-m null variables are referred to as **non-basic variables**.
The others are the **basic variables**.

The basic solutions can be:

feasible basic solutions: when all the basic variables are non-negative.
infeasible basic solutions: when at least one basic variable is negative.

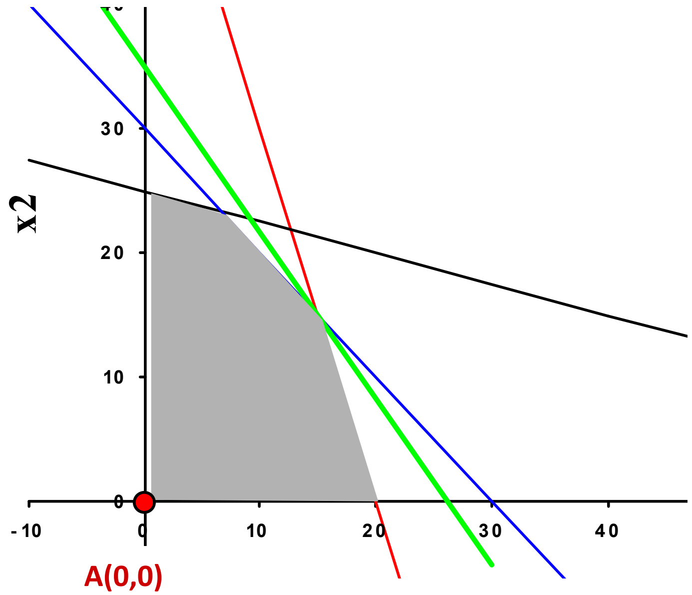
$$\max \text{ f} = 4x_1 + 3x_2$$

$$6x_1 + 2x_2 + s_1 = 120$$

$$x_1 + 4x_2 + s_2 = 100$$

$$5x_1 + 5x_2 + s_3 = 150$$

$$x_1 \geq 0, x_2 \geq 0$$



Tabular form

basis	X1	X2	S1	S2	S3	value
S1	6	2	1	0	0	120
S2	1	4	0	1	0	100
S3	5	5	0	0	1	150
f	4	3	0	0	0	0

Basic variables: S1 = 120, S2 = 100, S3 = 150

Non-basic variables: X1 = 0, X2 = 0

Point A

- The coefficient matrix of the basic variables is an <u>identity matrix</u> (or is convertible into one by row or column swaps)
- The coefficients of the basic variables in the objective function are <u>null</u>.

Simplex Method (step 3)

Step 3: Verify if the basic solution found is optimal:

For a maximization problem, if all the coefficients in the objective function are non-positive (<=0), then the problem is solved.

For a minimization problem, if all the coefficients in the objective function are non-negative (>=0), then the problem is solved.

basis	X1	X2	S1	S2	S3	value
S1	6	2	1	0	0	120
S2	1	4	0	1	0	100
S3	5	5	0	0	1	150
f	4	3	0	0	0	0

- At this moment, the value of x1 and x2 is zero (and also the o.f. value).
- As both coefficients in the objective function (o.f.) are positive numbers, if any of these variables becomes positive, the value of the o.f. would increase

  (f = 4 x1 + 3 x2).

Which of these two variables should be chosen to enter the basis?

Simplex Method (Step 4)

Step 4: Find a new basic solution that improves the objective function

To find a new basic solution, we will choose a **non-basic variable to enter the basis** and a **basic variable to leave the basis.**

Step 4.1: **Choose a non-basic variable to enter the basis** . The column corresponding to this variable is called *pivot column*.
In a maximization problem choose, amongst the variables with positive coefficients, the one with the highest positive value. In a minimization problem choose, amongst the variables with negative coefficients, the one with the highest negative value

Step 4.2 Choose a basic variable to leave the basis, The row corresponding to this variable is called *pivot row*.
Calculate the ratio between the right side members of the equations and the corresponding members in the pivot column. From the set of non-negative ratios, the row with the lowest ratio will be the pivot row.

	basis	X1	X2	S1	S2	S3	value		
L1	S1	6	2	1	0	0	120	120/6 =	20
L2	S2	1	4	0	1	0	100	100/1 =	100
L3	S3	5	5	0	0	1	150	150/5 =	30
f	f	4	3	0	0	0	0		

Pivot row

*Choose to leave the basis the variable with the lowest **non-negative** ratio.*

Pivot column

In a maximization problem, choose to enter the basis the variable with the highest positive value in the o.f.

