Duality – Revision from last lecture

Primal

$$\max x_1 + 2x_2$$
$$x_1 + x_2 = 4$$
$$x_2 \leq 2$$
$$x_1 \geq 1$$
$$x_1, x_2 \geq 0$$

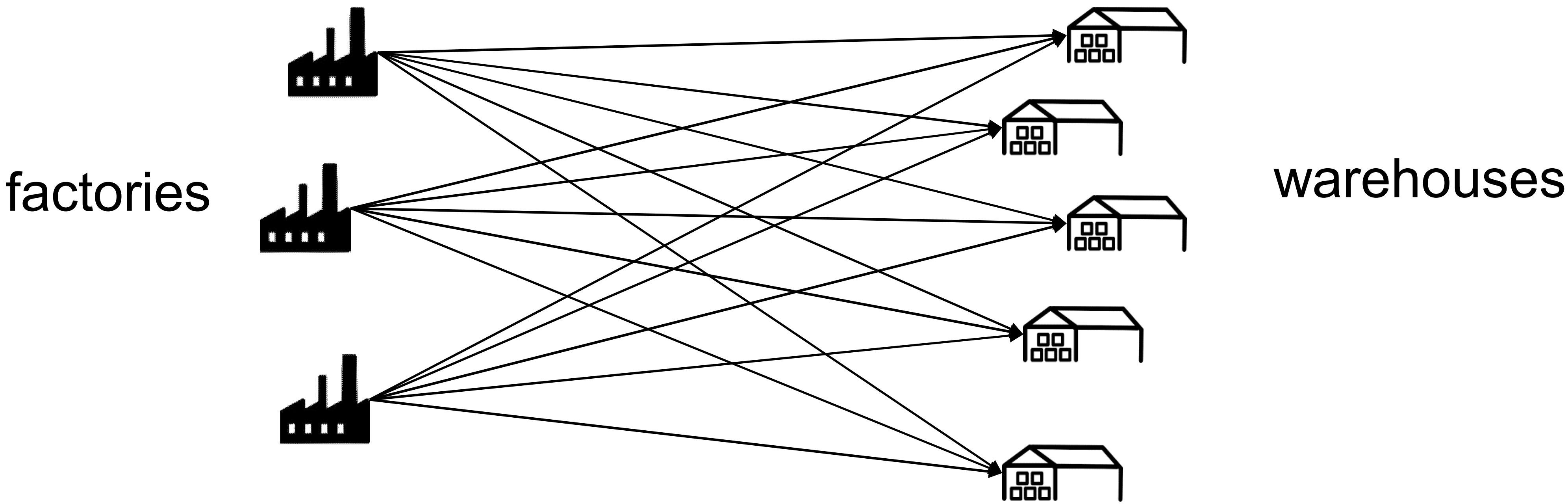
Primal standard form

$$\max x_1 + 2x_2$$
y_1 $$x_1 + x_2 \leq 4$$
y_2 $$-x_1 - x_2 \leq -4$$
y_3 $$x_2 \leq 2$$
y_4 $$-x_1 \leq -1$$
$$x_1, x_2 \geq 0$$

Dual

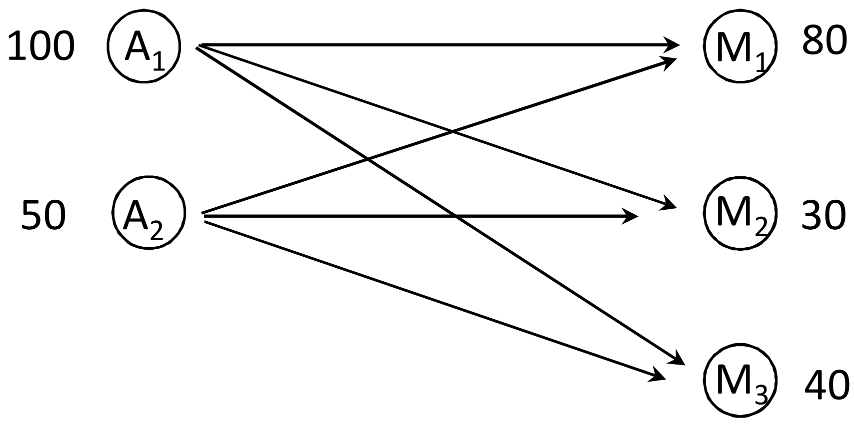
$$\min 4y_1 - 4y_2 + 2y_3 - y_4$$
$$y_1 - y_2 - y_4 \geq 1$$
$$y_1 - y_2 + y_3 \geq 2$$
$$y_1, y_2 \geq 0$$

The Transportation Problem



Transportation Problems

A company has two warehouses A1 and A2 that store 100 and 80 units, respectively, of a given product. From these two warehouses the company supplies three markets **M_1**, **M_2** and **M_3** consuming 80, 30 and 40 units of the product, respectively.



Transportation Costs

	M1	M2	M3
A1	5	3	2
A1	2	2	1

Decision variables

x_{ij}: amount of product to send from origin i to destination j

$$
\begin{array}{l}
\min\ z = 5x_{11} + 3x_{12} + 2x_{13} + 2x_{21} + 2x_{22} + x_{23} \\
\text{s.a}
\end{array}
$$

$$
\begin{array}{cccccccc}
x_{11} & +x_{12} & +x_{13} & & & & = & 100 \\
 & & & x_{21} & +x_{22} & +x_{23} & = & 50 \\
x_{11} & & & +x_{21} & & & = & 80 \\
 & x_{12} & & & +x_{22} & & = & 30 \\
 & & x_{13} & & & +x_{23} & = & 40 \\
x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23} & \geq & 0
\end{array}
$$

