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(1978-1979)

## Article Information

Journal Title: Contributions to atmospheric physics; Beiträge zur Physik der Atmosphäre.

Volume: 52 Issue:

Month/Year: 1979 Pages: 1-16

Article Author: Geleyn, J. F., and A. Hollingsworth

Article Title: An economical analytical method for the computation of the interaction between scattering and line absorption of radiation

## Loan Information

Loan Title:

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# An Economical Analytical Method for the Computation of the Interaction Between Scattering and Line Absorption of Radiation

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(Manuscript received 8.5.1978, in revised form 16.11.1978)

## Abstract:

We propose a method for short wave and long wave radiative calculations in a cloudy atmosphere taking account of grey absorption and scattering as well as line-type absorption by gases. The simplifying assumption is that the photon path lengths for gaseous absorption can be evaluated by a perturbation treatment of the "grey" case. Because of the analytical solution the cost of the computation is low and only slightly dependent on the number of parameters in the global transmission function for gases.

**Zusammenfassung:** Eine ökonomische analytische Methode zur Berechnung der Wechselwirkung zwischen Streuung und Linienabsorption der Strahlung

Wir schlagen eine Methode vor, die die kurzweligen und langwelligen Strahlungsflüsse in einer wolkenbedeckten Atmosphäre berechnet unter Berücksichtigung von grauer Absorption und Streuung sowie von linienförmiger Absorption durch Gase. Die vereinfachende Annahme ist, daß die Photonenweglängen für Gasabsorption sich durch eine Perturbationsbehandlung des „grauen“ Falls berechnen lassen. Wegen der analytischen Lösung ist der Rechenaufwand klein und nur in geringem Maße von der Anzahl der Parameter in der globalen Transmissionsfunktion für Gase abhängig.

**Résumé:** Une Méthode analytique économique pour le calcul de l'interaction entre la diffusion et l'absorption par raies du rayonnement

Nous proposons une méthode pour le calcul des flux radiatifs solaires et thermiques en atmosphère nuageuse qui tient compte à la fois de l'absorption et de la diffusion neutres et de l'effet des raies d'absorption des gaz. L'hypothèse simplificatrice est que les parcours des photons à travers les absorbants gazeux peuvent être évalués par un calcul de perturbation du cas „neutre“. Grâce à une solution analytique du problème le volume de calcul est faible et ne dépend que fort peu du nombre de paramètres entrant dans la fonction de transmission globale pour les gaz.

## 1 Introduction

In this paper we present an economical way of treating the interaction between grey extinction and line absorption. The basic hypothesis is that we can neglect the influence of gaseous absorption on the scattering processes. Numerical tests supporting this assumption will be presented in Section 8 below. The path lengths for gaseous absorption are estimated from the fluxes computed with only grey effects and from their analytical derivatives. We can therefore obtain fluxes taking both kinds of effects into account, with a low computational cost.

Provided that scattering and absorption can be considered as "grey" (i.e. that mean optical coefficients are accurate enough over given spectral intervals), one can easily compute radiative fluxes throughout the atmosphere using the standard procedures of the two-stream approximation for each layer and of the

adding method to combine the effects of all layers. The assumption of greyness can be accepted in practice (i.e. with few spectral intervals) for Rayleigh scattering (because it is a relatively small and therefore linear effect) and for extinction by cloud and aerosol (where our knowledge is anyhow very uncertain), but it cannot be applied to the highly non linear line-type gaseous absorption. Many radiative calculations tackle the problem either by neglecting scattering (as in emissivity-type methods) or by oversimplifying gaseous absorption. Two more complete approaches are possible. The first one is to express all gaseous transmission functions as sums of exponentials, thus dividing the spectral interval into a set of subintervals in which the absorption by gases is assumed to be grey; the second one is to evaluate the path lengths of all photons contributing to a flux. Besides using the mean path length, higher moments of the path length distribution may be considered for reasons of accuracy. It has been proven (see for example BAKAN et al. (1978)) that both methods are equivalent for an infinite number of terms (exponentials or moments). The second method gives better results for a small number of terms (RODGERS (1977)). Moreover, unlike the first method it is not limited to a single parameter transmission function. It can therefore be applied with the CURTIS-GODSON approximation for example, provided one computes path lengths for unreduced and reduced amounts of absorber. With three gases (water vapour, carbon dioxide and ozone), in the CURTIS-GODSON approximation, one has to calculate the path length six times for each moment. This can become very expensive if one uses the standard method (see for example BAKAN and QUENZEL (1976)) of recomputing the fluxes with a dummy grey absorption coefficient for gases to get each moment for each absorber. The method we are suggesting here (analytical differentiation of the fluxes with respect to these dummy absorption coefficients) removes this problem. The number of gases, and of parameters used in the gaseous transmission function, has little influence on the amount of computation. The higher degree of mathematical complexity in our method is only formal, the cumbersome calculations being done only "on paper". The actual programming remains quite simple and the results are obtained with a big reduction in the computational load.

For the sake of simplicity the method will first be presented for the first moment of the path length distribution (i.e. the mean encountered amount of absorber) and for solar radiation in a cloudless atmosphere. This will be done in Sections 2, 3 and 4. In the later sections we will indicate how to extend it to cloudy atmospheres (5), to thermal radiation (6) and to higher moments of the distributions (7).

## 2 Computations for solar radiation without gaseous absorption in a cloudless atmosphere

Here we will only indicate the mathematical aspects of the "two stream approximation" and of the "adding method" which are necessary for the understanding of the development in later sections. For a detailed discussion see ZDUNKOWSKI et al. (1974), or COAKLEY and CHYLEK (1975) to be read with WISCOMBE and GRAMS (1976).

Starting from the equation of radiative transfer and applying the two-stream method with the EDDINGTON approximation, we get a set of equations for fluxes in a horizontally homogeneous atmosphere.

$$\frac{dF^\uparrow(t)}{dt} = \alpha_1 \cdot F^\uparrow(t) - \alpha_2 \cdot F^\downarrow(t) - \alpha_3 \cdot S(t)/\mu_0 \quad (1a)$$

$$\frac{dF^\downarrow(t)}{dt} = \alpha_2 \cdot F^\uparrow(t) - \alpha_1 \cdot F^\downarrow(t) + \alpha_4 \cdot S(t)/\mu_0 \quad (1b)$$

$$\frac{dS(t)}{dt} = -S(t)/\mu_0 \quad (1c)$$

$$\text{with } \alpha_1 = 2 \cdot (1 - \omega \cdot (1 - \tilde{\beta})) \quad \alpha_2 = 2 \cdot \omega \cdot \tilde{\beta} \quad (2a, b)$$

$$\alpha_3 = \omega \cdot \beta(\mu_0) \quad \alpha_4 = \omega \cdot (1 - \beta(\mu_0)) \quad (2c, d)$$

In these equations  $F^\uparrow$  is the upward diffuse flux,  $F^\downarrow$  the downward diffuse flux and  $S$  the downward parallel flux of solar radiation ( $F^\uparrow$ ,  $F^\downarrow$  and  $S$  are fluxes across horizontal surfaces);  $t$  is the optical depth taken as vertical coordinate and increasing downwards from 0 at the top of the atmosphere to  $t_g$  at the earth's surface,  $\omega$  the single scattering albedo,  $\mu_0$  the cosine of the solar zenith angle,  $\beta(\mu_0)$  the upscattered fraction for the parallel solar beam and  $\tilde{\beta}$  the backscattered fraction for the diffuse beams.