-f = 0

This value is the simmetric of the o.f. value for the current solution.

x1 was chosen to enter the basis, keeping x2 = 0 (non-basic).
x1 should take the highest possible value while satisfying the constraints.

$$s_1 = 120 - 6x_1 \quad \text{Since } s_1 \geq 0, 120 - 6x_1 \geq 0 \Leftrightarrow \boxed{x_1 \leq \frac{120}{6} = 20}$$

Choose the most restrictive condition

$$s_2 = 100 - x_1 \quad \text{Since } s_2 \geq 0, 100 - x_1 \geq 0 \Leftrightarrow x_1 \leq 100$$

$$s_3 = 150 - 5x_1 \quad \text{Since } s_3 \geq 0, 150 - 5x_1 \geq 0 \Leftrightarrow x_1 \leq \frac{150}{5} = 30$$

When x1 = 20, S1 is null (leaves the basis)

	basis	X1	X2	S1	S2	S3	value		
L1	S1	6	2	1	0	0	120	120/6 =	20
L2	S2	1	4	0	1	0	100	100/1 =	100
L3	S3	5	5	0	0	1	150	150/5 =	30
f	f	4	3	0	0	0	0		

Pivot row

Choose to leave the basis the variable with the lowest non-negative ratio.

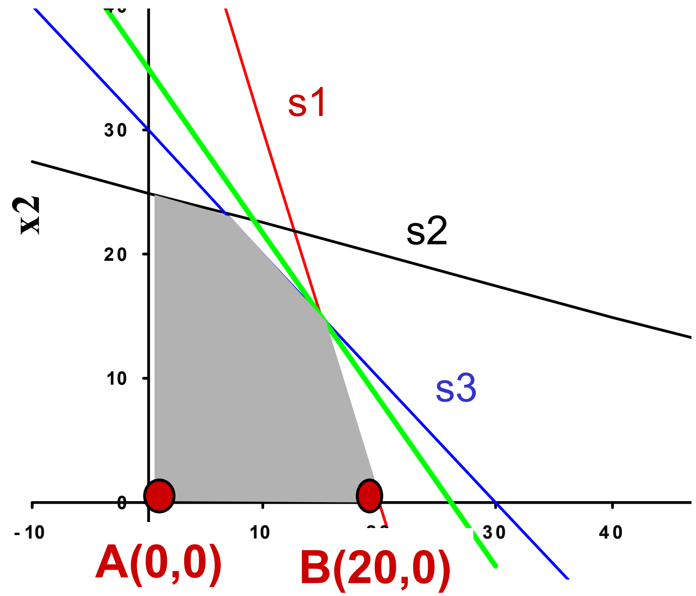
Pivot column

-f = 0

In a maximization problem, choose to enter the basis the variable with the highest positive value in the o.f.

X1 enters the basis, taking a non-negative value
S1 leaves the basis , taking the null value

Going from point A to Point B (20,0)



Simplex Method (step 5)

Step 5: Update Simplex tableaux to identify the new basic solution

The procedure is based in algebraic operations performed on the rows of the Simplex tableaux in order to build a new identity matrix with the rows and columns of the basic variables.

We perform algebraic operations in order to set the value 1 to the intersection of the pivot row and the pivot column and zero values in all the other coefficients of the pivot column (including the o.f.).

After identifying the new basic solution, go to step 3 to verify if the new solution is optimal.

Iteration 0 (point A)

	basis	X1	X2	S1	S2	S3	value		
L1	S1	6	2	1	0	0	120	120/6 =	20
L2	S2	1	4	0	1	0	100	100/1 =	100
L3	S3	5	5	0	0	1	150	150/5 =	30
f	f	4	3	0	0	0	0		

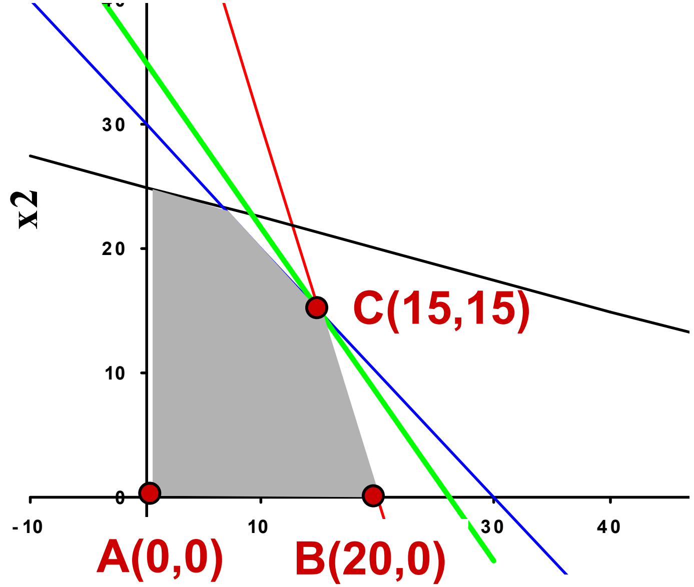
Divide by 6 all the values in this line

Iteration 1 (point B)

	basis	X1	X2	S1	S2	S3	value		
L'1 = L1/6	X1	1	0,3333	0,1667	0	0	20	20/0,33=	60
L'2 = L2 - L'1	S2	0	3,6667	-0,167	1	0	80	80/3,66=	21,82
L'3 = L3 - 5.L'1	S3	0	3,3333	-0,833	0	1	50	50/3,33=	15
f' = f' - 4.L'1	f	0	1,6667	-0,667	0	0	-80		