LP Formulation

$$\min \ z = \sum_i \sum_j c_{ij} \, x_{ij}$$

$$\text{s.a} \quad \sum_j x_{ij} = a_i \qquad (i = 1,\ldots, m) \qquad \text{supply constraints}$$

$$\sum_i x_{ij} = b_j \qquad (j = 1,\ldots, n) \qquad \text{demand constraints}$$

$$x_{ij} \geq 0, \qquad (i = 1,\ldots, m; \ j = 1,\ldots, n)$$

The particular structure of the coefficient matrix is characterized by the following:

- it only has 1's and 0's
- each **xij** variable appears only in two constraints (one is a supply constraint and the other is a demand constraint)
- $\sum_i a_i = \sum_j b_j \Rightarrow \sum_i \sum_j x_{ij} = \sum_j \sum_i x_{ij}$: one of the constraints is redundant because it is the linear combination of the others
- m x n variables
- **m+n-1 basic variables** (= number of independent constraints)
- (m x n) - (m+n-1) non-basic variables

$$\min z = 5x_{11} + 3x_{12} + 2x_{13} + 2x_{21} + 2x_{22} + x_{23}$$

s.a

$$\left.\begin{array}{llllllll}
x_{11} & +x_{12} & +x_{13} & & & & = & 100 \\
 & & & x_{21} & +x_{22} & +x_{23} & = & 50 \\
x_{11} & & & +x_{21} & & & = & 80
\end{array}\right\} \text{Supply constraints}$$

$$\left.\begin{array}{llllllll}
 & x_{12} & & & +x_{22} & & = & 30 \\
 & & x_{13} & & & +x_{23} & = & 40 \\
x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23} & \geq & 0
\end{array}\right\} \text{Demand constraints}$$

Formulation

$$\min \; c_{11}x_{11} + \dots + c_{1n}x_{1n} + \dots + c_{m1}x_{m1} + \dots + c_{mn}x_{mn}$$

$$\left.\begin{array}{lllll}
\boxed{x_{11} + x_{12} + \dots + x_{1n}} & & & & = a_1 \\
& \boxed{x_{21} + x_{22} + \dots + x_{2n}} & & & = a_2 \\
& & \dots & & \dots \\
& & & \boxed{x_{m1} + x_{m2} + \dots + x_{mn}} & = a_m
\end{array}\right\} \text{Supply constraints}$$

$$\left.\begin{array}{lllll}
x_{11} & + x_{21} & \dots & + x_{m1} & = b_1 \\
x_{12} & + x_{22} & \dots & + x_{m2} & = b_2 \\
\dots & \dots & & \dots & \dots \\
x_{1n} & + x_{2n} & \dots & + x_{mn} & = b_n
\end{array}\right\} \text{Demand constraints}$$

Example LP formulation in standard form

$$
\begin{array}{lllllllll}
 & \min \; z = 5x_{11} + 3x_{12} + 2x_{13} + 2x_{21} + 2x_{22} + x_{23} \\
 & \text{s.a} \\
u'_1 & x_{11} & + x_{12} & + x_{13} & & & & \geq & 100 \\
u''_1 & - x_{11} & - x_{12} & - x_{13} & & & & \geq & -100 \\
u'_2 & & & & x_{21} & + x_{22} & + x_{23} & \geq & 50 \\
u''_2 & & & & - x_{21} & - x_{22} & - x_{23} & \geq & -50 \\
v'_1 & x_{11} & & & x_{21} & & & \geq & 80 \\
v''_1 & - x_{11} & & & - x_{21} & & & \geq & -80 \\
v'_2 & & x_{12} & & & + x_{22} & & \geq & 30 \\
v''_2 & & - x_{12} & & & - x_{22} & & \geq & -30 \\
v'_3 & & & x_{13} & & & + x_{23} & \geq & 40 \\
v'_3 & & & - x_{13} & & & - x_{23} & \geq & -40 \\
 & x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23} & \geq & 0
\end{array}
$$

Dual Formulation in standard form

$$\max\ g = 100\,u_1' - 100\,u_1'' + 50\,u_2' - 50\,u_2'' +$$
$$+ 80\,v_1' - 80\,v_1'' + 30\,v_2' - 30\,v_2'' + 40\,v_3' - 40\,v_3''$$

s.a.

$$u_1' - u_1'' + v_1' - v_1'' \le 5$$
$$u_1' - u_1'' + v_2' - v_2'' \le 3$$
$$u_1' - u_1'' + v_3' - v_3'' \le 2$$
$$u_2' - u_2'' + v_1' - v_1'' \le 2$$
$$u_2' - u_2'' + v_2' - v_2'' \le 2$$
$$u_2' - u_2'' + v_3' - v_3'' \le 1$$
$$u_1', u_1'', u_2', u_2'', v_1', v_1'', v_2', v_2'', v_3', v_3'' \ge 0$$

Dual Formulation in standard form (simplified)

Let

$$u_i = u_i' - u_i'', i = 1,\dots,2$$

$$v_j = v_j' - v_j'', j = 1,\dots,3$$

$$\begin{aligned}
\max\ g &= 100u_1 + 50u_2 + \\
&\quad + 80v_1 + 30v_2 + 40v_3 \\
\text{s.a.}& \\
u_1 + v_1 &\le 5 \\
u_1 + v_2 &\le 3 \\
u_1 + v_3 &\le 2 \\
u_2 + v_1 &\le 2 \\
u_2 + v_2 &\le 2 \\
u_2 + v_3 &\le 1 \\
u_1, u_2, v_1, v_2, v_3 &\in \Re
\end{aligned}$$