The coefficients  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_3$  and  $\alpha_4$  vary with  $t$  through  $\omega$ ,  $\beta(\mu_0)$  and  $\tilde{\beta}$ . We divide the atmosphere into  $n$  layers bounded by the levels at which we want to compute the radiative fluxes. Within each layer we assume that the  $\alpha$  coefficients are constant. A given layer  $j$  (the subscript  $j$  will be used from now on to indicate a layer quantity) is then characterised by its optical depth  $\Delta t_j$  and its values for  $\alpha_{1j}$ ,  $\alpha_{2j}$ ,  $\alpha_{3j}$  and  $\alpha_{4j}$ . The set of Equations (1) can then be integrated analytically across this idealised layer. One can express the results of this integration as a linear dependency of the outgoing fluxes from the layer on the incoming fluxes to the layer. With subscripts  $t$  and  $b$  designating respectively the top and the bottom of the layer we have (see Figure 1):

$$\begin{bmatrix} S_b \\ F_b^\downarrow \\ F_t^\uparrow \end{bmatrix}_j = \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_4 \end{bmatrix}_j \cdot \begin{bmatrix} S_t \\ F_t^\downarrow \\ F_b^\uparrow \end{bmatrix} \quad (3)$$

$a_1$  is the transmission for the parallel beam,  $a_4$  the transmission for diffuse radiation,  $a_5$  the reflection for diffuse radiation,  $a_2$  the diffuse transmission for the parallel beam and  $a_3$  the diffuse reflection for the parallel beam.

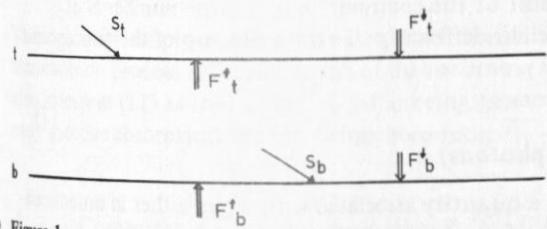


Figure 1  
Schematic representation of a layer showing the distinction between incoming and outgoing fluxes. A parallel flux is denoted by  $(\rightarrow)$  and a diffuse flux by  $(\Rightarrow)$ .

#### Bild 1

Schematische Darstellung einer Schicht zur Unterscheidung von ein- und ausfließenden Flüssen. Ein paralleler Fluß ist gekennzeichnet durch  $(\rightarrow)$  und ein diffuser Fluß durch  $(\Rightarrow)$ .

The coefficients  $a_{ij}$ , ( $i = 1, 5$ ) are analytical functions of  $\Delta t_j$ ,  $\alpha_{1j}$ ,  $\alpha_{2j}$ ,  $\alpha_{3j}$ ,  $\alpha_{4j}$  and  $\mu_0$ . For details of the analytical expressions see Appendix A.

Having computed the  $a_{ij}$  coefficients for all the  $n$  layers, we have  $3n$  linear equations for  $3n + 3$  unknown fluxes. The linear system is completed by three boundary conditions:  
at the top of the atmosphere

$$S(0) = \mu_0 \cdot I_0 \quad (4a)$$

$$F^\downarrow(0) = 0 \quad (4b)$$

and at the bottom of the atmosphere

$$F^\dagger(t_g) = A_g(\mu_0) \cdot S(t_g) + \tilde{A}_g \cdot F^\dagger(t_g) \quad (5)$$

( $I_0$  is the solar constant and  $A_g(\mu_0)$ ,  $\tilde{A}_g$  are the ground albedos for the parallel and the diffuse fluxes respectively).

Our linear system of  $3n + 3$  equations for  $3n + 3$  unknowns can then be written in matrix form as

$$\mathbf{A} \cdot \mathbf{F} = \mathbf{S} \quad (6)$$

$\mathbf{A}$  is a band structured matrix with all diagonal elements equal to one, the other non-zero elements being either  $-a_{ij}$  or  $-A_g(\mu_0)$  or  $-\tilde{A}_g$ .  $\mathbf{F}$  is the vector of the fluxes. It consists of the subvectors ( $S$ ,  $F^\dagger$ ,  $F^\dagger$ ) appropriate to each level.  $\mathbf{S}$  is the source vector with one single non-zero element equal to  $\mu_0 I_0$  (see Equation (4a)). The generation of Equation (6) is the first step in the solution of the problem by the "adding method"; it is similar to the corresponding step in the solution of a finite element problem.

With the methods described in this section we are able to compute fluxes throughout the atmosphere provided that averaged optical properties are given for each layer. In the case of the standard method referred to in the introduction, this would be done first without gaseous absorption and then repeated with different "dummy" grey gaseous absorption coefficients. In our method this need be done only once, as explained in the next sections.

### 3 Computation of the mean encountered amounts of absorber

The linear System (6) can easily be solved by a gaussian elimination-backsubstitution method. This is equivalent to the "adding method" if we arrange to do the minimum necessary computation by adapting the algorithm to the position of the non-zero entries of the matrix and by using the fact that all diagonal elements are one. Let  $\mathcal{F}_0$  be any element of the solution vector  $\mathbf{F}$  in the case without gaseous absorption.  $\mathcal{F}_0$  can be considered as the result of the contribution of a large number  $N$  of equi-energetical groups of photons which have travelled different paths from the top of the atmosphere to the level at which  $\mathcal{F}_0$  is defined:

$$\mathcal{F}_0 = N \cdot f \quad (7)$$

( $f$  is the flux associated with one group of photons)

Let us use the subscript  $m$  from now on to indicate a quantity associated with a gas either in unreduced or in reduced amounts; we shall for simplicity speak of an absorber, whether in unreduced or reduced amount, as a gas.

Given a certain gas, whose amount in each layer is defined as  $\Delta u_{jm}$ , we want to know the mean amount,  $u_m^{(1)}$ , of this gas encountered along their different paths by all the groups of photons contributing to  $\mathcal{F}_0$ . Assuming that this gas has an infinitesimal grey absorption coefficient  $k_m$  one can see that the introduction of its effect into the computation of the fluxes would give as a new result

$$\mathcal{F}_m = \sum_{l=1}^{l=N} f \cdot e^{-k_m \cdot u_{ml}} \quad (8)$$

In (8) the contribution of each group of photons,  $f$ , is multiplied by the transmission function for the amount of absorber  $u_{ml}$  encountered by the group of photons  $l$ . Since  $k_m$  is infinitesimally small  $u_m^{(1)}$  can be also defined as the quantity of encountered absorber which would give us the same result

$\tilde{\mathcal{F}}_m$  if all photons had the same path. In this case all the  $u_{ml}$  would be equal to  $u_m^{(1)}$  and  $-u_m^{(1)}$  would be the logarithmic derivative of  $\tilde{\mathcal{F}}_m$  with respect to  $k_m$ , as  $k_m$  goes to zero.

