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THE ROLE OF CORRELATED FACTORS IN FACTOR ANALYSIS

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The fundamental factor theorem is developed in matrix form for the case of correlated factors. The properties of the correlated factor system are discussed, and some effects of sampling error considered. The psychological meaning of correlated factors is discussed, and several mechanisms by which general factors may operate in the factorial system are indicated.

It is the purpose of this paper to develop a generalized factor theorem which applies directly to any set of correlated or uncorrelated factors and to consider the interpretation of correlated psychological factors.

The first and most basic assumption in factor analysis is that a test score may be expressed as a linear combination of scores on several factors. This assumption may be written in the form of an equation:

$$s_{ji} = a_{j1}x_{1i} + a_{j2}x_{2i} + a_{j3}x_{3i} + \cdots + a_{jt}x_{ti} , \tag{1}$$

where s_{ji} is the standard score of individual i on test j. The x's are the standard scores of the individuals on the factors $1, 2, 3, \cdots , t$; the a's are the weights to be applied to the factor scores in obtaining the individual's score on the test. The standardization of these scores is assumed to be based on an infinite population. This equation may be written in matrix form:

$$S = FP , \tag{2}$$

the entries in S being s_{ji}, in F being a_{jp}, and in P being x_{pi}, where p is the subscript used to designate the factors.

The correlations between two tests, j and k, may be expressed by the following formula:

$$r_{jk} = \frac{1}{N} \sum_{i=1}^{N} S_{ji} S_{ki} , \tag{3}$$

where N is the number of individuals in the sample used in obtaining the correlation. The standard deviations do not enter into the formula since they are unity for standard scores. Similarly, the correlation between the two factors, p and q, is

$$r_{pq} = \frac{1}{N} \sum_{i=1}^{N} x_{pi} x_{qi} \, . \tag{4}$$

The correlations between the factors will be allowed to take any values so long as the factors remain linearly independent. That is, although each factor must present some independent portion to justify its retention, some overlap will be allowed between the factors.

Equations (3) and (4) may be written in matrix form:

$$R_j = \frac{1}{N} SS' \, , \tag{5}$$

$$R_p = \frac{1}{N} PP' \, , \tag{6}$$

where R_j is the matrix of correlations between the tests and R_p is the matrix of correlations between the factors. When the value of S in equation (2) is substituted into equation (5), this latter equation becomes:

$$R_j = \frac{1}{N} FPP'F' \, ,$$

$$= F(\frac{1}{N} PP')F' \, ,$$

which becomes, by (6):

$$R_j = F \, R_p \, F' \, .^* \tag{7}$$

In equation (7) the fundamental factorial theorem of Thurstone (2) has been generalized to correlated factors.

It will be noted that if the factors are uncorrelated, R_p is an identity matrix, and equation (7) reduces to Thurstone's form of the theorem as a special case, namely,

$$R_j = FF' \, . \tag{8}$$

It is of interest that a transformation exists which carries the results of the one type of analysis to the other. This transformation makes it possible to use the present methods of factoring to obtain an arbitrary set of uncorrelated reference factors and then to transform them to a final set of factors which are either correlated or uncorrelated.

Since the factors have been assumed to be linearly independent,

* Since writing this manuscript, Professor Karl J. Holzinger has called the author's attention to a publication by himself and Mr. Harry H. Harman in which our matrix equation was written in expanded form (7, p. 324, eq. 4). The expanded form is, however, extremely laborious in use, both in theoretical and computational work.

the rank of R_p is equal to its order, and it possesses an inverse. R_p may now be reproduced by:

$$R_p \equiv HH' , \tag{9}$$

where H is a matrix of rank and order equal to the order of R_p. The matrix H also possesses an inverse. Substituting R_p of (9) in (7),

$$R_j = FHH'F' . \tag{10}$$

Then, if matrix F_u is defined as follows:

$$F_u \equiv FH , \tag{11}$$

(10) becomes

$$R_j = F_u F_u' . \tag{12}$$

By (8) it is seen that F_u is a factorial matrix for a set of uncorrelated reference factors. Thus, the transformation from the correlated factor case to the uncorrelated case has been completed. The reverse transformation is also possible. The transformation which may be applied to a matrix with uncorrelated factors is, by (11),

$$F = F_u H^{-1} . \tag{13}$$

Equation (9) gives the correlations between the factors when a matrix H is known. Thus it is possible to use any of the factorial methods that produce an uncorrelated set of reference factors and then to transform these factors to another set of factors which are either correlated or uncorrelated.

