② $Y_i \sim \text{Poisson}(\lambda_i) \quad \forall i = 1, \ldots \quad U = \sum Y_i$

$\lambda_i > 0$

$$m_U(t) = E\left[e^{tU}\right] = E\left[e^{t(\sum Y_i)}\right] = E\left[\prod_i e^{tY_i}\right] \overset{\text{ind}}{=} \prod_i E\left[e^{tY_i}\right]$$

$$= \prod_i m_{Y_i}(t) = \prod_i e^{\lambda_i(e^t - 1)} = e^{\sum_i \lambda_i (e^t - 1)}$$

$\Longleftrightarrow U \sim Poi(\sum \lambda_i)$

③ $$\int_0^\infty \frac{\lambda^x e^{-\lambda}}{x!} \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] \lambda^{\alpha-1} e^{-\lambda/\beta} \, d\lambda$$

$$= \int_0^\infty \frac{\lambda^{x+\alpha-1} e^{-(\lambda+\lambda/\beta)}}{x!\,(\alpha-1)!\,\beta^\alpha} d\lambda = \frac{(x+\alpha-1)!\left(\frac{\beta}{\beta+1}\right)^{x+\alpha}}{x!\,(\alpha-1)!\,\beta^\alpha} \cancel{\int_0^\infty \frac{\lambda^{(x+\alpha)-1} e^{-\lambda/\left(\frac{\beta}{\beta+1}\right)}}{\Gamma(x+\alpha)\left(\frac{\beta}{\beta+1}\right)^{x+\alpha}}}$$

$$= \frac{(x+\alpha-1)!}{x!\,(\alpha-1)!} \frac{\beta^x}{(\beta+1)^{x+\alpha}} = \frac{(x+\alpha-1)!}{x!\,(\alpha-1)!} \left(\frac{\beta}{\beta+1}\right)^x \left(1 - \frac{\beta}{\beta+1}\right)^\alpha$$