Writing first the definition of  $u_m^{(1)}$  as a mean value, using (7) and (8) and the fact that  $\mathcal{F}_m$  tends to  $\mathcal{F}_0$  as  $k_m$  goes to zero, we can show that this is still true even if the  $u_{ml}$  are different from each other:

$$u_m^{(1)} = \frac{1}{N} \sum_{l=1}^L u_{ml} = -\frac{f}{\mathcal{F}_0} \sum_{l=1}^L \frac{\partial(e^{-k_m \cdot u_{ml}})}{\partial k_m} \Big|_{k_m=0} = -\frac{\partial \mathcal{F}_m / \partial k_m}{\mathcal{F}_0} \Big|_{k_m=0}$$

$$= -\frac{\partial \mathcal{F}_0 / \partial k_m}{\mathcal{F}_0} \quad (9)$$

As mentioned in the introduction, we have here ignored the fact that, with a real absorption, some already absorbed photons cannot be further scattered, unlike the case treated above. To take this effect into account we would have to use the standard method of taking a realistic  $k_m$ , changing  $\Delta t_j$  and  $\omega_j$  for each layer (by increasing the absorption optical depth by the amount  $k_m \cdot \Delta u_{jm}$ ) and then repeating the whole process to compute  $\mathcal{F}_m$ . Then, using the second definition of  $u_m^{(1)}$  we would have:

$$u_m^{(1)} = -\ln(\mathcal{F}_m / \mathcal{F}_0) / k_m \quad (10)$$

The computational advantage of our method is twofold: firstly, as will be shown in the next section, the computation of  $\partial \mathcal{F}_0 / \partial k_m$  is less expensive than the computation of  $\mathcal{F}_m$ ; secondly the main part of the calculation of  $\partial \mathcal{F}_0 / \partial k_m$  is independent of the  $\Delta u_{jm}$  and so of the particular absorber.

Once the absorber amounts  $u_m^{(1)}$  for all gases have been obtained from Equation (9), they are used as input to compute a gaseous transmission function  $T$  by means of band model calculations. In our case, where the mean amounts are the only input information, this means that we compute the transmission by gases as if all photons had the same path. The final calculation of the flux  $\mathcal{F}$  taking both grey effects and gaseous absorption into account is given by:

$$\mathcal{F} = \mathcal{F}_0 \cdot T(u_m^{(1)}, m = 1, 6) \quad (11)$$

This operation simply applies the multiplication rule for transmissivities since  $\mathcal{F}_0$  is the result of a transmission process affecting a part of the incoming radiation at the top of the atmosphere. We can also interpret (11) as if the flux  $\mathcal{F}_0$ , after being generated by grey processes, had to cross a layer where only gaseous absorption occurs, before becoming  $\mathcal{F}$ .

#### 4 Computation of the derivative $\partial \mathcal{F}_0 / \partial k_m$

In analogy with  $\mathbf{IF}$  we denote by  $\mathbf{IF}'_m$  the corresponding vector of all  $\partial \mathcal{F}_0 / \partial k_m$ . The differentiation with respect to  $k_m$  of Equation (6) (using the fact that  $\mu_0 I_0$  is independent of  $k_m$ ) gives us a simple equation for  $\mathbf{IF}'_m$ :

$$\mathbf{A} \cdot \mathbf{IF}'_m = -\mathbf{A}'_m \cdot \mathbf{IF} \quad (12)$$

where  $\mathbf{A}'_m$  is the matrix derived from  $\mathbf{A}$  by replacing all the  $-a_{ij}$  elements by  $-\partial a_{ij} / \partial k_m$  and setting all other elements to zero ( $A_g(\mu_0)$  and  $\tilde{A}_g$  are independent of  $k_m$ ).

One can see that the form of Equation (12) for the flux derivatives is exactly the same as that of Equation (6) for the fluxes. There is of course a different right-hand side for each gas. Since Equation (6) has to be solved anyway, a careful storage of intermediate results will ensure that the extra cost of the solution of Equation (12) is small. We must first however compute the right-hand sides,  $-\mathbf{A}'_m \cdot \mathbf{IF}$ . We already know  $\mathbf{IF}$  and the only problem is to compute the  $-\partial a_{ij} / \partial k_m$  elements in  $\mathbf{A}'_m$ . But here again there is no necessity to perform a full computation for each gas.

We may write:

$$\frac{\partial a_{ij}}{\partial k_m} = \frac{\partial a_{ij}}{\partial (k_m \cdot \Delta u_{jm})} \cdot \Delta u_{jm} = \frac{\partial a_{ij}}{\partial \Delta t_j} \Big|_{(\omega_j \cdot \Delta t_j) = \text{constant}} \cdot \Delta u_{jm}$$

$$= \dot{a}_{ij} \cdot \Delta u_{jm}$$
(13)

The differentiation of the  $a_{ij}$  coefficients with respect to the local optical depth, keeping the product of optical depth and single scattering albedo constant, simply means that the introduction of gaseous absorption in a layer increases the absorption optical depth without modifying the scattering optical depth. The terms  $\dot{a}_{ij}$  do not depend on the particular absorber. Thus using (13) we can rewrite Equation (12):

$$\mathbb{A} \cdot \mathbf{IF}'_m = -\mathbb{A}'_m \cdot \mathbf{IF} = -\mathbb{U}_m \cdot \dot{\mathbb{A}} \cdot \mathbf{IF} = -\mathbb{U}_m \cdot \mathbb{G}$$
(14)

where  $\mathbb{U}_m$  is a diagonal matrix, the diagonal elements being the quantities of absorber  $\Delta u_{jm}$ . The vector  $\mathbb{G}$  is the product of  $\mathbf{IF}$  by the matrix  $\dot{\mathbb{A}}$  which is obtained by replacing the  $-a_{ij}$  by  $-\dot{a}_{ij}$  in  $\mathbb{A}$  and setting to zero all other elements.  $\mathbb{G}$  is independent of  $m$ . Thus the final row multiplication of  $\mathbb{G}$  by the local absorber amounts to calculate the right-hand sides is a trivial cost.

The analytical computation of  $\dot{a}_i$  is a simple application of differentiation rules to the analytical expressions given in Appendix A. The calculations of both the  $a_{ij}$  and the  $\dot{a}_{ij}$  for one layer are done in parallel to take advantage of some common intermediate results. For the detailed expressions see Appendix B.

To summarise the previous two sections, we have proposed a method to calculate the mean path length of all photons contributing to a flux. The method is economical for two reasons. Firstly the cost of the calculation of the right-hand sides of Equation (12) is almost independent on the number of gases (see Equation (14)); secondly, the solution of the linear System (12) has already been done for Equation (6) and so the additional cost of solving Equation (12) is not important. Actual timings to verify these points are given in Section 8 below.