In practice this transformation may be simplified by the definition of the matrix Λ, as follows. Let

$$\underline{\Lambda \equiv H^{-1}}D , \tag{14}$$

where D is a diagonal matrix such that the columns of Λ are unit vectors (that is, the sums of the squares of the entries in the columns of Λ are unity). Then F_u is postmultiplied by Λ to obtain a matrix V,

$$V \equiv F_u \Lambda . \tag{15}$$

But, by (14),

$$V = F_u H^{-1} D ,$$

and, by (13),

$$V = FD , \tag{16}$$

or

$$F = VD^{-1} . \tag{17}$$

The factorial matrix F is proportional by columns to V.
By (14),

$$H = D \Lambda^{-1} , \tag{18}$$

and, by (9)

$$R_p = D\,\Lambda^{-1}(\Lambda^{-1})'D' ,$$

$$R_p = D(\Lambda'\,\Lambda)^{-1}D' . \tag{19}$$

Thus, if Λ and V are known, F , H , and R_p may be found by equations (17), (18) and (19), the condition on D being that the diagonal entries in R_p must be unity.

The reason for the use of Λ in performing the transformation from an original set of uncorrelated factors to the final set of factors can be discussed best after the investigation of the special case of uncorrelated transformed factors. In this case,

$$R_p = I \; ; \tag{20}$$

then, by (9),

$$HH' = I , \tag{21}$$

and

$$H' = H^{-1} . \tag{22}$$

But the sums of the squares of the entries in the columns of H' are already unity, as shown by equation (21) and therefore D , in equation (14), is an identity matrix, that is,

$$D = I . \tag{23}$$

Then by (14) and (22),

$$\Lambda = H' , \tag{24}$$

and by (21),

$$\Lambda'\Lambda = I . \tag{25}$$

By (15), (17), and (23),

$$F = V = F_u\,\Lambda . \tag{26}$$

The matrix V is the transformed factorial matrix F in this special case instead of being merely proportional by columns to F , as shown for the general case in equation (17). But this result may be used in the interpretation of any V , whether the factors are correlated or uncorrelated. If a single column of a V is considered at a time, the corresponding column of Λ may be placed in a new matrix Λ_1 , which satisfies equation (25), and therefore is a transformation to a set of uncorrelated factors. That column of V then becomes a column of factor loadings in the new matrix V_1 to which it has been transferred. Thus, the entries in any column of V may be considered to be loadings on a factor which is independent of the rest of the system.

The advantages of using Λ in transforming an F_u to a final F are that its use not only gives entries in V proportional by colmuns to the entries in F , but that these entries in V have a special significance, and that the condition that each column of Λ is to be a unit vector applies directly to that column without any reference to its relation to the other columns of Λ. In order to know H^{-1}, the entire matrix H must be known. The use of Λ allows each factor to be determined independently. Since the entries in a column of V are proportional to the entries in the corresponding column of F, zero entries are not altered in using V instead of F, and the same simple structure will exist in the two matrices. But some other condition could be placed on the columns of Λ instead of their being unit vectors. Such a condition might be that the largest entry in each column of Λ be unity. The condition of unit vectors has been chosen because of the special significance of the entries in V, as previously developed, when this condition is used.

The final transformation, Λ, may be built up either graphically (1, 2, 3, 4) or by means of analytic criteria (2, 5). The matrix V may then be found by equation (15) and R_p by equation (19). H and F are seldom found, since V is proportional by columns to F and therefore may be used directly in the interpretation of the factors.

