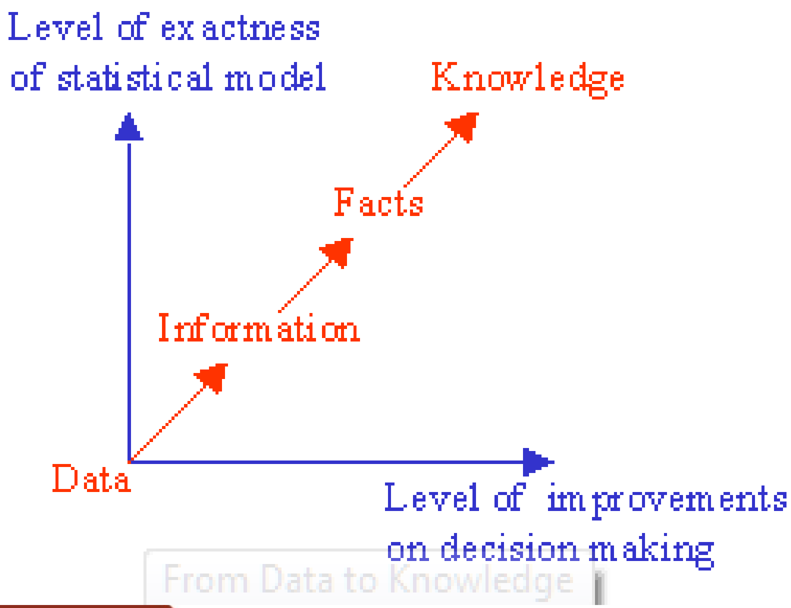
Decision Theory

From Data to a Decisive Knowledge

- Data is known to be crude information and not knowledge by itself.
- Data becomes information, when it becomes relevant to your decision problem.
- Information becomes fact, when the data can support it.
- Fact becomes knowledge, when it is used in the successful completion of a decision process.



Decision Analysis: Making Justifiable, Defensible Decisions

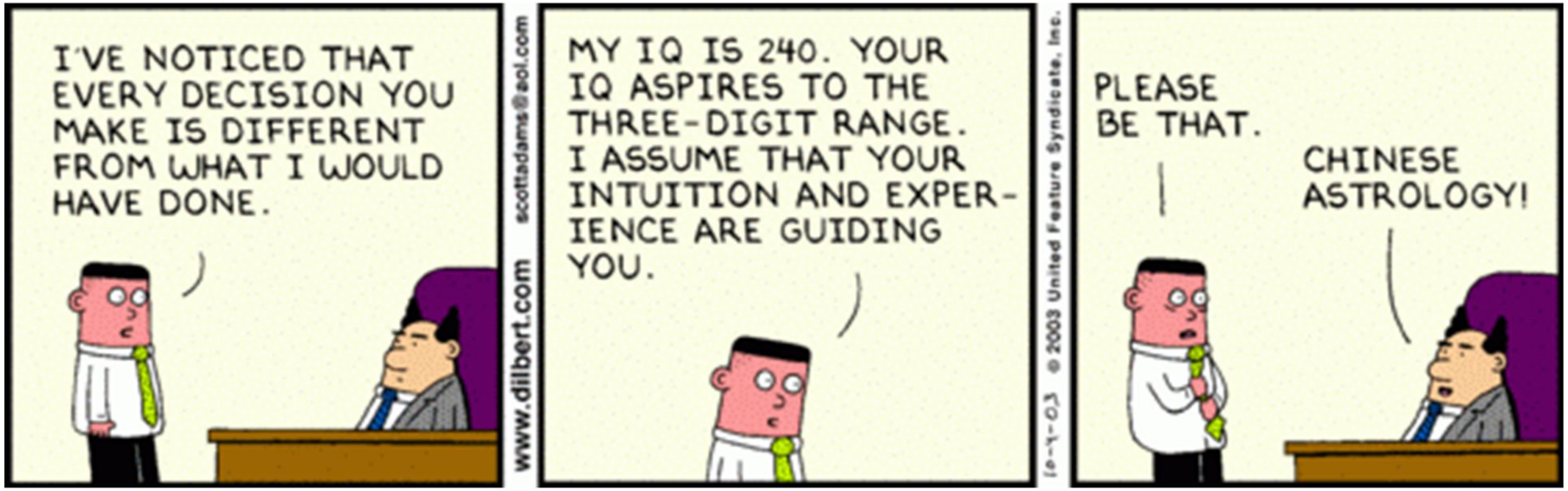
- Decision analysis is the discipline of evaluating complex alternatives in terms of values and uncertainty.
- Values are generally expressed monetarily because this is a major concern for management.
- Complexity in the modern world, along with information quantity, uncertainty and risk, make it necessary to provide a rational decision making framework.
- The goal of decision analysis is to give guidance, information, insight, and structure to the decision-making process in order to make better, more 'rational' decisions.