## 5 Introduction of cloudiness

Considering a partially covered layer we can assume that it is characterised by two sets of  $(\Delta t, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ : one for the cloudy part (superscript c) and one for the cloud-free part (superscript f). Thus we get two sets of  $a_{ij}$  coefficients  $a_{ij}^c$  and  $a_{ij}^f$ . We consider that each flux  $\mathcal{F}_0$  is the sum of two parts  $\mathcal{F}_0^c$  and  $\mathcal{F}_0^f$  depending on whether the photons are coming out of a cloudy or out of a cloud-free area. Assuming that the scattering and absorption processes inside both parts of a layer produce horizontally homogeneous outgoing radiation fields from composite incoming ones, the set of Equations (3) can be replaced by:

$$\begin{bmatrix} S_b^f \\ F_b^{tf} \\ F_t^{tf} \end{bmatrix}_j = \begin{bmatrix} a_1^f & 0 & 0 \\ a_2^f & a_4^f & a_5^f \\ a_3^f & a_5^f & a_4^f \end{bmatrix}_j \cdot \begin{bmatrix} b_1 \cdot S_t^f + (1 - b_3) \cdot S_t^c \\ b_1 \cdot F_t^{tf} + (1 - b_3) \cdot F_t^{tc} \\ b_2 \cdot F_b^{tf} + (1 - b_4) \cdot F_b^{tc} \end{bmatrix}_j$$
(15a)

$$\begin{bmatrix} S_b^c \\ F_b^{tc} \\ F_t^{tc} \end{bmatrix}_j = \begin{bmatrix} a_1^c & 0 & 0 \\ a_2^c & a_4^c & a_5^c \\ a_3^c & a_5^c & a_4^c \end{bmatrix}_j \cdot \begin{bmatrix} (1 - b_1) \cdot S_t^f + b_3 \cdot S_t^c \\ (1 - b_1) \cdot F_t^{tf} + b_3 \cdot F_t^{tc} \\ (1 - b_2) \cdot F_b^{tf} + b_4 \cdot F_b^{tc} \end{bmatrix}_j$$
(15b)

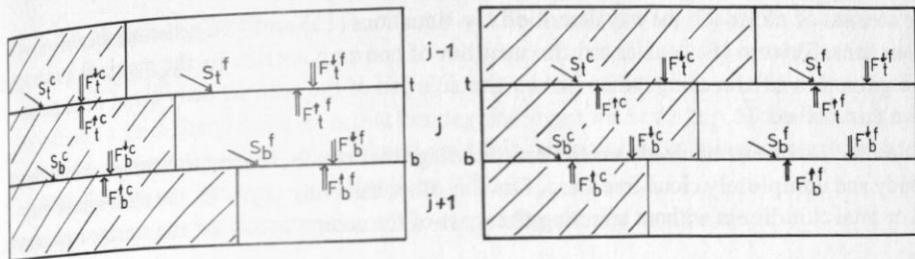


Figure 2  
Schematic representation of some possible cloud geometries: one can, for example, verify that the contribution of  $F_t^{tc}$  to  $F_t^{tc}$  has to be weighted by the factor  $C(j)/C(j-1)$  on the right hand side and by the factor 1 on the left hand side. Thus we derive the analytical expression of  $b_{3j}$ .

Bild 2  
Schematische Darstellung einiger möglicher Wolkenbilder: man kann zum Beispiel verifizieren, daß der Beitrag von  $F_t^{tc}$  zu  $F_t^{tc}$  auf der rechten Seite durch den Faktor  $C(j)/C(j-1)$  und auf der linken Seite durch den Faktor 1 gewichtet werden muß. So erhalten wir den analytischen Ausdruck für  $b_{3j}$ .

so that the outgoing "clear" and "cloudy" fluxes are linearly dependent on both "clear" and "cloudy" incoming fluxes.

If we assume that cloudy areas in adjacent layers have the maximum overlap, then it is readily shown (see Figure 2 where both possible geometries at the top and at the bottom of the layer  $j$  are represented) that, with  $C$  representing the amount of cloud:

$$b_{1j} = \frac{1 - \max(C_j, C_{j-1})}{1 - C_{j-1}} \quad b_{2j} = \frac{1 - \max(C_j, C_{j+1})}{1 - C_{j+1}} \quad (16 \text{ a,b})$$

$$b_{3j} = \frac{\min(C_j, C_{j-1})}{C_{j-1}} \quad b_{4j} = \frac{\min(C_j, C_{j+1})}{C_{j+1}} \quad (16 \text{ c,d})$$

When indeterminate according to Equation (16) the coefficients  $b_{1j}, b_{2j}, b_{3j}$  and  $b_{4j}$  are set equal to one, which is the value of the limit of the indeterminacy.

This treatment of partial cloudiness assumes maximum overlapping of adjacent cloudy layers but total independence in position between cloudy parts separated by cloud free air.

We think it is more reliable than assuming independence in position between all cloudy layers (MANABE and STRICKLER, (1964)). With this latter proposal the effect of a vertically homogeneous cloud depends on the number of layers it is occupying in the model and is therefore quite arbitrary. Of course, this latter proposal is cheaper because one defines

$$a_{ij} = C_j \cdot a_{ij}^c + (1 - C_j) \cdot a_{ij}^f \quad (17)$$

and the separation into two sets of fluxes is not necessary. We think that the amount of supplementary computation required by our treatment is worth doing because it allows a definition of partial cloudiness without any geometrical restriction.

The equivalent of Equation (9) must now become

$$v_m^{(1)} = -\frac{\partial \mathcal{F}_0^c / \partial k_m + \partial \mathcal{F}_0^f / \partial k_m}{\mathcal{F}_0^c + \mathcal{F}_0^f} \quad (18)$$

the rest of the algorithm remaining unchanged.

When we take account of clouds in the way described by Equations (15) and (16), the number of equations in our linear System (6) doubles and the number of non-zero entries in the matrix  $\mathbb{A}$  quadruples; but a careful organisation of the computation makes the solution of the system only about three times more expensive than before.

This approach to the treatment of clouds has the disadvantage that one increases the computational load equally in cloudy and completely cloud-free areas. On the other hand any layer in the atmosphere may have a partial or total cloudiness without affecting the cost of the computation or the storage requirement.

## 6 Application of the method to long wave fluxes

In the case of thermal radiation the system of Equations (1) is replaced by:

$$\frac{dF^\uparrow(t)}{dt} = \alpha_1 \cdot (F^\uparrow(t) - \pi \cdot B(t)) - \alpha_2 \cdot (F^\downarrow(t) - \pi \cdot B(t)) \quad (19a)$$

$$\frac{dF^\downarrow(t)}{dt} = \alpha_2 \cdot (F^\uparrow(t) - \pi \cdot B(t)) - \alpha_1 \cdot (F^\downarrow(t) - \pi \cdot B(t)) \quad (19b)$$

where  $B(t)$  is the spectrally integrated PLANCK function,  $\pi B(t)$  being therefore the black-body flux at the level defined by  $t$ .