The correlations between the tests and the factors are often desired for predictive purposes. The formula for these correlations is:

$$r_{jp} = \frac{1}{N} \sum_{i=1}^{N} s_{ji} x_{pi} \ ,$$

or, in matrix form:

$$R_{jp} = \frac{1}{N} SP' \ ,$$

where R_{jp} is the matrix of correlations of the tests j with the factors p.

By (2)

$$R_{jp} = \frac{1}{N} FPP' \ ;$$

by (6)

$$R_{jp} = F\, R_p \ ;^*$$

by (9)

$$R_{jp} = FHH' \ ;$$

and by (11)

$$R_{jp} = F_u H' \ . \tag{27}$$

It will be noted that only when the factors are uncorrelated and R_p an identity matrix, is the matrix F also the matrix of correlations between the tests and the factors.

* This equation was written in expanded form by Professor Karl J. Holzinger and Mr. Harry H. Harman (7, p. 322, eq. 2).

But the development of the mathematical properties of the factorial system is not sufficient for a complete discussion of the effect of allowing correlated factors; the psychological limitations of the system must also be considered. The simplest case, which is to be considered first, is for an infinite population when no sampling errors have been introduced into the system. The effects of sampling will be considered later.

Thurstone (2) recognized three types of factors when dealing with uncorrelated factors. He assumed that there were factors common to two or more tests, other meaningful factors present in only one test, and still other factors present in each test which arose from the chance errors of measurement. East test could have loadings on any common factor, but only on the specific factor and measurement error factor which were associated with the test. The only revision necessary in the present context is that the common factors may be correlated. The specific factors may be defined as being uncorrelated with all other factors. This definition throws the burden of description of all the correlations between the tests upon the common factors. Further, the variations of the scores on any test which depend upon the test's specific factor most surely should be independent of all other tests given in the battery, or that factor would not be specific to that test. In an infinite sample the variable errors of measurement should be uncorrelated with everything. The division of the factors into these three categories is based entirely on psychological reasons. If it is the operational unities of the mind that are to be discovered by the use of factor analysis, then these divisions are required. They are part of the restrictions placed upon the system by the science of psychology. No statistical or mathematical considerations can remove them; rather, they are to be dealt with mathematically and statistically as a part of the problem.

But we must take into consideration what occurs to the system when a finite sample is used rather than the infinite population used above. It will be assumed that, for any particular subject tested, equation (1) still holds as long as that subject's scores are expressed as standard scores in terms of the entire infinite population. While these scores are impossible to obtain, they are a good starting point in the investigation of the effects of testing only a sample of the population.

If S_s and P_s are the sections of the S and P matrices containing the scores for the subjects used in the sample, then in accordance with equation (2):

$$S_s = FP_s\,. \tag{28}$$

But these scores are not standard scores with reference to the sample tested, for the means and standard deviations for the sample are different from the means and standard deviations for the entire population. The first problem is to adjust these scores to standard scores in terms of the sample used. Equation (28) may be written in summation form:

$$s_{ji} = \sum_{p=1}^{t} a_{jp} x_{pi} \ , \tag{29}$$

t being the number of factors. The equations for the means of the scores on the tests and on the factors are:

$$m_j = \frac{1}{N} \sum_{i=1}^{N} s_{ji} \ ; \tag{30}$$

$$m_p = \frac{1}{N} \sum_{i=1}^{N} x_{pi} \ , \tag{31}$$

m_j being the mean for test j , and m_p the mean for the factor p . Substituting the value of s_{ji} from equation (29) into equation (30) and then rearranging:

$$\begin{aligned} m_j &= \frac{1}{N} \sum_{i=1}^{N} \sum_{p=1}^{t} a_{jp} x_{pi} \ , \\ &= \sum_{p=1}^{t} a_{jp} \frac{1}{N} \sum_{i=1}^{N} x_{pi} \ . \end{aligned}$$