Generalization

Dual of a Transportation Problem

$$\max\ a_1(U'_1 - U''_1) + \ldots + a_m(U'_m - U''_m) + b_1(V'_1 - V''_1) + \ldots + b_n(V'_n - V''_n)$$

$$(U'_1 - U''_1) + (V'_1 - V''_1) \leq c_{11}$$

$$(U'_1 - U''_1) + (V'_2 - V''_2) \leq c_{12}$$

$$(U'_2 - U''_2) + (V'_1 - V''_1) \leq c_{21}$$

$$(U'_m - U''_m) + (V'_n - V''_n) \leq c_{mn}$$

$$U'_i, U''_i, V'_j, V''_j \geq 0$$

$$U_i = U'_i - U''_i, i = 1,\ldots,m$$
$$V_j = V'_j - V''_j, j = 1,\ldots,n$$

$$\boxed{\begin{array}{lll} \max & a_1U_1 + \ldots + a_mU_m + b_1V_1 + \ldots + b_nV_n & \\ \text{s.a} & U_i + V_j \leq c_{ij} & \\ & U_i, V_j \in \mathbb{R} & \end{array}}$$

Primal formulation

$$\min z = \sum_i \sum_j c_{ij}\, x_{ij}$$

$$\text{s.a} \quad \sum_j x_{ij} = a_i \qquad (i = 1,\ldots,m)$$

$$\sum_i x_{ij} = b_j \qquad (j = 1,\ldots,n)$$

$$x_{ij} \geq 0, \qquad (i = 1,\ldots,m;\, j = 1,\ldots,n)$$

Dual formulation

$$\max \; a_1U_1 + \ldots + a_mU_m + b_1V_1 + \ldots + b_nV_n$$

$$\text{s.a} \quad U_i + V_j \leq c_{ij}$$

$$U_i,\ V_j \in |R$$

$$i=1,\ldots m,\ j=1..n$$

Dual in canonic form:

$$\max \; a_1U_1 + \ldots + a_mU_m + b_1V_1 + \ldots + b_nV_n$$

$$\text{s.a} \quad \mathbf{U_i + V_j + S_{ij} = c_{ij}}$$

$$U_i,\ V_j \in |R$$

$$i=1,\ldots m,\ j=1..n$$

Relationship between the primal optimal solution and the dual optimal solution

From duality theory we know that in the optimal solution there is a correspondence between the primal and the dual variables:

Primal		Dual
If $\mathbf{x}_{ij}$ is a decision variable	=>	$\mathbf{s}_{ij}$ is a slack variable
If $\mathbf{x}_{ij}$ is a basic variable	=>	$\mathbf{s}_{ij}$ is a non-basic variable (=0)
If $\mathbf{x}_{ij}$ is a non-basic variable (=0)	=>	$\mathbf{s}_{ij}$ is a basic variable

If $\mathbf{x}_{ij}$ **>0** , i.e., if x_{ij} is basic, then:

- In the final Simplex tableau, the $\mathbf{x}_{ij}$ **coefficient** in the objective function is **0**.
- In the corresponding dual final tableau, the slack variable $\mathbf{s}_{ij}$ (corresponding to $\mathbf{x}_{ij}$) is non-basic, so $\mathbf{s}_{ij}$ **= 0**.

Hence, $$u_i + v_j \le c_{ij} \Leftrightarrow u_i + v_j + \underbrace{s_{ij}}_{0} = c_{ij} \Leftrightarrow u_i + v_j = c_{ij}$$

Relationship between the primal optimal solution and the dual optimal solution

From duality theory we know that in the optimal solution there is a correspondence between the primal and the dual variables:

Primal		Dual
If $\mathbf{x}_{ij}$ is a decision variable	=>	$\mathbf{s}_{ij}$ is a slack variable
If $\mathbf{x}_{ij}$ is a basic variable	=>	$\mathbf{s}_{ij}$ is a non-basic variable (=0)
If $\mathbf{x}_{ij}$ is a non-basic variable (=0)	=>	$\mathbf{s}_{ij}$ is a basic variable

If $\mathbf{x}_{ij} = 0$, (either $\mathbf{x}_{ij}$ is non-basic or the optimal solution is degenerated):

- In the final Simplex tableau, the $\mathbf{x}_{ij}$ **coefficient** in the objective function is **positive** (or zero) since this is a minimization problem.
- In the corresponding dual final tableau, the slack variable $\mathbf{s}_{ij}$ (corresponding to $\mathbf{x}_{ij}$) is **positive** (or zero).

Hence $$u_i + v_j \le c_{ij} \Leftrightarrow u_i + v_j + s_{ij} = c_{ij} \Leftrightarrow s_{ij} = \Delta_{ij} = c_{ij} - (u_i + v_j) \ge 0$$

Transportation Algorithm

Destination

c_{ij}

Origin	1	2	3	Origin supply
1	5 X_{11}	3 X_{12}	2 X_{13}	100
2	2 X_{21}	2 X_{22}	1 X_{23}	50
Destination demand	80	30	40	**150**

1st phase: Find a basic feasible solution

- North West Corner Rule
- Least Cost Rule

1st phase: Find a basic feasible solution

North West Corner Rule

	1	2	3	
1	80			~~100~~ 20
2	0			50
	~~80~~	30	40	

	1	2	3	
1	80	20	0	~~100~~ ~~20~~
2	0			50
	80	~~30~~ 10	40	

	1	2	3	
1	80	20	0	~~100~~ ~~20~~
2	0	10		~~50~~ 40
	~~80~~	~~30~~ ~~10~~	40	

	1	2	3	
1	80	20	0	~~100~~ ~~20~~
2	0	10	40	~~50~~ ~~40~~
	~~80~~	~~30~~ ~~10~~	~~40~~	

1st phase: Find a basic feasible solution

Least Cost Rule

	1	2	3	
1	5	3	2 0	100
2	2	2	1 40	~~50~~ 10
	80	30	~~40~~	

	1	2	3	
1	5	3	2 0	100
2	2 0	2 10	1 40	~~50~~ 0
	80	~~30~~ 20	~~40~~ 0	

	1	2	3	
1	5	3 20	2 0	~~100~~ 80
2	2 0	2 10	1 40	~~50~~ 0
	80	~~30~~ 0	~~40~~ 0	

	1	2	3	
1	5 80	3 20	2 0	~~100~~ ~~80~~
2	2 0	2 10	1 40	~~50~~ ~~0~~
	~~80~~	~~30~~ ~~10~~	~~40~~ 0	