If we set:

$$F^{\uparrow*} = \pi \cdot B - F^\uparrow \text{ and } F^{\downarrow*} = \pi \cdot B - F^\downarrow \quad (20a,b)$$

the System (19) becomes:

$$\frac{dF^{\uparrow*}(t)}{dt} = \alpha_1 \cdot F^{\uparrow*}(t) - \alpha_2 \cdot F^{\downarrow*}(t) + \pi \cdot \frac{dB(t)}{dt} \quad (21a)$$

$$\frac{dF^{\downarrow*}(t)}{dt} = \alpha_2 \cdot F^{\uparrow*}(t) - \alpha_1 \cdot F^{\downarrow*}(t) + \pi \cdot \frac{dB(t)}{dt} \quad (21b)$$

We can integrate this system for each layer as before, provided we assume that  $B$  is a linear function of the optical depth  $t$  inside each layer. Given that we only know  $B$  at the top and bottom of a layer in most problems, this is the simplest and most reasonable assumption we can make. The result of the integration is

$$\begin{bmatrix} F_b^{\downarrow*} \\ F_t^{\uparrow*} \end{bmatrix}_j = \begin{bmatrix} a_4 & a_5 \\ a_5 & a_4 \end{bmatrix}_j \cdot \begin{bmatrix} F_t^{\downarrow*} \\ F_b^{\uparrow*} \end{bmatrix}_j + \begin{bmatrix} a_6 - a_6 \\ -a_6 & a_6 \end{bmatrix}_j \cdot \begin{bmatrix} \pi \cdot B_b \\ \pi \cdot B_t \end{bmatrix}_j \quad (22)$$

with

$$a_6 = \frac{1 - a_4 + a_5}{(\alpha_1 + \alpha_2) \cdot \Delta t} \quad (23)$$

( $a_4$  and  $a_5$  are defined in Appendix A).

The boundary conditions replacing the Equations (4) and (5) are:

$$\text{at the top of the atmosphere } F^{\downarrow*}(0) = \pi \cdot B(0) \quad (24)$$

$$\text{and at the bottom of the atmosphere } F^{\uparrow*}(t_g) = (1 - E_g) \cdot F^{\downarrow*}(t_g) \quad (25)$$

where  $E_g$  is the ground emissivity.

The matrix Equation (6) becomes:

$$\mathbf{A} \cdot \mathbf{F}^* = \mathbf{S}^* \quad (26)$$

The difference from the solar case is that besides the input  $\pi B$  at the top of the atmosphere there are other non-zero elements in  $\mathbf{S}^*$ : temperature gradients in atmospheric layers generate source terms for Equation (26), as may be seen from Equation (21).

From the solution of the linear System (26) we get fluxes:

$$\mathcal{F}_0 = \pi \cdot B_I - \mathcal{F}_0^* \quad (27)$$

where  $B_I$  is the local PLANCK function corresponding to  $\mathcal{F}_0$  and  $\mathcal{F}_0^*$  is any element of the solution  $\mathbf{F}^*$  obtained by the gaussian algorithm.

Unfortunately we cannot estimate the path lengths by differentiating  $\mathcal{F}_0^*$  with respect to  $k_m$  because we have no simple relation between flux and path length equivalent to Equation (8). The reason is that there is no single source term in Equation (26).

If the atmosphere were isothermal we would not have this difficulty. An isothermal earth-atmosphere system transmits and reflects the difference between the black-body flux and the actual flux, the only source term for this difference being at the top of the atmosphere. In such an atmosphere the terms in  $dB/dt$  vanish in Equation (21); and in Equation (26) the source vector  $\mathbf{S}^*$  has only one non-zero entry,  $\pi B$ , at the top of the atmosphere. We then have an exact analogue for the case of solar radiation treated earlier.

We therefore propose to base our estimate for the absorber path lengths on the path lengths encountered in an isothermal earth-atmosphere system. This is in a certain sense analogous to the "cooling to space" approximation.

If we denote by  $(\bar{\ })$  the values appropriate for an isothermal earth-atmosphere system, then Equation (26) becomes in such a case:

$$\mathbf{A} \cdot \bar{\mathbf{F}}^* = \bar{\mathbf{S}}^* \quad (28)$$

We can now repeat all the steps described in section 3 and 4 until we get a gaseous transmission function  $\bar{T}$  computed in isothermal conditions.

The easiest way of applying  $\bar{T}$  to the final calculation of  $\mathcal{F}$  would be to make a total "cooling to space" assumption: we compute each flux as if the earth-atmosphere system were isothermal with the temperature of the level at which it is computed.

$$\mathcal{F} = \pi \cdot B_I + (\mathcal{F}_0 - \pi \cdot B_I) \cdot \bar{T} \quad (29)$$

Just as for the "cooling to space" method (GINZBURG and FEIGELSON (1970)) one may go further and introduce the effect of temperature lapse rates. The ratio

$$\bar{E}_0 = \bar{\mathcal{F}}_0 / (\pi \cdot \bar{B}) \quad (30)$$

is a background emissivity for "grey" effects in the isothermal case. Let us assume that this emissivity can also be applied to the computation of  $\mathcal{F}$ . Then, by analogy to the discussion at the end of section 3, we posit that: a flux  $\mathcal{F}_0$  emitted by a grey body of emissivity  $\bar{E}_0$  (the associated black-body flux being therefore  $\mathcal{F}_0/\bar{E}_0$ ) has to cross a layer in which only gaseous absorption and the simultaneous emission take place; the black-body flux varies across the layer from the value given above to the local value  $\pi B_I$ ; the resulting flux will be  $\mathcal{F}$ .

If for consistency we assume again that the PLANCK function depends linearly on the optical depth (i.e. in this case the logarithm of the transmission), the final result is:

$$\begin{aligned}\mathcal{F} = \pi \cdot B_l + (\mathcal{F}_0 - \mathcal{F}_0/\bar{E}_0) \cdot \bar{T} \\ + (\mathcal{F}_0/\bar{E}_0 - \pi \cdot B_l) \cdot (\bar{T} - 1)/\ln(\bar{T})\end{aligned}\quad (31)$$

One can verify that this result is what one would get from Equations (21) in the case of zero scattering with the appropriate changes in the notation.

The method for calculation of the path lengths is apparently singular when  $\mathcal{F}_0$  is zero in Equation (9) or  $\bar{\mathcal{F}}_0^*$  is zero in the corresponding equation for long wave radiation.

In the case of solar radiation this is unimportant since, after the introduction of gaseous absorption,  $\mathcal{F}$  will be zero anyhow if  $\mathcal{F}_0$  is zero in Equation (11).

In the case of thermal radiation  $\bar{\mathcal{F}}_0^*$  being zero means that the flux  $\mathcal{F}_0$  is equal to the black-body flux  $\pi \bar{B}$  of the isothermal earth-atmosphere system. Our method gives no information to resolve the ambiguity; but this situation can only arise in an idealised system where there is no scattering or reflection process in the atmosphere or at the ground. Provided we take the physically realistic point of view that the ground emissivity  $E_g$  is never one and/or that there are always scattering particles in the atmosphere, the singularity is avoided.

## 7 The calculation of higher moments of the path length distribution

By extending the arguments used in section 3 one can show that the  $n^{\text{th}}$  moment  $u_m^{(n)}$  of the distribution of path lengths through the absorber  $m$  is given by

$$u_m^{(n)} = (-1)^n \frac{\partial^n \mathcal{F}_0 / \partial k_m^n}{\mathcal{F}_0} \quad (32)$$

The method described in earlier sections can be applied to derive the higher moments of the path length distribution by repeated differentiation of the Equation (6) for the fluxes:

$$\mathbb{A} \cdot \mathbb{F}_m^{(n)} = - \sum_{k=1}^{k=n} B_k^n \cdot \mathbb{A}_m^{(k)} \cdot \mathbb{F}_m^{(n-k)} \quad (33)$$

where  $B_k^n$  are the binomial coefficients.