This becomes, upon substituting for $(\frac{1}{N} \sum_{i=1}^{N} x_{pi})$ its equivalent, m_p , as given in (31):

$$m_j = \sum_{p=1}^{t} a_{jp} m_p \ . \tag{32}$$

The scores may now be expressed as standard scores with respect to the sample used, u_{ji} being the standard score on test j with respect to the sample, and y_{pi} being the standard score on the factor p with respect to the sample:

$$u_{ji} = \frac{1}{\sigma_j} (s_{ji} - m_j) \ , \tag{33}$$

$$y_{pi} = \frac{1}{\sigma_p} (x_{pi} - m_p) \ , \tag{34}$$

where σ_j is the standard deviation for test j , and σ_p is the standard deviation for factor p . When the value of s_{ji} in equation (29) and the values of m_j in equation (32) are substituted into equation (33),

$$u_{ji} = \frac{1}{\sigma_j}\left(\sum_{p=1}^{t} a_{jp}x_{pi} - \sum_{p=1}^{t} a_{jp}m_p \right) ,$$
$$= \frac{1}{\sigma_j} \sum_{p=1}^{t} a_{jp}(x_{pi} - m_p) ,$$

which becomes, upon consideration of equation (34),

$$u_{ji} = \sum_{p=1}^{t} \frac{1}{\sigma_j} a_{jp}\sigma_p y_{pi} .$$

Or, when

$$b_{jp} \equiv \frac{1}{\sigma_j} a_{jp}\sigma_p , \tag{35}$$

then

$$u_{ji} = \sum_{p=1}^{t} b_{jp} y_{pi} : \tag{36}$$

The importance of equations (35) and (36) is that the linear combination of equation (1) still holds when a sample group is tested instead of the whole population. The only change is in the weights used, and this change is brought about by multiplying factors only. Thus, the simple structure is not changed by these effects of sampling, for zero weights must remain zero.

But the sampling has affected the correlations between the factors. The specific factors and measurement error factors are no longer uncorrelated. It is now a statistical problem to separate these factors from the common factors as well as possible, and thus to obtain as close an approximation to the true common factor matrix as possible. One way open is to eliminate the specific variance and measurement error variance from the diagonal entries of the experimental correlation matrix and then to factor the resulting matrix. This elimination separates out a diagonal factorial matrix whose factors are uncorrelated in the sample used. But the side entries of the correlation matrix have been affected by the sampling and have not been adjusted. The factors obtained from these correlations will also be affected; but this is not all — a new section has been added to the factorial matrix. This new section is needed to complete the description of the side entries of the correlation matrix. It is rarely explicitly determined, its effects being left as residuals when the factoring has been completed. Since the sampling also affects the correlations among the common factors, it would be indeed surprising to find any set of uncorrelated factors which represent operational units in the minds of the subjects tested. Thus, some of the correlation between factors may be attributed to sampling error, but some of the correlations between factors may arise from psychological phenomena.

Psychologically, the factors may be neither ultimate nor indivisible. Further study of them probably will show that each factor depends upon the coordinated activity of many mental elements. Some factors may be primarily determined by the structure and dynamic activity of the human body. Other factors may originate in the training and past experience of the individuals. Thus, one may think of different domains of factors. The factors in one domain may be affected by factors in other domains. The effect of these connections between factors in different domains is to leave the factors correlated.

