Recitation 4

B Trees

Question

What advantage does balanced BST provide over storing data in an unsorted array?
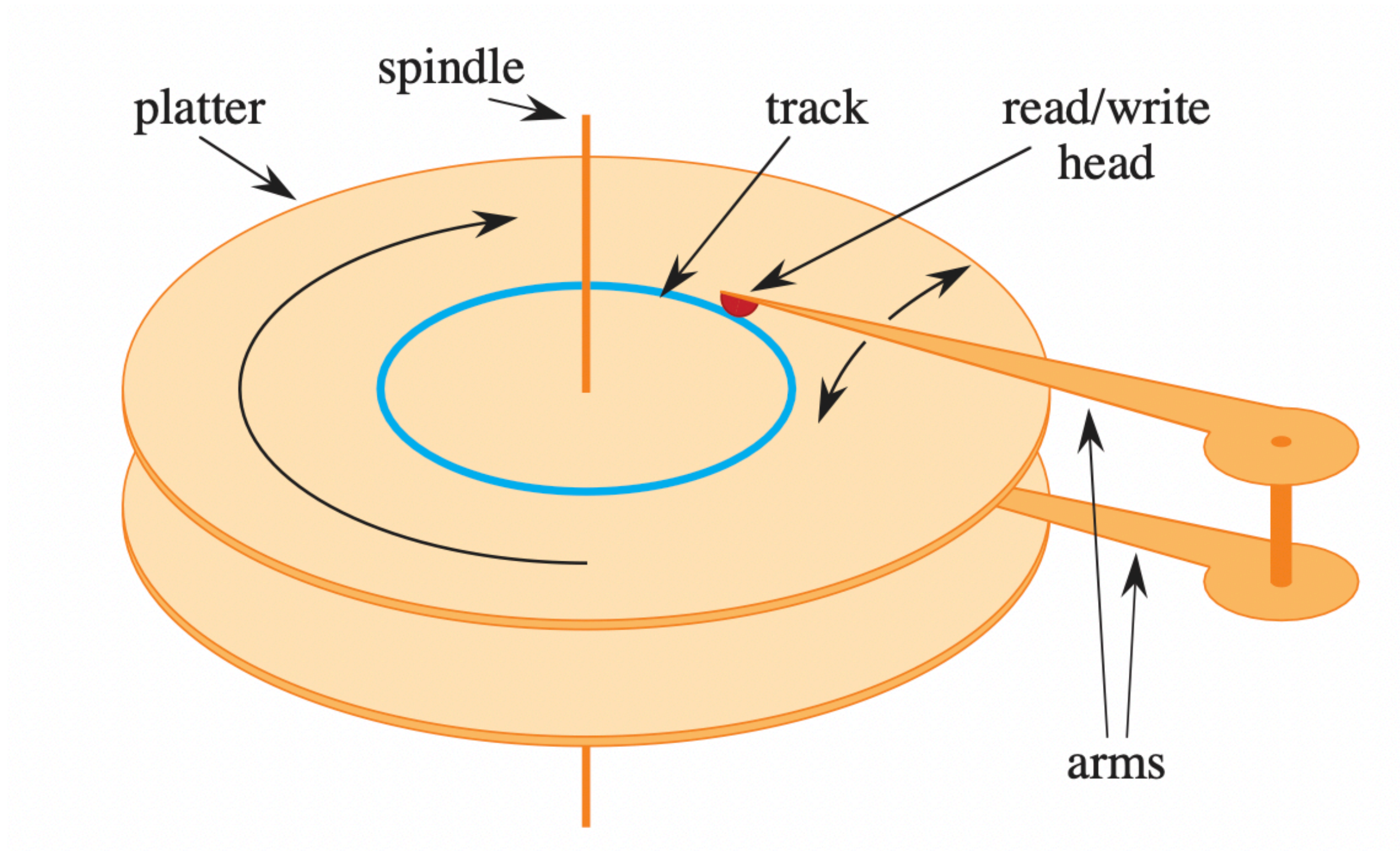
Question

What advantage does balanced BST provide over storing data in an unsorted array?

Access time.

As you move forward with the course you will notice computer scientists really care a lot about access time. In your lectures after trees you are going to learn about hash tables which has O(1) access time on average.