2nd phase: Iteratively, improve the current solution until the optimal solution is found

Initial feasible basic solution

5 **80**	3 **20**	2 0	u1= 0*
2 0	2 **10**	1 **40**	u2 = -1
v1 = 5	v2 = 3	v3 = 2	

1st step: For m origins and n destinations, define m+n values for $\mathbf{u_i}$ (i=1,…,m) and $\mathbf{v_j}$ (j=1,…,n) such that, when $\mathbf{x_{ij}}$ is basic, then $\mathbf{u_i + v_j = c_{ij}}$

Primal variable	**Dual constraint**
x11 = 80	u1 + v1 = c11 = 5
x12 = 20	u1 + v2 = c12 = 3
x22 = 10	u2 + v2 = c22 = 2
x23 = 40	u2 + v3 = c23 = 1

Arbitrarily set u1 = 0*, then v1=5
v2=3
u2=-1
v3 = 2

2nd phase: Iteratively, improve the current solution until the optimal solution is found

5 / 80	3 / 20	2 / 0 (0)	u1= 0*
2 / 0 (-2)	2 / 10	1 / 40	u2 = -1
v1 = 5	v2 = 3	v3 = 2	

2nd step: verify if the solution is optimal. Compute **Δij = cij – ui – vj** for all non-basic variables **Xij**.

The solution is optimal if all **Δij** are non-negative.

(*Note: If all values of Δij are positive, the optimal solution is unique; if any Δij is null, there are alternative optimal solutions.*

Non-basic variable	**Dual constraint**
x13 = 0	Δ_{13} = 2-u1-v3 = 0
x21 = 0	Δ_{21} = 2-u2-v1 = -2

3rd step (2nd phase): Choose a variable to enter the basis: choose the one with the most negative Δij

In this example, choose X_{21}, since $\Delta_{21} = -2$

5 / 80	3 / 20	2 / 0 / **0**	u1= 0*
2 / 0 / **-2**	2 / 10	1 / 40	u2 = -1
v1 = 5	v2 = 3	v3 = 2	

4º step (2nd phase): The variable to enter the basis must be incremented of a positive amount θ ;

To choose the value for θ , we must guarantee that:

-none of the variables will be negative;

- a single non-basic variable becomes basic;

- in order to satisfy the demand and supply constraints, for each variable that has an increment of $+\theta$ in a row (column) , there is another variable in the same row (columm) that has a decrement of θ .

The value of θ will be the minimum of the values associated to $-\theta$

(One of those variables will become non-basic).

5 / $80^{-\theta}$	3 / $20^{+\theta}$	2 / 0 / **0**	u1= 0*
2 / $0^{+\theta}$ / **-2**	2 / $10^{-\theta}$	1 / 40	u2 = -1
v1 = 5	v2 = 3	v3 = 2	

$$\boxed{\theta = \min\{10,80\} = 10}$$

5 / 70	3 / 30	2 / 0	u1=
2 / 10	2 / 0	1 / 40	u2 =
v1 =	v2 =	v3 =	

2nd iteration: Go to the **1st step** of the 2nd phase

1st step: For m origins and n destinations, define m+n values for $\mathbf{u_i}$ (i=1,…,m) and $\mathbf{v_j}$ (j=1,…,n) such that, when $\mathbf{x_{ij}}$ is basic, then $\mathbf{u_i + v_j = c_{ij}}$

Primal Variable	Dual Constraint	
x11 = 70	u1 + v1 = c11 = 5	*Arbitrarily set* u1 = 0*, then v1=5
x12 = 30	u1 + v2 = c12 = 3	v2=3
x21 = 10	u2 + v1 = c21 = 2	u2=-3
x23 = 40	u2 + v3 = c23 = 1	v3 = 4

5 / 70	3 / 30	2 / 0	u1 = 0*
2 / 10	2 / 0	1 / 40	u2 = -3
v1 = 5	v2 = 3	v3 = 4	

2nd iteration :

2nd step: verify if the solution is optimal. Compute **Δij = cij – ui – vj** for all **non-basic variables Xij**. The solution is optimal if all **Δij** are non-negative.

5 / 70	3 / 30	2 / 0 (-2)	u1 = 0*
2 / 10	2 / 0 (2)	1 / 40	u2 = -3
v1 = 5	v2 = 3	v3 = 4	

Non-basic variable	Dual constraint
x13 = 0	Δ_{13} = 2-u1-v3 = -2
x22 = 0	Δ_{22} = 2-u2-v2 = 2

3rd step: Choose a variable to enter the basis: choose the one with the most negative Δij

In this example, we choose X_{13}, since Δ_{13} = -2.

4th step: The variable to enter the basis must be incremented of a positive amount θ;

5 / $70^{-\theta}$	3 / 30	2 / $0^{+\theta}$	u1 = 0*
2 / $10^{+\theta}$	2 / 0	1 / $40^{-\theta}$	u2 = -3
v1 = 5	v2 = 3	v3 = 4	

$$\boxed{\theta = \min\{40, 70\} = 40}$$

→

5 / 30	3 / 30	2 / 40	u1 =
2 / 50	2 / 0	1 / 0	u2 =
v1 =	v2 =	v3 =	

3rd iteration: Go to the **1st step** of the 2nd phase

1st step: For m origins and n destinations, define m+n values for $\mathbf{u_i}$ (i=1,…,m) and $\mathbf{v_j}$ (j=1,…,n) such that, when $\mathbf{x_{ij}}$ is basic, then $\mathbf{u_i + v_j = c_{ij}}$