The author has found that the comprehension of the above-de-

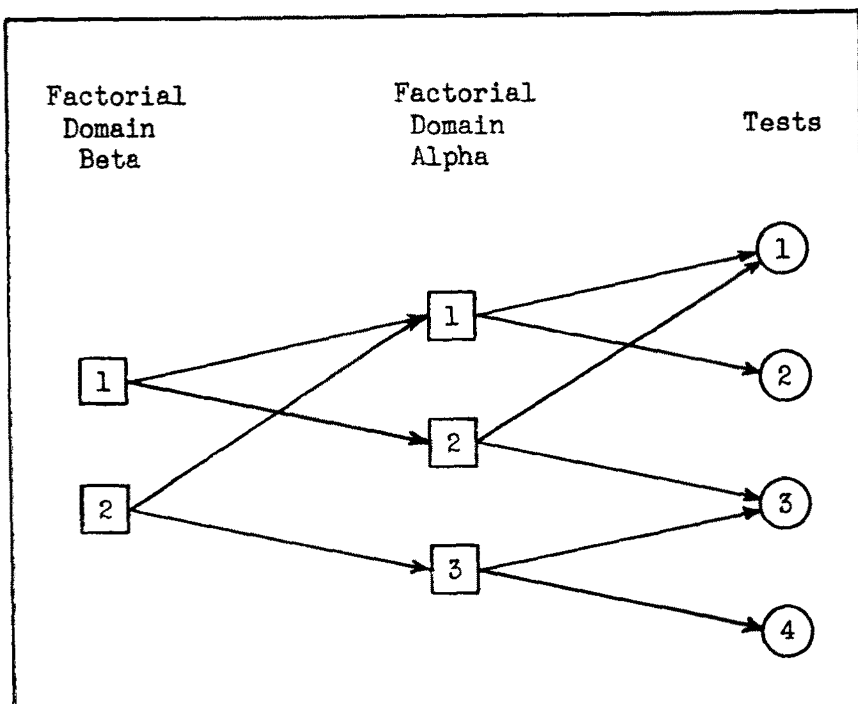

FIGURE 1

scribed relations and their implications could be simplified by the adaptation of a diagrammatic schema originally devised by Sewall Wright in connection with "path coefficients" (6). An example of this schema is given in *Figure 1.* The circles are used to designate the tests. The squares are used to designate the factors. Only the common

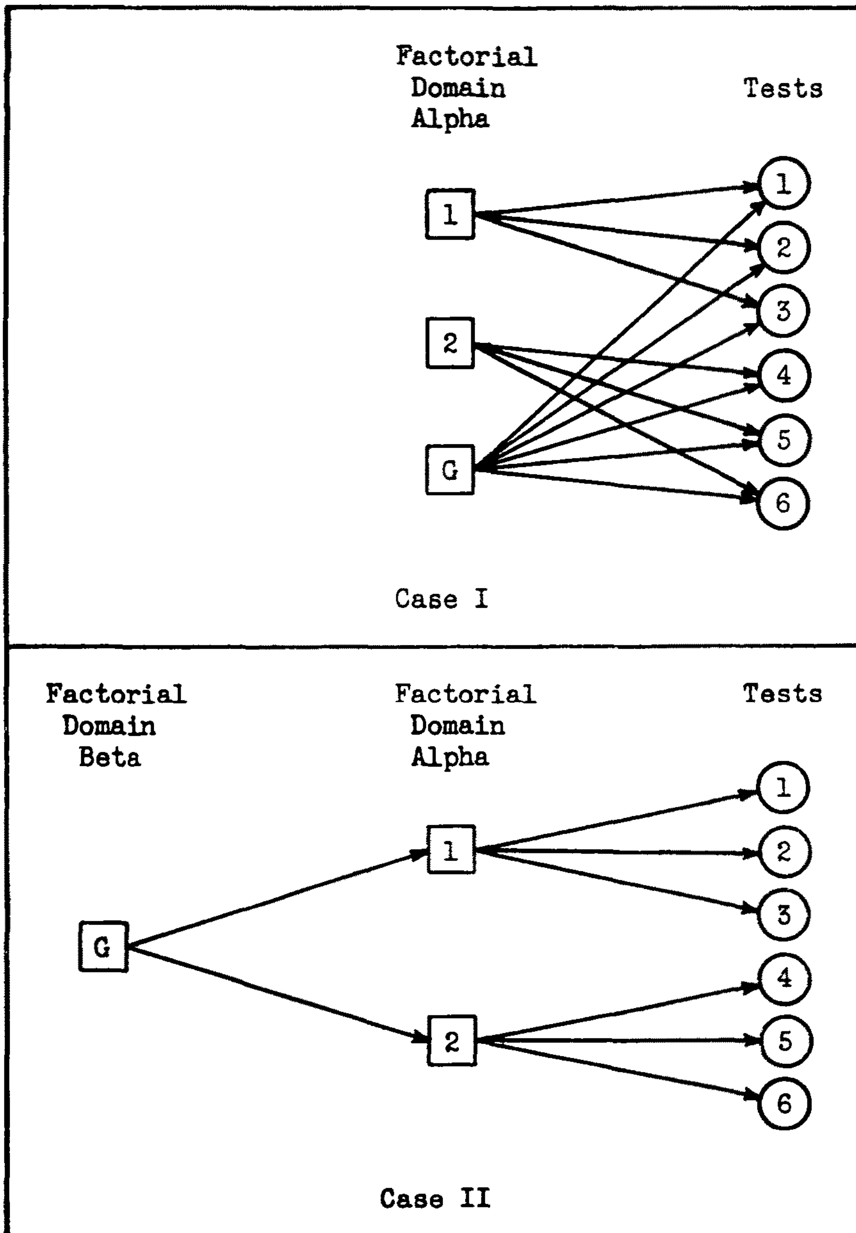
Factorial
Domain
Alpha
Tests
1
2
G
1
2
3
4
5
6
Case I
Factorial
Domain
Beta
Factorial
Domain
Alpha
Tests
G
1
2
1
2
3
4
5
6
Case II