Decisions are difficult

- Complexity
  Number of decision makers, objectives, options, consequences, constraints,…
- Uncertainty
  Values, unclear objectives, unclear trade-offs
  External factors
- Several objectives
- Different perspectives

What is a good decision?

When, afterwards, we can say that we would make the same decision taking into account the available information at the time the decision was made.



Structured decision processes versus unstructured decision processes

Structured decision processes

- It is possible to define clear and objective rules to process the available information and choose among the possible alternatives.
- It is possible (theoretically) to build an algorithm to replace the decision agent.
- Normative approach

Unstructured decision processes

- Objectives, constraints, information and methodology not previously determined.
- The experience and intuition of the decision maker are fundamental.

Hopwood Matrix(1980) ...

Uncertainty on the objectives' definition

Negotiation **Management control** Semi-structured decision process	**Inspiration** Unstructured decision process
Computation **Operational control** Structured decision process	**Judgement formulation** **Strategic planning** Unstructured decision process

Uncertainty of decision consequences

Decision Theory

- Scientific methods to organize the decision process
- Allows to identify and represent the uncertainty
- Methodology to establish values and preferences
- Transparency of the decision process
- Prescriptive approach

Preference relations

It is important to establish a preference relation among the different alternatives.

Strong preference (A is better than B) $\mathrm{A} > \mathrm{B}$

Weak preference (A is at least as good as B) $A \geq B$

Indifference (A is equal in value to B) $A \equiv B$

Axioms:

$$A > B \Leftrightarrow A \geq B \wedge \neg B \geq A$$

$$A \equiv B \Leftrightarrow A \geq B \wedge B \geq A$$

Example:
"My car is better than your car.“ is equivalent to

"My car is at least as good as your car, but yours is not at least as good as mine."

Properties of preference relations

Any preference relation is established for a set of entities (alternatives), designated by Domain.

Completeness: The relation >= is complete if and only if for any elements A and B of its domain, either A>=B or B>=A.

Example 1 : *if you want to choose between three cereal brands (A, B, C) and you clearly prefer A brand on the others, you do not need to establish any preference relation between B and C : the relation is incomplete.*

Example 2: *a voter in a multi-party election can do without ranking the parties or candidates that he does not vote for: the relation is incomplete.*

We can often live happily with incomplete preferences, even when our preferences are needed to guide our actions.

In decision theory, we assume that the preference relations are complete

Properties of preference relations

Transitivity

A strong (strict) preference relation (>) is transitive if and only if for all the elements in its domain

if A > B and B > C, then A > C.

In decision theory, transitivity also applies to weak preference and to indifference.

These properties are generally considered to be more controversial than the transitivity of strict preference.

Transitivity, just like completeness, is a common but problematic assumption in decision theory.

Is indifference a transitive relation?

- Consider 1000 cups of coffee, numbered C0, C1, C2,... up to C999.
- Cup C0 contains no sugar, cup C1 one grain of sugar, cup C2 two grains, etc.
- Since I cannot taste the difference between C0 and C1, they are equally good in my taste, C0 ≡C1. For the same reason, we have C1 ≡ C2, C2 ≡ C3, etc, all the way up to C998 ≡ C999.
- If indifference is transitive, then it follows from C0 ≡ C1 and C1 ≡ C2 that C0 ≡ C2.
- Furthermore, it follows from C0 ≡ C2 and C2 ≡ C3 that C0 ≡ C3.
- Continuing the procedure we obtain C0 ≡ C999.