X2 enters the basis with a non-negative value
S3 leaves the basis, with null value

Going from point B to Point C (15,15)




BRACE YOURSELVES
(ANOTHER) ITERATION IS COMING

Iteration 1 (point B)

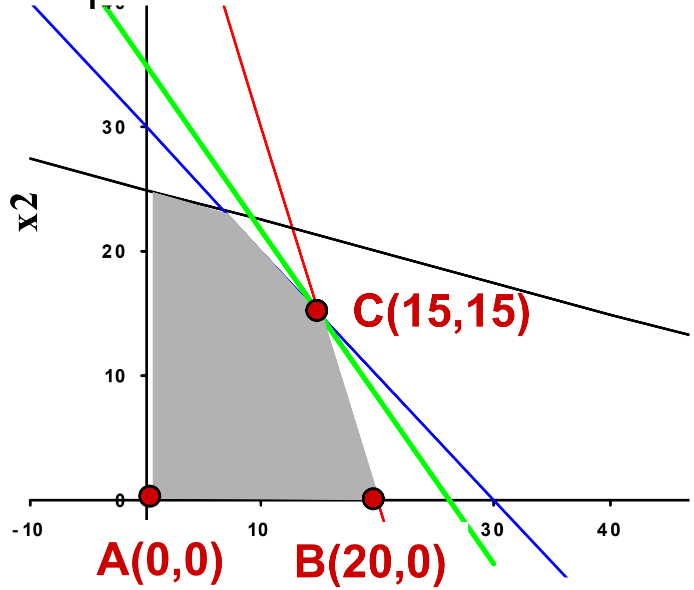
basis	X1	X2	S1	S2	S3	value
X1	1	0,3333	0,1667	0	0	20
S2	0	3,6667	-0,167	1	0	80
S3	0	3,3333	-0,833	0	1	50
f	0	1,6667	-0,667	0	0	-80

Iteration 2 (point C)

	basis	X1	X2	S1	S2	S3	value
L"1 = L'1 - 0,33.L"3	X1	1	0	0,25	0	-0,1	15
L"2 = L'2 - 3,67.L"3	S2	0	0	0,75	1	-1,1	25
L"3 = L'3 / 3,33	X2	0	1	-0,25	0	0,3	15
f" = f' - 1,67.L"3	f	0	0	-0,25	0	-0,5	-105

***Point C is the optimal solution**, since none of the coefficients in the o.f is positive (we are solving a maximization problem)*

f = 4*X1+3*X2 = 4*15+3*15 = 60+45=**105**



Exercise 1

$$\max f = -x_1 + 2x_2 + x_3$$
$$2x_1 + x_2 - x_3 \leq 2$$
$$2x_1 - x_2 + 5x_3 \leq 6$$
$$4x_1 + x_2 + x_3 \leq 6$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Iteration 0 : Point (0,0,0)

	basis	X1	X2	X3	S1	S2	S3	value		
L1	S1	2	1	-1	1	0	0	2	2/1 =	2
L2	S2	2	-1	5	0	1	0	6		
L3	S3	4	1	1	0	0	1	6	6/1 =	6
f	f	-1	2	1	0	0	0	0		

Variable X2 enters the basis and variable S1 leaves the basis

Iteration 1 : Point (0,2,0)

	basis	X1	X2	X3	S1	S2	S3	value		
L'1 = L1	X2	2	1	-1	1	0	0	2		
L'2 = L2+L'1	S2	4	0	4	1	1	0	8	8/4 =	2
L'3 =L3-L'1	S3	2	0	2	-1	0	1	4	4/2 =	2
f' = f-2*L'1	f	-5	0	3	-2	0	0	-4		

Variable X3 enters the basis and variable S2 leaves the basis

Iteration 1 : Point (0,2,0)

	basis	X1	X2	X3	S1	S2	S3	value		
L'1 = L1	X2	2	1	-1	1	0	0	2		
L'2 = L2+L'1	S2	4	0	4	1	1	0	8	8/4 =	2
L'3 =L3-L'1	S3	2	0	2	-1	0	1	4	4/2 =	2
f' = f-2*L'1	f	-5	0	3	-2	0	0	-4		

Variable X3 enters the basis and variable S2 leaves the basis

Iteration 2 : Point (0,4,2)

	basis	X1	X2	X3	S1	S2	S3	value
L"1 =L'1+L"2	X2	3,00	1	0	1,25	0,25	0	4,00
L"2 = L'2/4	X3	1,00	0	1,00	0,25	0,25	0	2,00
L"3 =L'3-2L"2	S3	0	0	0	-1,5	-0,5	1	0
f" = f' - 3*L"2	f	-8	0	0	-2,75	-0,75	0	$\boxed{-10}$

Optimal solution: X1 = 0
X2 = 4
X3 = 2

Optimal value of o.f. = 10