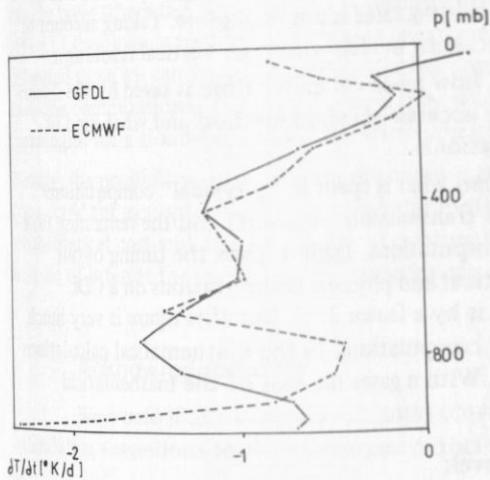
This is a recursion relation for the  $n^{\text{th}}$  derivative of the fluxes in terms of their lower derivatives. The methods used earlier can therefore be applied again to calculate  $\mathbb{F}_m^{(n)}$ . Thus our method may be generalised to take account of higher moments of the path length distribution in the computation of the transmission function. The only difficulty is the analytical calculation of the derivatives of the  $a_i$  coefficients. This must be done manually and probably involves increasingly cumbersome computations as the order of the derivative increases, but it need only be done once and for all.

## 8 Results and conclusions

The method described in this paper has been pre-operationaly implemented in the framework of a new parameterization package designed at ECMWF for medium range forecasting (GELEYN (1977)). The implementation treats clouds and long-wave radiation as described in Section 6 and 7 (of course, the cloud treatment is also valid for thermal radiation). Only the first moment of the path length distribution

is used in a CURTIS-GODSON approximation for  $H_2O$  vapour,  $CO_2$  and  $O_3$ . In Figure 3 we have a comparison of the vertical distributions of the net radiative heating rates averaged for the earth over two ten-day forecasts, one using the ECMWF parameterization (see TIEDTKE et al. (1979)) and the other the GFDL parameterization with the radiative scheme of MANABE and STRICKLER (1964). The resolution of the integration was  $1^\circ \times 1^\circ \times 0.875$  with 15 levels. For more details see HOLLINGSWORTH et al. (1979).. The differences in the boundary layer can be explained by a different treatment on continents of the interaction between surface temperature and radiation; the differences in the stratosphere are probably due to deficiencies in our treatment of the top boundary condition for temperature. The temperature at zero pressure has proven to be an important parameter whose formulation is still being studied. The overall agreement between both curves is, of course, no proof of the validity of our method, but it shows that it can give reliable results in the context for which it has been designed.

The case in which our method is likely to give its worst results is that of long wave radiation in a cloudless atmosphere. The deficiencies of our pseudo "cooling to space" approximation (which is necessary to include multiple scattering, but is not a consequence of linearization) should be noticeable in this case, where emissivity methods are quite accurate, because there is hardly any scattering. Figure 4 shows a comparison

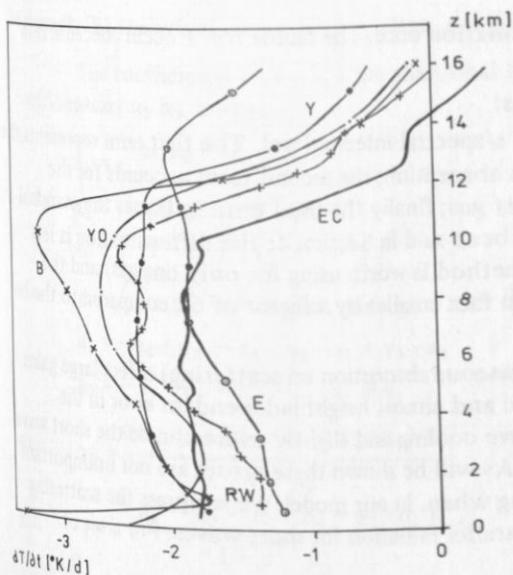


● Figure 3

Comparison of the global net radiative heating rates for two ten day forecasts differing only through their parametrization scheme for physical processes. Initial conditions from 22-2-1976.

● Bild 3

Vergleich globaler Nettostrahlungserwärmungsraten für zwei zehntägige Vorhersagen, die nur durch ihre Parametrisierung von physikalischen Prozessen unterschiedlich sind. Anfangsdaten vom 22.2.1976.



● Figure 4

Comparison of the long wave heating rates computed for the cloudless tropical atmosphere of LONDON (1952). Letters correspond to: E-ELASSER (1942), B-BROOKS (1950), Y-YAMAMOTO (1952), YO-YAMAMOTO and ONISHI (1953), RW-RODGERS and WALSHAW (1966) and EC-ECMWF.

● Bild 4

Vergleich langwelliger Erwärmungsraten, die für die wolkenfreie Atmosphäre von LONDON (1952) berechnet worden sind. Die Buchstaben entsprechen: E - ELSASSER (1942), B - BROOKS (1950), Y - YAMAMOTO (1952), YO - YAMAMOTO und ONISHI (1953), RW - RODGERS and WALSHAW (1966) and EC - ECMWF.

■ Table 1 Time costs in  $10^{-3}$  s per spectral interval and per level on the CDC CYBER 175 computer.

■ Tabelle 1 Zeitaufwand in  $10^{-3}$  s pro Spektral-Intervall und pro Fläche auf dem CDC CYBER 175 Computer.

	mathematical part	physical part	sum
our method	0.3083	0.4605	0.7688
standard method	1.3461	0.4605	1.8066

of our results with those of emissivity-type methods for the vertical profiles of cooling rates in the cloudless LONDON tropical atmosphere. Our averaging of the photon path lengths is probably the reason why our method gives overall smooth cooling rates. Nevertheless, the main feature of a more or less constant cooling rate in the lower troposphere and a rapid decrease of the cooling rate in the upper troposphere and lower stratosphere are reasonably well represented. The cost of our method for thermal radiation is linear in the number of levels, while the cost of emissivity methods is quadratic. As will be shown below, the influence of scattering on long wave heating rates is not negligible. Taking account of both these points, our scheme offers advantages in forecasting models with high vertical resolution. But we want now to concentrate on the essential issue: how much computer time is saved by the linearization technique we are using, compared with the more accurate standard method and what kind of errors are introduced in the results through the linearization?

Let us first make a distinction in computing time between what is spent in "physical" computations (optical depths, single scattering albedos ... and gaseous transmission functions) and the remaining time which we shall designate as spent in "mathematical" computations. Table 1 gives the timing of our method and of the standard method for both mathematical and physical computations on a CDC CYBER 175. One can see that the overall improvement is by a factor 2.35. But this figure is very much dependent on the level of sophistication of the physical computations. In the mathematical calculations the improvement factor is 4.37 for the case of six gases. With  $n$  gases the cost of the mathematical calculations in the standard method is given by

$$[(n + 1) \times 0.1923] \times 10^{-3} \text{ s/spectral interval/level};$$

$0.1923 \times 10^{-3}$  s is the cost of doing the full flux calculation once. The factor  $n + 1$  occurs because it is done once without gases and once for each gas.