Primal variable	**Dual constraint**	
x11 = 30	u1 + v1 = c11 = 5	*Arbitrarily set* u1 = 0*, then v1=5
x12 = 30	u1 + v2 = c12 = 3	v2=3
x13 = 40	u1 + v3 = c13 = 2	v3=2
x21 = 50	u2 + v1 = c21 = 2	u2 = -3

5 / 30	3 / 30	2 / 40	u1 = 5
2 / 50	2 / 0	1 / 0	u2 = -3
v1 = 5	v2 = 3	v3 = 2	

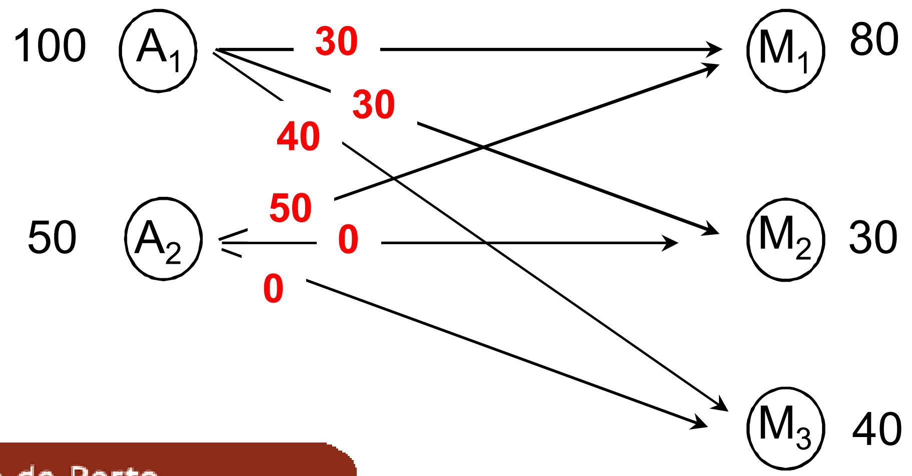
3ª iteração:

2nd step: verify if the solution is optimal. Compute **Δij = cij – ui – vj** for all **non-basic variables Xij**. The solution is optimal if all **Δij** are non-negative.

5 — 30	3 — 30	2 — 40	u1 = 0*
2 — 50	2 — 0	1 — 0	u2 = -3
v1 = 5	v2 = 3	v3 = 2	

Non-basic variable	**Dual constraint**
x22 = 0	Δ_{22} = 2-u2-v2 = 2
x23 = 0	Δ_{23} = 1-u2-v3 = 2

The solution is optimal and unique, because all Δij are positive



Exercise

Consider the FBS (feasible basic solution), obtained by the Least Cost Rule

	1	2	3	4	5
1	3 50	2 0	3 0	4 0	1 25
2	4 35	1 0	2 40	4 75	2 0
3	1 15	0 60	5 0	3 0	2 0

Basic variables: x_{11}, x_{15}, x_{21}, x_{23}, x_{24}, x_{31}, x_{32}

1st step: Compute $\mathbf{U_i}$ e $\mathbf{V_j}$, for the basic variables

Since we have 7 equations (m+n-1) and 8 variables (m+n), one of U_i , V_j values can be arbitrarily set.
Let $\mathbf{U_1 = 0}$ and compute the remaining values.

Usually, U_i and V_j are written directly on the tableau.

x_{11}:	$U_1+V_1 = c_{11} = 3$
x_{15}:	$U_1+V_5 = c_{15} = 1$
x_{21}:	$U_2+V_1 = c_{21} = 4$
x_{23}:	$U_2+V_3 = c_{23} = 2$
x_{24}:	$U_2+V_4 = c_{24} = 4$
x_{31}:	$U_3+V_1 = c_{31} = 1$
x_{32}:	$U_3+V_2 = c_{32} = 0$

$\mathbf{U_1 = 0}$ $\mathbf{V_1 = 3}$
$\mathbf{V_2 = 2}$
$\mathbf{U_2 = 1}$
$\mathbf{U_3 = -2}$ $\mathbf{V_3 = 1}$
$\mathbf{V_4 = 3}$
$\mathbf{V_5 = 2}$

Exercise (contd)

	1	2	3	4	5
1	3 50	2 0 0	3 2 0	4 1 0	1 25
2	4 35	1 -2 0	2 40	4 75	2 -1 0
3	1 15	0 60	5 6 0	3 2 0	2 2 0

$U_1 = 0^*$

$U_2 = 1$

$U_3 = -2$

$V_1 = 3$ $V_2 = 2$ $V_3 = 1$ $V_4 = 3$ $V_5 = 1$

cij	
	0
Δ_{ij}	

2nd step: For the non-basic variables (=0), compute:

$$\Delta_{ij} = \mathbf{c_{ij} - (U_i + V_j)}$$

Since there are $\Delta_{ij} < 0$, the solution is not optimal.

3rd step: Choose the non-basic variable with the most negative Δ_{ij}, which is $\mathbf{x_{22}}$

Exercise (contd)

4th step: the variable that will enter the basis, $\mathbf{x_{22}}$, will have a positive value;

	1	2	3	4	5
1	50	0	0	0	25
2	$35^{-\theta}$	$0^{+\theta}$	40	75	0
3	$15^{+\theta}$	$60^{-\theta}$	0	0	0

$\theta = \min\{35, 60\} = 35$

Notes:
1. A single variable enters the basis and a single variable leaves the basis.
2. There always exists a tour for θ an this tour unique.