FIGURE 2

factors are represented; all specific and measurement error factors are omitted. The lines are used to represent a postulated "flow of variance," the arrowhead indicating the direction of effect. The factors are divided into two domains. The tests are affected by the factors in factorial domain α which are, in turn, affected by the factors in factorial domain β. Factor 1 in the factorial domain α affects both tests 1 and 2, which are consequently correlated. Factor 2 in the factorial domain β affects both factors 1 and 3 in the factorial domain α, which are, therefore, correlated. If the test intercorrelations were factored, the factors in factorial domain α would be obtained. Similarly, if the correlations between the factors in domain α were factored, the factors in domain β would be obtained. The rank of the matrix of common factors in the tests is the number of factors in domain α, which in the example is three. When factoring the correlations between the factors in domain α, the rank of the matrix of common factors would be the number of factors in domain β, which is two in the example. This rank may be less than the rank of the matrix of common factors in the tests. But each factor in domain α has its own specific factor. This specific factor affects the corresponding factor in domain α and, thus, several of the tests. It therefore indirectly affects the correlations between these tests. Thus, the number of factors in domain β which affect the intercorrelations of the tests is equal to the number of common factors in domain β plus as many specific factors as there are common factors in domain α. Therefore, the factors in domain β are not found when the test intercorrelations are factored, the rank of this matrix of correlations being equal to the number of factors in domain α alone.

The question of a general factor may now be discussed in terms of correlated factors and the schema described previously. Two possible cases present themselves. Is the general factor one of the factors in domain α, or is it one of the factors in domain β? Figure 2 presents these two cases. A simple example of six tests, two group factors, and a general factor has been chosen. In each case, tests 1, 2, and 3 are affected by group factor 1; and tests 4, 5, and 6 are affected by group factor 2. In case I the general factor, G, affects all tests directly and independently of the group factors. In this case it is placed in domain α. In case II the general factor affects the group factors and then the tests. It is placed in domain β in this case. In case I the rank of the matrix of common factors in the tests is three. One of these factors, G, may not be found by the criterion of simple structure; but it can be set uncorrelated with the group factors which are found by this criterion. In case II the rank of the matrix of common factors in the tests is two, one less than in case I. The group factors are correlated in this case, and the general factor may be found by factoring the matrix of correlations between these factors.

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