- However, this is absurd, since I can clearly taste the difference between C0 and C999, and like the former much better.

Using preferences in decision-making

In decision-making, preference relations are used to find the best alternative.

The following simple rule can be used for this purpose:

An alternative is (uniquely) best if and only if it is better than all other alternatives. If there is a uniquely best alternative, choose it.

There are cases in which no alternative is uniquely best, since the highest position is "shared" by two or more alternatives. More generally, the following rule can be used:

An alternative is (among the) best if and only if it is at least as good as all other alternatives. If there are alternatives that are best, pick one of them.

However, preferences that violate rationality criteria such as transitivity are often not useful to guide decisions

Utility function

- It is usual to assign a numerical value to each alternative, trying to represent the preference relation by a numeric relation.
- The preference relations obtained this way are always complete and transitive.
- In decision theory it is common to use the concept of utility, that tries to translate the value/utility of a given decision and may be related to monetary values or happiness.

Representation of decision problems

- A set of alternatives (options): a_i
  - It can be an open or a closed set (but in decision theory we consider it is closed)
  - Mutually exclusive
- A set of states of nature (scenarios): θ_j
  - External factors that influence the outcome of the decision
- A set of results
  - The combined action of the chosen alternative and the actual state of nature

		States of nature	
		It rains	**It does not rain**
Alternatives	**Umbrella**	Dry clothes, heavy suitcase	Dry clothes, heavy suitcase
	No umbrella	Soaked clothes, light suitcase	Dry clothes, light suitcase

Utility Matrix

Mainstream decision theory is almost exclusively devoted to problems that can be expressed in utility matrices. In order to use a matrix to analyze a decision, we need to know:

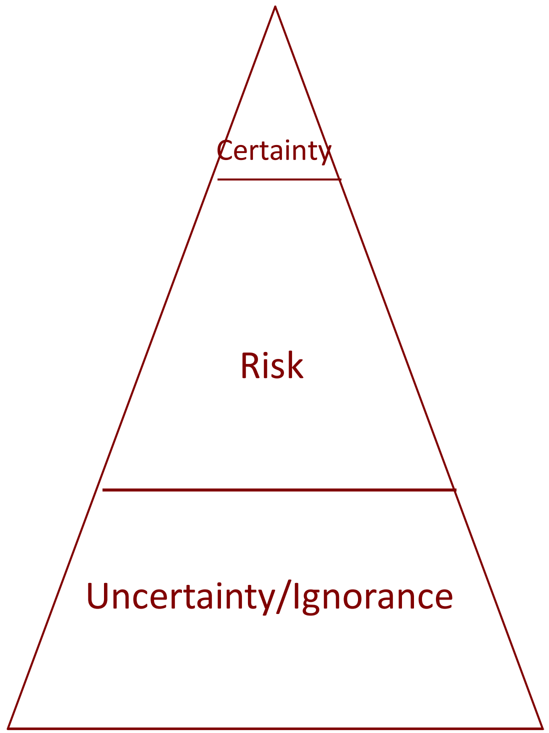
1. how the outcomes are valued, and
2. the degree of uncertainty of the decision context (certainty, risk, ignorance).

$$U_{ij} = U(a_i, \theta_j)$$

		States of nature	
		It rains	**It does not rain**
Alternatives	**Umbrella**	15	10
	No umbrella	0	18

Information about states of nature (Luce and Raiffa,1957)

- Certainty
  - if each action is known to lead invariably to a specific outcome.
- Risk
  - if each action leads to one of a set of possible specific outcomes, and each outcome occurs with a known probability
- Uncertainty/Ignorance
  - if either action leads to a set of possible specific outcomes, but where the probabilities of these outcomes are completely unknown or are not even meaningful.



Decision with certainty

- Certainty (or perfect information)
  - if each action is known to lead invariably to a specific outcome.

		States of nature	
		It rains	**It does not rain**
Alternatives	**Umbrella**	15	10
	No umbrella	0	18

- If the decision maker knows that it will rain, he will bring the umbrella.
- If the decision maker knows that it will not rain, he will not bring the umbrella.