Exercise (contd)

2nd solution

	1	2	3	4	5	
1	^{3} 50 $^{-\theta}$	$^{2}_{0}$ 0	$^{3}_{0}$ 0	$^{4}_{-1}$ 0 $^{+\theta}$	^{1} 25	$U_1 = 0^*$
2	$^{4}_{2}$ 0	^{1} 35 $^{+\theta}$	^{2} 40	^{4} 75 $^{-\theta}$	$^{2}_{2}$ 0	$U_2 = -1$
3	^{1} 50 $^{+\theta}$	^{0} 25 $^{-\theta}$	$^{5}_{4}$ 0	$^{3}_{0}$ 0	$^{2}_{3}$ 0	$U_3 = -2$
	$V_1 = 3$	$V_2 = 2$	$V_3 = 3$	$V_4 = 5$	$V_5 = 1$	

$$\theta = \min\{25, 50, 75\} = 25$$

3rd solution

	1	2	3	4	5	
1	^{3} 25	$^{2}_{1}$ 0	$^{3}_{1}$ 0	^{4} 25	^{1} 25	$U_1 = 0^*$
2	$^{4}_{1}$ 0	^{1} 60	^{2} 40	^{4} 50	$^{2}_{1}$ 0	$U_2 = 0$
3	^{1} 75	$^{0}_{1}$ 0	$^{5}_{5}$ 0	$^{3}_{1}$ 0	$^{2}_{3}$ 0	$U_3 = -2$
	$V_1 = 3$	$V_2 = 1$	$V_3 = 2$	$V_4 = 4$	$V_5 = 1$	

Single optimal solution:

$\forall$ i, j, $\Delta_{ij} > 0$

CT = 3x25+4x25+1x25+1x60+2x40+4x50+1x75
= **615**

Special cases in Transportation Problems

- The supply and the demand are not equal
  - The total demand is higher than the total supply
  - The total supply is higher than the total demand
- Some transportation routes are not allowed
- Maximization of the objective function
- Degeneracy

The supply and the demand are not equal

- **The total demand is higher than the total supply**: insert an artificial row with unitary costs equal to zero. The values for the variables in this row correspond to unsatisfied demand.

  Example: Total demand = 200; Total supply = 150

	P1	P2	
O1	2	4	80
O2	3	5	70
	100	100	

	P1	P2	
O1	2	4	80
O2	3	5	70
O*	0	0	50
	100	100	

- **The total supply is higher than the total demand**: insert an artificial column with unitary costs equal to zero (if no storage costs are included)

  Example: Total supply= 100; Total demand= 70

	P1	P2	
O1	1	5	50
O2	2	3	50
	20	50	

	P1	P2	P*	
O1	1	5	0	50
O2	2	3	0	50
	20	50	30	

Impossible transportation routes

Let O_i be an origin and D_j a destination such that no transportation flow is allowed between them.

To guarantee that $\mathbf{x_{ij} = 0}$ in the optimal solution, we consider an infinite unitary transportation cost between O_i and D_j ($c_{ij}=\infty$).

Example: A company wants to supply two clients (C1 and C2) whose demand is **75** units (for each client) of a given product.

- Plant **F1** produces **100** units and can supply both clients (the unitary cost for client **C1** is **1** and for client **C2** is **4**).
- Plant **F2** produces **80** units and can only supply client **C2**, with a unitary cost of 2.

	C1	C2	C*	
F1	1	4	0	100
F2	∞	2	0	80
	75	75	30	

Maximization of the objective function

If we intend to maximize an objective function, instead of mimimizing the transportations costs, we have two options:

(i) maximize f $\Leftrightarrow$ minimize (-f)

switch the signs of the objective function coefficients and apply the transportation algorithm..

(ii) keep the objective function coefficients (maximization) and change the following criteria in the application of the transportation algorithm:

- Optimality criterion: a solution is optimal when for all the non-basic variables,

  $\Delta_{ij} = \mathbf{c_{ij} - (U_i + V_j)}$ are non-positive

- Choosing the variable to enter the basis: if the solution is not optimal the solution tha will enter the basis will be the one with the highest value for Δ_{ij} .

Degeneracy

- Degeneracy occurs when in a feasible basic solution, one or more of the basic variables are null.
- We may have a degeneracy:
  - In the definition of the initial feasible basic solution
  - During the application of the transportation algorithm.

Example 1: Find an initial FBS for the following problem using the North West Corner Rule:

	C1	C2	C3	
F1	**2**	**1**	**0**	100
F2	**1**	**0**	**3**	60
	100	20	40	

	C1	C2	C3	
F1	**2** 100	**1** 0	**0** 0	~~100~~
F2	**1** 0	**0**	**3**	60
	~~100~~	20	40	

	C1	C2	C3	
F1	**2** 100	**1** 0	**0** 0	~~100~~
F2	**1** 0	**0** 20	**3**	60
	~~100~~	~~20~~	40	

	C1	C2	C3	
F1	**2** 100	**1** 0	**0** 0	~~100~~
F2	**1** 0	**0** 20	**3** 40	~~60~~
	~~100~~	~~20~~	~~40~~	

Assigning x_{11}=100, we satify 2 constraints at the same time.

The number of basic variables is m+n-1 = 2+3-1=**4** and only **3** of them are positive.
The basic solution is **degenerated**, since one of the basic variable is null.

Degeneracy (cont.)

Example 1: Apply the Transportation Algorithm to the following non-degenerated FBA

	C1	C2	C3	
F1	**2** 80	**2** 20	**5** 0	100
F2	**3** 20	**0** 0	**3** 40	60
	100	20	40	

	C1	C2	C3	
F1	**2** 80 $+\theta$	**2** 20 $-\theta$	**5** **3** 0	u1=0
F2	**3** 20 $-\theta$	**0** **-3** 0 $+\theta$	**3** 40	u2=1
	v1=2	v2=2	v3=2	

$$\theta = \min\{20,20\} = 20$$

	C1	C2	C3	
F1	**2** 100	**2** 0	**5** 0	100
F2	**3** 0	**0** 20	**3** 40	60
	100	20	40	

The number of basic variables is m+n-1 = 2+3-1=**4** and only 3 of them are positive !!.
The new basic solution is **degenerated**, since one of the basic variables is null.

