3) Realiza el Calculo analitico para demostrar la identidad de la distribución binomial negativa como mezcla de Poisson-gamma

Tenemos que $N|X \sim Poisson(n|x)$ y $X \sim Gama(x|\lambda,\alpha)$

Veamos como se distribuye N

$$\Rightarrow P(N=n) = \int_0^\infty P_o(n|x)\, Ga(x|\lambda,\alpha)\,dx = \int_0^\infty \frac{e^{-x} x^n}{n!} \cdot \lambda e^{-\lambda x} \frac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}\,dx$$

$$\Rightarrow = \frac{\lambda \lambda^{\alpha-1}}{n!\,\Gamma(\alpha)} \int_0^\infty e^{-x} e^{-\lambda x} x^n x^{\alpha-1}\,dx = \frac{\lambda^\alpha}{n!\,\Gamma(\alpha)} \int_0^\infty e^{-(\lambda+1)x} x^{n+\alpha-1}\,dx$$

$$\Rightarrow = \frac{\lambda^\alpha}{n!\,\Gamma(\alpha)} \int_0^\infty \frac{(\lambda+1)}{(\lambda+1)} e^{-(\lambda+1)x} \frac{(\lambda+1)^{(n+\alpha)-1} x^{[n+\alpha]-1}}{\Gamma(n+\alpha)} \cdot \frac{\Gamma(n+\alpha)}{(\lambda+1)^{(n+\alpha)-1}}\,dx$$

(*) completamos una nueva gama.

$$\Rightarrow = \frac{\lambda^\alpha}{n!\,\Gamma(\alpha)} \cdot \frac{\Gamma(n+\alpha)}{(\lambda+1)(\lambda+1)^{n+\alpha-1}} \cancel{\int_0^\infty (\lambda+1) e^{-(\lambda+1)x} \frac{[(\lambda+1)x]^{(n+\alpha)-1}}{\Gamma(n+\alpha)}\,dx} \to 1$$

$$= \frac{\Gamma(\alpha+n)}{n!\,\Gamma(\alpha)} \frac{\lambda^\alpha}{(\lambda+1)^{n+\alpha}} = \frac{\Gamma(\alpha+n)}{n!\,\Gamma(\alpha)} \left(\frac{\lambda}{\lambda+1}\right)^\alpha \left(\frac{1}{\lambda+1}\right)^n$$

si $P = \frac{\lambda}{\lambda+1}$ y $(1-P) = 1 - \frac{\lambda}{\lambda+1} = \frac{1}{\lambda+1}$

$N \sim BinNegativa(n|\alpha)$, donde $f_N(n) = \frac{\Gamma(\alpha+n)}{n!\,\Gamma(\alpha)} p^\alpha (1-p)^n$.

donde