If θ is the state of nature that will occur, $a_0 = \max U(a_i, \theta_0)$

Decision with uncertainty/ignorance

- Uncertainty/Ignorance
  - if either action leads to a set of possible specific outcomes, but where the probabilities of these outcomes are completely unknown or are not even meaningful.

		States of nature	
		It rains	**It does not rain**
Alternatives	**Umbrella**	Dry clothes, heavy suitcase	Dry clothes, heavy suitcase
	No umbrella	Soaked clothes, light suitcase	Dry clothes, light suitcase

Let us first see what we can do with only a preference relation (i.e., with no information about utilities). As before, your preferences are:

Dry clothes, light suitcase is better than

Dry clothes, heavy suitcase is better than *Soaked clothes, light suitcase*

The Maximin rule

The maximin principle was first proposed by von Neumann as a strategy against an intelligent opponent.

Wald (1950) extended its use to games against nature.

For each alternative, we define its *security level* as the worst possible outcome with that alternative. The maximin rule urges us to choose the alternative that has the maximal security level. In other words, *maximize the minimal outcome*.

		States of nature		
		It rains	It does not rain	
Alternatives	Umbrella	Dry clothes, heavy suitcase	Dry clothes, heavy suitcase	Dry clothes, heavy suitcase
	No umbrella	Soaked clothes, light suitcase	Dry clothes, light suitcase	Soaked clothes, light suitcase

If we assign numerical values to the possible outcomes, we will have the following utility matrix:

		States of nature		
		It rains	It does not rain	$\min(a_i,\theta_j)$
Alternatives	Umbrella	15	10	10
	No umbrella	0	18	0

Using the maximin rule we should **bring the umbrella**.

The maximin rule is often said to represent extreme prudence or **pessimism**.

The Maximax rule

The maximax rule says that we must choose the alternative whose hope level (best possible outcome) is best. To apply this rule we can use the preference relations or a utility matrix.

		States of nature		
		It rains	**It does not rain**	**$\max(a_i,\theta_j)$**
Alternatives	**Umbrella**	15	10	15
	No umbrella	0	18	18

A maximaxer will not bring his umbrella! He is an optimistic!

It is in general "difficult to justify the maximax principle as rational principle of decision, reflecting, as it does, wishful thinking“ (Rapoport, 1989).

Nevertheless, life would probably be duller if not at least some of us were maximaxers on at least some occasions…

The Hurwicz $\propto$ -index (Hurwicz,1951)

The Hurwicz $\propto$ -index is also called the *optimism-pessimism index.*

The decision-maker is required to choose an index $\propto$ between 0 and 1, that reflects his degree of optimism or pessimism. *0* means the maximum optimism and *1* the maximum pessimism

For each alternative A, let min(A) be its security level, i.e. the lowest utility to which it can give rise, and let max(A) be the hope level, i.e., the highest utility level that it can give rise to.

The $\propto$ -index of A is calculated according to the formula:

$$\propto \times \min(A) + (1 - \propto) \times \max(A)$$

		States of nature			maximax	maximin
		It rains	It does not rain	$\propto$	$\propto$=0	$\propto$=1
Alternatives	Umbrella	15	10	$\propto$x10+(1-$\propto$)x15	15	10
	No umbrella	0	18	$\propto$x0+(1-$\propto$)x18	18	0

Which is the $\propto$ that compells us to bring the umbrella?

$$\propto \times 10 + (1 - \propto) \times 15 > \propto \times 0 + (1 - \propto) \times 18$$

$$\propto > \frac{3}{13} = \mathbf{0,23}$$

The Laplace (average) Rule

This rule uses the *principle of insufficient reason*, that was first formulated by Jacques Bernoulli (1654-1705).

This principle states that if there is no reason to believe that one event is more likely to occur than another, then the events should be assigned equal probabilities.

The principle is intended for use in situations where we have an exhaustive list of alternatives, all of which are mutually exclusive. In our umbrella example, it leads us to assign the probability 1/2 to rain.

Each state of naure has probability 1/n

We should choose the alternative that maximizes $\frac{1}{n}\left(\sum_{j=1}^{n} U(a_i, \theta_j)\right)$

	Sates of nature		
Alternatives	**It rains**	**It does not rain**	$\frac{1}{2}\left(\sum_{j=1}^{2} U(a_i, \theta_j)\right)$
Umbrella	15	10	12,5
No umbrella	0	18	9

The problem with Laplace Rule

One of the problems with Laplace rule is that it is extremely dependent on the partitioning of the alternatives.

