Tarea 3 28/02/19

$X, Y \sim (\mu, \Sigma)$ con $\mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}$ $\Sigma = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}$

$$f_{y|x} = \frac{f_{xy}}{f_x} = \frac{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right)\right)}{\frac{1}{\sqrt{2\pi}\sigma_x} \exp\left(\frac{-1}{2\sigma_x^2}(x-\mu_x)^2\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left(\frac{(y-\mu_y)^2}{\sigma_y^2} - 2\frac{\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right) - \frac{1}{2\sigma_x^2}\left(\frac{(x-\mu_x)^2}{(1-\rho^2)} - (x-\mu_x)^2\right)\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left(\frac{(y-\mu_y)^2}{\sigma_y^2} - 2\frac{\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right) - \frac{1}{2\sigma_x^2}\left(\frac{\cancel{(x-\mu_x)^2} - \cancel{(x-\mu_x)^2} + \rho^2(x-\mu_x)^2}{(1-\rho^2)}\right)\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{(y-\mu_y)^2}{\sigma_y^2} - 2\frac{\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{\rho^2(x-\mu_x)^2}{\sigma_x^2}\right)\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left(\left(\frac{(y-\mu_y)}{\sigma_y} - \frac{\rho(x-\mu_x)}{\sigma_x}\right)^2\right)\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)\sigma_x\sigma_y}\left((y-\mu_y)\sigma_x - \rho(x-\mu_x)\sigma_y\right)^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)\sigma_x\sigma_y}\left(y\sigma_x - \mu_y\sigma_x - \rho x\sigma_y + \rho\mu_x\sigma_y\right)^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{\sigma_x^2}{2(1-\rho)\sigma_x\sigma_y}\left(y - \left(\mu_x + \frac{\rho x\sigma_y}{\sigma_x} - \frac{\rho\mu_x\sigma_y}{\sigma_x}\right)\right)^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho)\sigma_y}\left(y - \left(\mu_x + \frac{\rho\sigma_y}{\sigma_x}(x-\mu_x)\right)\right)^2\right\}$$

$$\therefore f_{y|x} \sim N\left(\mu_x + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x),\ \underbrace{\sigma_y^2(1-\rho)}_{\text{Varianza}}\right)$$ ■

donde $\rho = \frac{\mathrm{cov}(x,y)}{\sigma_x\sigma_y}$

Tare 3

(2) $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ $X \sim N_p(x \mid \mu, \Sigma)$ $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$

donde $X_1 \in M^{p_1 \times p}$ $X_2 \in M^{p_2 \times p}$ $p_1 + p_2 = n$

$$f(X) = f(x_1, x_2) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left[-\tfrac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right] = c \exp\left[-\tfrac{1}{2} Q(x_1, x_2)\right]$$

donde $Q(x_1, x_2) = (x-\mu)^T \Sigma^{-1} (x-\mu) = [(x_1-\mu_1), (x_2-\mu_2)]^T \begin{pmatrix} \Sigma_{11}^* & \Sigma_{12}^* \\ \Sigma_{21}^* & \Sigma_{22}^* \end{pmatrix} \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}$

$= (x_1-\mu_1)^T \Sigma_{11}^* (x_1-\mu_1) + 2(x_1-\mu_1)^T \Sigma_{12}^* (x_2-\mu_2) + (x_2-\mu_2^T) \Sigma_{22}^* (x_2-\mu_2)$

$$\Sigma_{11}^* = \Sigma_{11}^{-1} + \Sigma_{11}^{-1} \Sigma_{12} (\Sigma_{22} - A_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1} \Sigma_{12}^T \Sigma_{11}^{-1}$$

$$\Sigma_{22}^* = \Sigma_{22}^{-1} + \Sigma_{22}^{-1} \Sigma_{12}^T (\Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}^T)^{-1} \Sigma_{12} \Sigma_{22}^{-1}$$

$$\Sigma_{12}^* = \Sigma_{21}^{*T}$$

$Q(x_1, x_2) = (x_1-\mu_1)^T (\Sigma_{11}^{-1} + \Sigma_{11}^{-1} \Sigma_{12} (\Sigma_{22} - A_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1} \Sigma_{12}^T \Sigma_{11}^{-1})(x_1-\mu_1)$

$-2(x_1-\mu_1)^T (\Sigma_{11}^{-1} \Sigma_{12} (\Sigma_{22} - \Sigma_{12}^T \Sigma_{11} \Sigma_{12})^{-1} (x_2-\mu_2)$

$+ (x_2-\mu_2)^T ((\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1} (x_2-\mu_2)$

$= (x_1-\mu_1)^T \Sigma_{11}^{-1} (x_1-\mu_1) + (x_1-\mu_1)^T \Sigma_{11}^{-1} \Sigma_{12} (\Sigma_{22} - A_{12}^T \Sigma_{12})^{-1} \Sigma_{12}^T \Sigma_{11}^{-1} (x_1-\mu_1)$

$-2(x_1-\mu_1)^T (\Sigma_{11}^{-1} \Sigma_{12} (\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1}) (x_2-\mu_2$

$+ (x_2-\mu_2))^T ((\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1}) (x_2-\mu_2)$

$= (x_1-\mu_1)^T \Sigma$