Using our method for  $n$  gases the corresponding cost is:

$[0.1923 + 0.1005 + (n - 1) \times 0.0031] \times 10^{-3}$  s/spectral interval/level. The first term represents the cost of the computation of the fluxes without gaseous absorption; the second term accounts for the formal differentiation and the computation for the first gas; finally the third term indicates the marginal cost for any supplementary gas. This verifies what has been said in Section 4: the differentiation is less expensive than the basic computation (therefore the method is worth using for only one gas) and the marginal cost of any supplementary gas is negligible, in fact smaller by a factor of 62 compared to the standard method.

Our linearization (i.e. our neglect of the influence of gaseous absorption on scattering) offers large gains in computational efficiency at the price of a systematic and almost height independent error in the heating rates: we somewhat underestimate the long wave cooling and slightly overestimate the short wave heating. The effect on the surface fluxes is very small. As will be shown these errors are not unimportant but it is worth comparing them with the errors occurring when, in our model, we suppress the scattering of diffuse fluxes but retain the primary scattering of parallel radiation for short waves. For a set of

Table 2 Root mean square differences in  $\text{W/m}^2$  for fluxes and in  $^\circ\text{K/d}$  for heating rates.

Tabelle 2 Streuungen der Flüsse in  $\text{W/m}^2$  und der Erwärmungsraten in  $^\circ\text{K/d}$ .

	flux			heating rate		
	long wave	short wave	net	long wave	short wave	net
standard method v. linearization method	3.23	2.19	3.80	0.0860	0.0592	0.104
multiple scattering v. no multiple scattering	8.64	208.	208.	0.360	0.647	0.691

realistic test atmospheres generated with random deviations around mean profiles (see GELEYN (1977)) we show in Table 2 both root-mean-square differences in the fluxes and in the heating rates. In all cases the differences due to the linearization are about one order of magnitude smaller than the differences due to the neglect of multiple scattering. In the long wave domain these results show that the hypothesis of black or grey non-reflecting clouds is sometimes unreliable. This is especially true above the uppermost cloudy layer where the upward flux is smaller and therefore the overall cooling stronger if the reflective effect of the cloud is included. Thus our linearization error is not completely negligible, but can be tolerated since we can include in the radiation computation the important effect of multiple scattering and keep the computational costs to the level they would be without it. We believe that this choice is especially meaningful for a cloud-radiation interactive scheme such as we are using for medium range forecasts.

Finally, the possibility of increasing the number of parameters in the gaseous transmission function (one could for example use a four parameter method: RODGERS (1967)) without any important extra mathematical cost and the possibility of treating higher moments of the distribution of these parameters may be of interest for more specific radiation studies.

## Acknowledgements

We would like to thank Prof. ZDUNKOWSKI and Dr. KORB of the university of Mainz for their stimulating suggestions in the early stages of this work.

## Appendix A Analytical expressions of the $a_i$ coefficients

The coefficients  $a_i$ ,  $i = 1, 5$  are analytical functions of the optical depth  $\Delta t$  and of the four coefficients  $\alpha_1 \alpha_2 \alpha_3$  and  $\alpha_4$ .

$$a_1 = e^{-\Delta t/\mu_0} \quad (\text{A } 1 \text{ a})$$

$$a_4 = \tau \cdot (1 - \rho^2) / (1 - \tau^2 \cdot \rho^2) \quad (\text{A } 1 \text{ b})$$

$$a_5 = \rho \cdot (1 - \tau^2) / (1 - \tau^2 \cdot \rho^2) \quad (\text{A } 1 \text{ c})$$

$$\text{with } \epsilon = \sqrt{\alpha_1 - \alpha_2} \quad \rho = \alpha_2 / (\alpha_1 + \epsilon) \quad \tau = e^{-\epsilon \cdot \Delta t} \quad (\text{A } 2 \text{ a, b, c})$$

$$(\text{A } 1 \text{ d})$$

$$a_2 = -a_4 \cdot \gamma_2 - a_5 \cdot \gamma_1 \cdot a_1 + \gamma_2 \cdot a_1 \quad (\text{A } 1 \text{ e})$$

$$(\text{A } 1 \text{ e})$$

$$a_3 = -a_5 \cdot \gamma_2 - a_4 \cdot \gamma_1 \cdot a_1 + \gamma_1 \quad (\text{A } 3 \text{ a})$$

$$\text{with } \gamma_1 = \frac{\alpha_3 - \mu_0 \cdot (\alpha_1 \cdot \alpha_3 + \alpha_2 \cdot \alpha_4)}{1 - \epsilon^2 \cdot \mu_0^2} \quad (\text{A } 3 \text{ a})$$

$$\text{and } \gamma_2 = \frac{-\alpha_4 - \mu_0 \cdot (\alpha_1 \cdot \alpha_4 + \alpha_2 \cdot \alpha_3)}{1 - \epsilon^2 \cdot \mu_0^2} \quad (\text{A } 3 \text{ b})$$

There are two special cases where the above results become indeterminate and other expressions have to be given:

(i) for  $\omega = 1$  (pure scattering case, i.e. no absorption)

$$a_4 = \frac{1}{1 + \alpha_0 \cdot \Delta t} \quad a_5 = \frac{\alpha_0 \cdot \Delta t}{1 + \alpha_0 \cdot \Delta t} \quad (\text{A } 4 \text{ a, b})$$

$$\text{with } \alpha_0 = \alpha_1 = \alpha_2$$

(ii) for  $\epsilon\mu_0 = 1$  (so called resonance case)

$$a_2 = -a_4 \cdot \gamma_2^* - a_5 \cdot \left( \gamma_1^* + \gamma_1^{**} \cdot \frac{\Delta t}{\mu_0} \right) \cdot a_1 + \left( \gamma_2^* + \gamma_2^{**} \cdot \frac{\Delta t}{\mu_0} \right) \cdot a_1 \quad (\text{A } 5 \text{ a})$$

$$a_3 = -a_5 \cdot \gamma_2^* - a_4 \cdot \left( \gamma_1^* + \gamma_1^{**} \cdot \frac{\Delta t}{\mu_0} \right) a_1 + \gamma_1^* \quad (\text{A } 5 \text{ b})$$

$$\text{with } \gamma_1^* = \mu_0 \cdot \frac{\alpha_1 \cdot \alpha_3 + \alpha_2 \cdot \alpha_4}{2} \quad \gamma_2^* = \mu_0 \cdot \frac{\alpha_1 \cdot \alpha_4 + \alpha_2 \cdot \alpha_3}{2} \quad (\text{A } 6 \text{ a, b})$$

$$\text{and } \gamma_1^{**} = \gamma_1^* - \alpha_3/2 \quad \gamma_2^{**} = \gamma_2^* + \alpha_4/2 \quad (\text{A } 6 \text{ c, d})$$