In our umbrella example, we might divide the "rain" state of nature into two or more sub-states, such as "*it rains a little*" and "*it rains a lot*".

This simple reformulation reduces the probability of no rain from 1/2 to 1/3.

To be useful, the principle of insufficient reason must be combined with symmetry rules for the structure of the states of nature.

The basic problem with the principle of insufficient reason, its arbitrariness, has not been solved. (Seidenfeld 1979, Harsanyi 1983) ☹

The Minimax Regret Rule (Savage, 1951)

The Minimax Regret Rule or **Savage Rule** is based on a moderated pessimist approach.

The regret occurs when the decision maker compares the decision he made with the decision that he could have made after knowing the state of nature. The objective of Savage Rule is to minimize the regret.

$$\text{Regret}(a_i, \boldsymbol{\theta}_j) = \max_i U(a_i, \boldsymbol{\theta}_j) - U(a_i, \boldsymbol{\theta}_j)$$

In other words, it is the difference between the maximum utility observed for state of nature $\boldsymbol{\theta}_j$*, for all the possible alternatives and the utility associated to pair* $U(a_i, \boldsymbol{\theta}_j)$.

A ***regret matrix*** may be derived from the above utility matrix: assign to each outcome the difference between the utility of the maximal outcome in its column and the utility of the outcome itself.

Utility Matrix

		States of nature	
		It rains	**It does not rain**
Alternatives	**Umbrella**	15	10
	No umbrella	0	18

Regret Matrix

	States of nature		
Alternatives	**It rains**	**It does not rain**	
Umbrella	0	8	8
No umbrella	15	0	15

Example (1)

Both the maximin criterion and the minimax regret criterion are rules for the cautious who do not want to take risks. However, the two criteria do not always make the same recommendation.

This can be seen from the following example.

Three methods are available for the storage of nuclear waste. There are only three relevant states of nature. One of them is *stable rock*, the other is a *geological catastrophy* and the third is *human intrusion* into the depository. (For simplicity, the latter two states of affairs are taken

to be mutually exclusive.)

To each combination of depository and state of nature, a utility level is assigned, perhaps inversely correlated to the amount of human exposure to ionizing radiation that will follow:

		States of nature		
		Stable rock	**Geological catastrophy**	**Human intrusion**
Alternatives	**Method 1**	-1	-100	-100
	Method 2	0	-700	-900
	Method 3	-20	-50	-110

Example (2)

Utility matrix

		States of nature					
		Stable rock	Geological catastrophy	Human intrusion	Maximin	Maximax	Minmax regret
Alternatives	Method 1	-1	-100	-100	**-100**	-1	50
	Method 2	0	-700	-900	-900	**0**	800
	Method 3	-20	-50	-110	-110	-20	**20**

Regret matrix

		States of nature		
		Stable rock	Geological catastrophy	Human intrusion
Alternatives	Method 1	1	50	0
	Method 3	0	650	800
	Method 3	20	0	10

The major decision rules for uncertainty/ignorance

Decision rule	Value information needed	Character of the rule
maximin	preferences	pessimism
maximax	preferences	optimism
optimism-pessimism index	utilities	varies with index
minimax regret (Savage)	utilities	cautiousness
insufficient reason (Laplace)	utilities	depends on partitioning

Decision with risk

The dominating approach for decision making under risk is based on the concept of expected utility.

Maximum Expected Utility criterion

To each alternative is assigned a weighted average of its utility values under different states of nature, and the probabilities of these states are used as weights. Then, choose the decision alternative that has the largest expected utility.

	States of nature		EU
Alternatives	**It rains**	**It does not rain**	
Umbrella	15	10	**=0.1*15+0.9*10 =10.5**
No umbrella	0	18	**=0.1*0+0.9*18 =16.2**
Probabilities	**p=0.1**	**p=0.9**	

In this case, we should not bring the umbrella!

Value of perfect information

Is it possible to know which state of nature will occur?