Solution: Among the variables that were set to zero, arbitrarily choose one to be handled as a basic variable.

Degeneracy (contd)

Example 2: Solve the following transportation problem using the Least Cost Rule to find the initial feasible basic solution.

0	2	2	7	50
0	5	1	2	50
0	1	4	3	50
50	25	45	30	

0 50	2 0*	2 0	7 0	50
0 0	5	1	2	50
0 0	1	4	3	50
50	25	45	30	

0 50	2 0*	2 0	7 0	50
0 0	5 0	1 45	2 5	50
0 0	1 25	4 0	3 25	50
50	25	45	30	

0 50	2 −θ 0*	2 +θ -1 0	7 3 0	u1=0*
0 2 0	5 5 0	1 −θ 45	2 +θ 5	u2=-2
0 1 0	1 +θ 25	4 2 0	3 −θ 25	u3=-1
v1=0	v2=2	v3=3	v4=4	

Basic var: x11, **x12**, x23, x24, x32, x34

$$\theta = \min\{0^*, 25, 45\} = 0^*$$

0 50	2 0	2 0*	7 0
0 0	5 0	1 45	2 5
0 0	1 25	4 0	3 25

Basic var: x11, **x13**, x23, x24, x32, x34

0 50	2 1 0	2 0*	7 4 0	u1=0
0 1 0	5 5 0	1 45	2 5	u2=-1
0 0 0	1 25	4 2 0	3 25	u3=0
v1=0	v2=1	v3=2	v4=3	

Optimal solution

Sensitivity analysis in Transportation Problem

Changing the values of supply/demand

The u's and v's can be considered the shadow prices of the constraints. If the increase in the supply and the increase on the demand is denoted by Δ, the value of the new objective function value will be

$$new\ z\ value = \sum_{i}^{m}\sum_{j}^{m} c_{ij}x_{ij} + \Delta u_i + \Delta v_j$$

- If x_{ij} is a basic variable, the amount Δ is added to x_{ij}
- If x_{ij} is non basic, we have to find a loop involving a basic variable in row i, we add Δ to that basic variable and add/subtract Δ to the basic variables in the loop.

As long as Δ does not change the basis, we can analyse the effect of changing supply and demand.

Example (for a non-basic variable)

	1	2	3	4	5
1	3 25	2 0 1	3 0 1	4 25	1 25
2	4 0 1	1 60	2 40	4 50	2 0 1
3	1 75	0 0 1	5 0 5	3 0 1	2 0 3

$U_1= 0^*$

$U_2= 0$

$U_3= -2$

Optimal solution with cost= 615

$V_1= 3$ $V_2= 1$ $V_3= 2$ $V_4= 4$ $V_5= 1$

What's the impact of adding more supply to origin 1 and more demand to destination 2?

	1	2	3	4	5
1	25	0	0	25+Δ	25
2	0	60+Δ	40	50-Δ	0
3	75	0	0	0	0

	1	2	3	4	5
1	25	0	0	50	25
2	0	85	40	25	0
3	75	0	0	0	0

If, for example, Δ =25

New cost = 615 + Δ *u1 + Δ *v2 = 615 + 25*0+ 25*1 = 615+25 = 640

How much can I increase X to maintain the basis of the optimal solution?

The Transportation Paradox

- The **transportation paradox** is related to the classical transportation problem. For certain instances of this problem an increase in the amount of goods to be transported may lead to a decrease in the optimal total transportation cost. Thus this phenomenon has also been named the **more-for-less-paradox**.

Consider the following problem:

Problem A

50	300	5
320	60	10
7	8	

Optimal solution of A:

50 **5**	300 90 **0**	5	u1=0*
320 **2**	60 **8**	10	u2=270
7	8		
v1=50	v2=-210		

Cost_A= 1370

The Transportation Paradox

Consider now that we increase a_1 and b_2 by one unit:

Problem B

50	300	**5+1=6**
320	60	10
7	**8+1=9**	

Optimal solution of B:

50 **6**	300 90 **0**	5	u1=0*
320 **1**	60 **9**	10	u2=270
7	8		
v1=50	v2=-210		

Cost_B= 1160 < Cost_A = 1370

So…one more unit transported will reduce the optimal cost by 210!!!
This is the **Transportation Paradox** or **More for Less Paradox**

The Transportation Paradox
Some historical facts

- It is not quite clear when and by whom this paradox was first discovered.
- Apparently, several researchers have discovered the paradox independently from each other. But most papers on the subject refer to the papers by Charnes and Klingman, and Szwarc as the initial papers.
- The transportation paradox is known as Doigs paradox at the London School of Economics, named after Alison Doig who used it in exams etc. around 1959 (However, Doig did not publish any paper on it).
- Since the transportation paradox seems not to be known to the majority of those who are working with (or teaching) the transportation problem, one may be tempted to believe that this phenomenon is only an academic curiosity which will most probably not occur in any practical situation.
- But that seems not to be true: The necessary and sufficient conditions for a problem to be immune against the transportation paradox are rather restrictive…

Transportation Paradox
When will the paradox not occur?

Definition: An immune cost matrix satisfies $z(C, a, b) \leq z(C, a', b')$ for all supply vectors a and a' with $a \leq a'$ and for all demand vectors b and b' with $b \leq b'$.

Theorem 1: A *mxn* cost matrix $C = [c_{ij}]$ is immune against the transportation paradox if and only if, for all integers q, r, s, t with $1 \leq q, s \leq m$, $1 \leq r, t \leq n$, $q \neq s$, $r \neq t$, the inequality

$$c_{qr} \leq c_{qt} + c_{sr}$$

is sattisfied,

50	300
320	60

In this problem, $c_{21} \geq c_{22}+c_{11}$,
so this cost matrix in not immune to transportation paradox

Transportation Paradox
When will the paradox occur?