### Appendix B Analytical expressions of the $\dot{a}_i$ coefficients

The analytical computations of the  $\dot{a}_i$  in the general case is a simple application of the differentiation rules to the Expressions (A1). With the same notations as in Appendix A we have:

$$\dot{a}_1 = -a_1/\mu_0 \quad (\text{B } 1 \text{ a})$$

$$\dot{a}_4 = \left( 4 \cdot \tau \cdot \rho \cdot \frac{a_5}{\Delta t} - 2 \cdot (1 + \tau^2 \cdot \rho^2) \cdot \alpha_1 \cdot a_4 \right) / (\epsilon \cdot (1 - \tau^2 \cdot \rho^2)) \quad (\text{B } 1 \text{ b})$$

$$\dot{a}_5 = \left( 4 \cdot \tau \cdot \rho \cdot \alpha_1 \cdot a_4 - 2 \cdot (1 + \tau^2 \cdot \rho^2) \cdot \frac{a_5}{\Delta t} \right) / (\epsilon \cdot (1 - \tau^2 \cdot \rho^2)) \quad (\text{B } 1 \text{ c})$$

$$\begin{aligned} \dot{a}_2 &= -\dot{a}_4 \cdot \gamma_2 - a_4 \cdot \dot{\gamma}_2 - \dot{a}_5 \cdot \gamma_1 \cdot a_1 - a_5 \cdot (\dot{\gamma}_1 \cdot a_1 + \gamma_1 \cdot \dot{a}_1) \\ &\quad + \dot{\gamma}_2 \cdot a_1 + \gamma_2 \cdot \dot{a}_1 \end{aligned} \quad (\text{B } 1 \text{ d})$$

$$\dot{a}_3 = -\dot{a}_5 \cdot \gamma_2 - a_5 \cdot \dot{\gamma}_2 - \dot{a}_4 \cdot \gamma_1 \cdot a_1 - a_4 \cdot (\dot{\gamma}_1 \cdot a_1 + \gamma_1 \cdot \dot{a}_1) + \dot{\gamma}_1 \quad (\text{B } 1 \text{ e})$$

with the derivatives of  $\gamma_1$  and  $\gamma_2$  given by:

$$\begin{aligned} \dot{\gamma}_1 &= \gamma_1 \cdot (2 \cdot \mu_0^2 \cdot (2 \cdot \alpha_1 - \epsilon^2) / (1 - \epsilon^2 \cdot \mu_0^2) - 2) / \Delta t \\ &\quad - \alpha_3 \cdot (2 \cdot \mu_0 - 1) / (\Delta t \cdot (1 - \epsilon^2 \cdot \mu_0^2)) \end{aligned} \quad (\text{B } 2 \text{ a})$$

$$\begin{aligned} \dot{\gamma}_2 &= \gamma_2 \cdot (2 \cdot \mu_0^2 \cdot (2 \cdot \alpha_1 - \epsilon^2) / (1 - \epsilon^2 \cdot \mu_0^2) - 2) / \Delta t \\ &\quad - \alpha_4 \cdot (2 \cdot \mu_0 + 1) / (\Delta t \cdot (1 - \epsilon^2 \cdot \mu_0^2)) \end{aligned}$$

In the two special cases the computation is more complicated. We cannot here apply the differentiation rules because the Expressions (A4) and (A5) are only valid for one value of  $\Delta t$  or  $\omega$ .

In the pure scattering case we must do a limited expansion of the results in terms of a small variation of  $\Delta t$ . We then obtain:

$$\dot{a}_4 = -(1 + a_4^2 - a_5^2/3) \quad (B 3 a)$$

$$\dot{a}_5 = -(1 - a_4^2 + a_5^2/3) \quad (B 3 b)$$

In the resonance case we have again to do a limited expansion but the computation is easier since we can do it in terms of the variation of a quantity unchanged in the differentiation process:  $\mu_0$ . Even so the results are rather complicated:

$$\begin{aligned} \dot{a}_2 = & -a_4 \cdot \delta_2^* - \dot{a}_4 \cdot \left( \frac{\gamma_2^{**}}{2} + \gamma_2^* \right) - a_5 \cdot \left[ a_1 \cdot (\delta_1^* + (\delta_1^{**} - \delta_1^{***})) \cdot \frac{\Delta t}{\mu_0} \right. \\ & \left. + \frac{\delta_1^{***}}{2} \cdot \frac{\Delta t^2}{\mu_0^2} \right] + \dot{a}_1 \cdot \left( \gamma_1^* - \frac{\gamma_1^{**}}{2} + \gamma_1^{**} \cdot \frac{\Delta t}{\mu_0} \right) - \dot{a}_5 \cdot a_1 \cdot \left( \gamma_1^* \right. \\ & \left. + \frac{\gamma_1^{**}}{2} + \gamma_1^{**} \cdot \frac{\Delta t}{\mu_0} \right) + a_1 \cdot (\delta_2^* + (\delta_2^{**} - \delta_2^{***})) \cdot \frac{\Delta t}{\mu_0} \\ & + \frac{\delta_2^{***}}{2} \cdot \frac{\Delta t^2}{\mu_0^2} \end{aligned} \quad (B 4 a)$$

$$\begin{aligned} \dot{a}_3 = & -a_5 \cdot \delta_2^* - \dot{a}_5 \cdot \left( \frac{\gamma_2^{**}}{2} + \gamma_2^* \right) - a_4 \cdot \left[ a_1 \cdot (\delta_1^* + (\delta_1^{**} - \delta_1^{***})) \cdot \frac{\Delta t}{\mu_0} \right. \\ & \left. + \frac{\delta_1^{***}}{2} \cdot \frac{\Delta t^2}{\mu_0^2} \right] + \dot{a}_1 \cdot \left( \gamma_1^* - \frac{\gamma_1^{**}}{2} + \gamma_1^{**} \cdot \frac{\Delta t}{\mu_0} \right) - \dot{a}_4 \cdot a_1 \cdot \left( \gamma_1^* \right. \\ & \left. + \frac{\gamma_1^{**}}{2} + \gamma_1^{**} \cdot \frac{\Delta t}{\mu_0} \right) + \delta_1^* \end{aligned} \quad (B 4 b)$$

with

$$\delta_1^* = -((4 \cdot \gamma_1^* + 2 \cdot \gamma_1^{**}) \cdot \alpha_1 \cdot \mu_0^2 - \alpha_3 \cdot (6 \cdot \mu_0 - 1)/4) / \Delta t \quad (B 5 a)$$

$$\delta_2^* = -((4 \cdot \gamma_2^* + 2 \cdot \gamma_2^{**}) \cdot \alpha_1 \cdot \mu_0^2 - \alpha_4 \cdot (6 \cdot \mu_0 + 1)/4) / \Delta t \quad (B 5 b)$$

$$\delta_1^{**} = -((2 \cdot \gamma_1^* + 4 \cdot \gamma_1^{**}) \cdot \alpha_1 \cdot \mu_0^2 - \alpha_3 \cdot (2 \cdot \mu_0 - 1)/2 - \gamma_1^*) / \Delta t \quad (B 5 c)$$

$$\delta_2^{**} = -((2 \cdot \gamma_2^* + 4 \cdot \gamma_2^{**}) \cdot \alpha_1 \cdot \mu_0^2 - \alpha_4 \cdot (2 \cdot \mu_0 + 1)/2 - \gamma_2^*) / \Delta t \quad (B 5 d)$$

$$\delta_1^{***} = -\gamma_1^{**} \cdot (2 \cdot \alpha_1 \cdot \mu_0^2 - 1) / \Delta t \quad (B 5 e)$$

$$\delta_2^{***} = -\gamma_2^{**} \cdot (2 \cdot \alpha_1 \cdot \mu_0^2 - 1) / \Delta t \quad (B 5 f)$$

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