If possible, which is the value of such additional information or , in other words, how much am I willing to pay for it? This corresponds to the Value of Perfect Information.

Alternatives	States of nature: It rains	States of nature: It does not rain	EU
Umbrella	15	10	**10.5**
No umbrella	0	18	**16.2**
Probabilities	p=0.1	p=0.9	

Optimal decision a priori

Imagine that it is possible to know that it will rain. In this case, the optimal decision is to bring the umbrella, with outcome = 15. The outcome of making an *a priori* decision is 0.

The value of the perfect information is the difference between the outcome after we know the information and the outcome that we would have if we have made the decision a priori for the state of nature that has occurred. **VPI = 15 – 0 = 15**

Decision with risk
Expected value of perfect information

The expected value of perfect information is the difference between the expected utility of the outcome when the decision was made with perfect information and the expected utility of the outcome when the decision was made with risk.

Decision with risk

Alternatives	States of nature: It rains	States of nature: It does not rain	EU
Umbrella	15	10	**10.5**
No umbrella	0	18	**16.2**
Probabilities	p=0.1	p=0.9	

Decision with perfect information

Alternatives	States of nature: It rains	States of nature: It does not rain	EU
$\max(a_i,\theta_j)$	15	18	**17.7**
Probabilities	p=0.1	p=0.9	

Expected value of perfect information (EVPI) = 17,7-16,2 = 1,5

Decision with risk
Some considerations

- Features of Maximum Expected Utility criteria
  - It accounts for all the states of nature and their probabilities.
  - The expected payoff can be interpreted as what the *average* payoff would become if the same situation were repeated many times. Therefore, *on average*, repeatedly applying Maximum Expected Value decision rule to make decisions will lead to larger payoffs in the long run than any other criterion.

- Criticisms of Maximum Expected Utility criteria
  - There usually is considerable uncertainty involved in assigning values to the prior probabilities.
  - Prior probabilities inherently are at least largely subjective in nature, whereas sound decision making should be based on objective data and procedures.
  - It ignores typical aversion to risk. By focusing on average outcomes, expected (monetary) payoffs ignore the effect that the amount of variability in the possible outcomes should have on decision making.

Expected utility theory
Some considerations

Expected utility theory is as old as mathematical probability theory (although the phrase "expected utility" is of later origin).

They were both developed in the 17th century in studies of parlour-games. According to the *Port-Royal Logic* (1662),

"to judge what one ought to do to obtain a good or avoid an evil, one must not only consider the good and the evil in itself, but also the probability that it will or will not happen and view geometrically the proportion that all these things have together."
(Arnauld and Nicole [1662] 1965, p. 353 [IV:16])

Maximizing expected utility is a safe method to maximize the outcome in the long run

Suppose, for instance, that the expected number of deaths in traffic accidents in a region will be 300 per year if safety belts are compulsory and 400 per year if they are optional.

If we aim at reducing the number of traffic casualties, then this can, due to the law of large numbers, safely be achieved by maximizing the expected utility (i.e., minimizing the expected number of deaths).

The validity of this argument depends on the large number of road accidents, that levels out random effects in the long run.

However….

The argument is not valid for case-by-case decisions on unique or very rare events.

Suppose, for instance, that we have a choice between a probability of .001 of an event that will kill 50 persons and the probability of .1 of an event that will kill one person.

Here, random effects will not be leveled out as in the traffic belt case. In other words, we do not know, when choosing one of the options, whether or not it will lead to fewer deaths than the other option.

In such a case, taken in isolation, there is no compelling reason to maximize expected utility.

St. Petersburg Paradox (Nicolas Bernoulli -1687-1759)

Proposed by Nicolas Bernoulli in 1713

A fair coin is tossed until the first head occurs.

If the first head comes up on the first toss, then you receive 1 gold coin.

If the first head comes up on the second toss, you receive 2 gold coins.

If it comes up on the third toss, you receive 4 gold coins.

In general, if it comes up on the *n*'th toss, you will receive 2*n* gold coins.

The probability that the first head will occur on the n'th toss is $1/2^n$.