Theorem 2: Assume that indexes *p* and *q* exist, $1 \le p \le m$; $1 \le q \le n$, such that $u_p + v_q < 0$.

Assume further that a positive number exists, such that when supply *ap* is replaced by $\widehat{ap} = ap + \theta$, and demand *bq* is replaced by $\widehat{bq} = bq + \theta$, a basic feasible solution for the new instance can be found which is optimal and has the same set B of basic variables. Then the paradox will occur.

Consider the following example:

						ai
	4	15	6	13	14	7
	16	9	22	13	16	18
	8	5	11	4	5	6
	12	4	18	9	10	15
bj	4	11	12	8	11	

c14 > c11+c34

According to Theorem 1, this problem is not immune.

Let's see if the paradox will occur…

Transportation Paradox

The optimal solution, after 5 iterations (it's up to you to confirm …as homework ;))

iteration 5

						ui
	4 3 **0**	15 21 **0**	6 **7**	13 15 **0**	14 14 **0**	0*
	16 **4**	9 **6**	22 1 **0**	13 **8**	16 1 **0**	15
	8 2 **0**	5 6 **0**	11 **5**	4 1 **0**	5 **1**	5
	12 1 **0**	4 **5**	18 2 **0**	9 1 **0**	10 **10**	10
vj	1	-6	6	-2	0	

Since $u1+v4=-2<0$, according to Theorem 2, the paradox will occur!

Transportation Paradox

According to Theorem 2, let us see if it is possible to increase a1 = 7 and b4 = 8 by a number $\Theta > 0$ such that the present optimal basic feasible solution can be modified to become optimal for the new instance with the same set of basic variables.

						a_i
	4 / **0**	15 / **0**	6 / **7+Θ**	13 / **0**	14 / **0**	7+Θ
	16 / **4**	9 / **6-Θ**	22 / **0**	13 / **8+Θ**	16 / **0**	18
	8 / **0**	5 / **0**	11 / **5-Θ**	4 / **0**	5 / **1+Θ**	6
	12 / **0**	4 / **5+Θ**	18 / **0**	9 / **0**	10 / **10-Θ**	15
b_j	4	11	12	8+Θ	11	

Θ may be selected as any number $0 < \Theta \leq 5$

Transportation Paradox

Let’s choose ϴ = 4

						a_i
	4 **0**	15 **0**	6 **11**	13 **0**	14 **0**	**11**
	16 **4**	9 **2**	22 **0**	13 **12**	16 **0**	18
	8 **0**	5 **0**	11 **1**	4 **0**	5 **5**	6
	12 **0**	4 **9**	18 **0**	9 **0**	10 **6**	15
b_j	4	11	12	**12**	11	

The cost of this solution is 444 + 4(-2) = 436 < 444 !!!
So shipping 4 additional units will reduce the total transportation cost by 8 units!!

The Transhipment problem

We are given *m* pure supply nodes with demand a_i, *n* pure demand nodes with demand b_j and *l* transhipment nodes. Suppose the unit transportation cost from supply node *i* to transhipment node *k* is c_{ik} and the unit transportation cost form transhipment node *k* to demand node *j* is c_{kj}. The transhipment problem can be formulated as

$$\min \sum_{i=1}^{m}\sum_{k=1}^{l} c_{ik}x_{ik} + \sum_{k=1}^{l}\sum_{j=1}^{n} c_{kj}x_{kj}$$

$$\sum_{k=1}^{l} x_{ik} = a_i, i = 1,2, \dots, m$$

$$\sum_{i=1}^{m} x_{ik} - \sum_{j=1}^{n} x_{kj} = 0, i = 1,2, \dots, l$$

$$\sum_{k=1}^{l} x_{kj} = b_j, j = 1,2, \dots, n$$

$$x_{ik}, x_{kj} \geq 0, i = 1, \dots, m; k = 1, \dots, l; j = 1, \dots n$$

Capacitated Transportation Problem

Objective function $\min z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Supply constraints $\sum_{j=1}^{n} x_{ij} = a_i$

Demand constraints $\sum_{i=1}^{m} x_{ij} = b_j$

Capacity constraints $x_{ij} \leq u_{ij}, i = 1, .., m; j = 1, .., n$

$x_{ij} \geq 0, i = 1, .., m; j = 1, .., n$

Capacitated Transportation Problem with bounds on rim conditions

$$(P1):\ \min\{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{i\in I} F_i\}$$

subject to

$$a_i \le \sum_{j\in J} x_{ij} \le A_i \qquad \forall\, i \in I$$

$$b_j \le \sum_{i\in I} x_{ij} \le B_j \qquad \forall\, j \in J$$

$$l_{ij} \le x_{ij} \le u_{ij} \ \text{ and integers } \forall\, i \in I,\ j \in J$$

Bottleneck Capacitated Transportation Problem

$$\min T = \left\{ \max_{(i,j)} t_{ij} \mid x_{ij} > 0 \right\}$$

$$a_i \leq \sum_{j \in J} x_{ij} \leq A_i \qquad \forall\, i \in I$$

$$b_j \leq \sum_{i \in I} x_{ij} \leq B_j \qquad \forall\; j \in J$$

$$l_{ij} \leq x_{ij} \leq u_{ij} \quad \text{and integers} \quad \forall\, i \in I,\, j \in J$$

Solid Transportation Problem

$$\text{Minimize} \quad z = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} c_{ijk} x_{ijk}$$

subject to

$$\sum_{j=1}^{n} \sum_{k=1}^{l} x_{ijk} = a_i, \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} \sum_{k=1}^{l} x_{ijk} = b_j, \quad j = 1, 2, \ldots, n$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} = e_k, \quad k = 1, 2, \ldots, l$$

$$x_{ijk} \geq 0, \text{ for all } i, j \text{ and } k$$