Your **expected wealth** after having played the game is (considering **h** for head and **t** for tail):

$$EV = p(h) \times v(h) + p(t\_h) \times v(t\_h) + p(t\_t\_h) \times v(t\_t\_h) + \ldots =$$

$$= 1/2 \times 2^1 + (1/2)^2 \times 2^2 + (1/2)^3 \times 2^3 + \ldots =$$

$$= 1 + 1 + 1 + 1 + \ldots$$

This sum is equal to infinity.

Thus, **according to the rule of maximization of the expected wealth a rational agent should be prepared to pay any finite amount of money for the opportunity to play this game**.

St. Petersburg paradox (Nicholas Bernoulli -1687-1759)

In 1738, **Daniel Bernoulli** (1700-1782, a cousin of Nicholas') proposed what is still the conventional solution to the St. Petersburg puzzle.

His basic idea was to replace the maximization of expected wealth by the maximization of the expected (**subjective**) utility.

The utility attached by a person to wealth does not increase in a linear fashion with the amount of money, but rather increases at a decreasing rate.

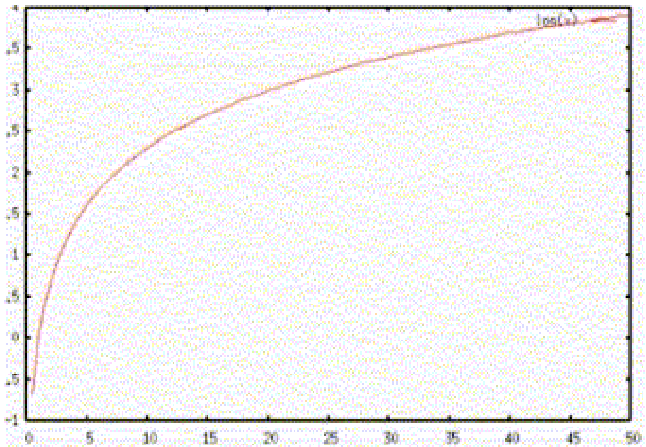
Your first $1000 is more worth to you than is $1000 if you are already a millionaire.

(More precisely, Daniel Bernoulli proposed that the utility of the next increment of wealth is inversely proportional to the amount you already have, so that the utility of wealth is a logarithmic function of the amount of wealth.)

A person with such a utility function may very well be unwilling to put his savings at stake in the St. Petersburg game.

u(w) = k log(w) + constant,

where w represents the amount of money and k is a parameter



And…Prospect Theory (Kahneman and Tversky, 1979)

Consider the following games:

Situation 1

A - you win 2000 € for certain

or

B - You may win 4000 € with 60% probability or nothing at all (40%).

Which alternative do you prefer?

Situation 2

A - you loose 2000 € for certain

or

B - You may loose 4000 € with 60% probability or you may loose nothing at all (40%).

Which alternative do you prefer?

https://www.youtube.com/watch?v=DUd8XA-5HEk

Consider the following two lotteries:

Lottery A:	1 million €	- 11% of the time
	0 €	- 89% of the time
Lottery B:	5 million €	- 10% of the time
	0 €	- 90% of the time

Think for a moment about which you prefer.
Write your answer down.

Now consider these two other lotteries:

Lottery C:	1 million €	- guaranteed
Lottery D:	5 million €	- 10% of the time
	1 million €	- 89% of the time
	0 €	- 1% of the time.

Of these two lotteries, which do you prefer?

Let's analyse lotteries **A** and **B** first, considering that U(0€)= 0

If you prefer **B**:

EU(**A**) < EU(**B**) ⇔ U(1) x 0,11 < U(5) x 0,1

Let's analyse lotteries **C** and **D**

If you prefer **C**:

EU (**C**) > EU (**D**) ⇔ U(1) x 1 > U(5) x 0,1 + U(1) x 0,89

⇔ U(1) x 1 - U(1) x 0,89 > U(5)

⇔ U(1) x 0,11 > U(5) (???)

So choosing **B** and **C** (or **A** and **D**) is inconsistent.

While choosing **A** and **C** (or **B** and **D**) is consistent.

What have been your choices?

This is called the Allais paradox (Maurice Allais was a Nobel prize winning economist) and shows that sometimes we are not able (or we don't have time enough) to make consistent choices....