Access time in Memory
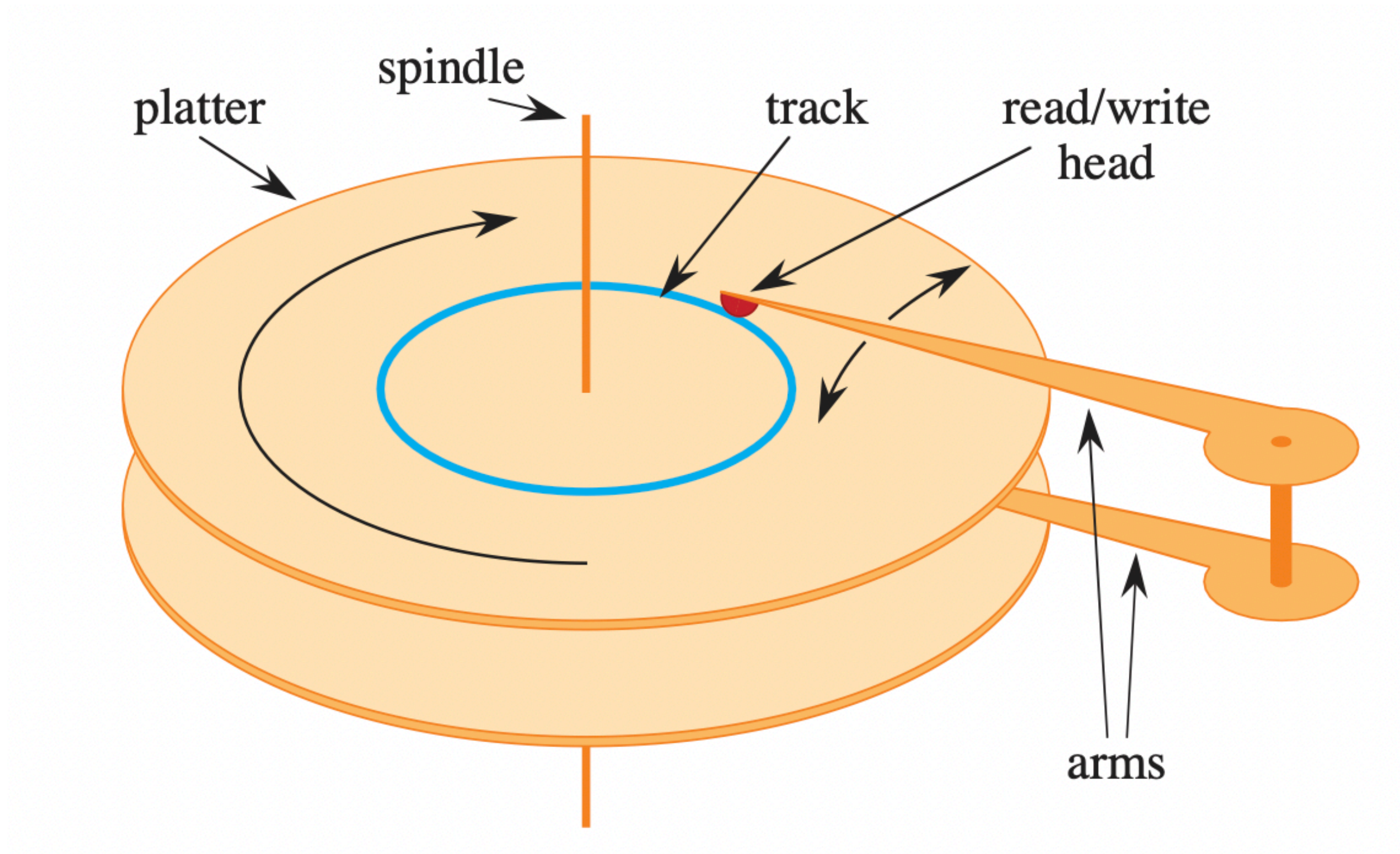

Typical Image of a Hard Disk Drive (HDD)

Data are stored in the track.

To access a data from the HDD, the platter rotates towards the read/write head. So, that the read/write head, (which can only move towards and backwards from the spindle) can read the data stored in the track when it comes under the read write head.
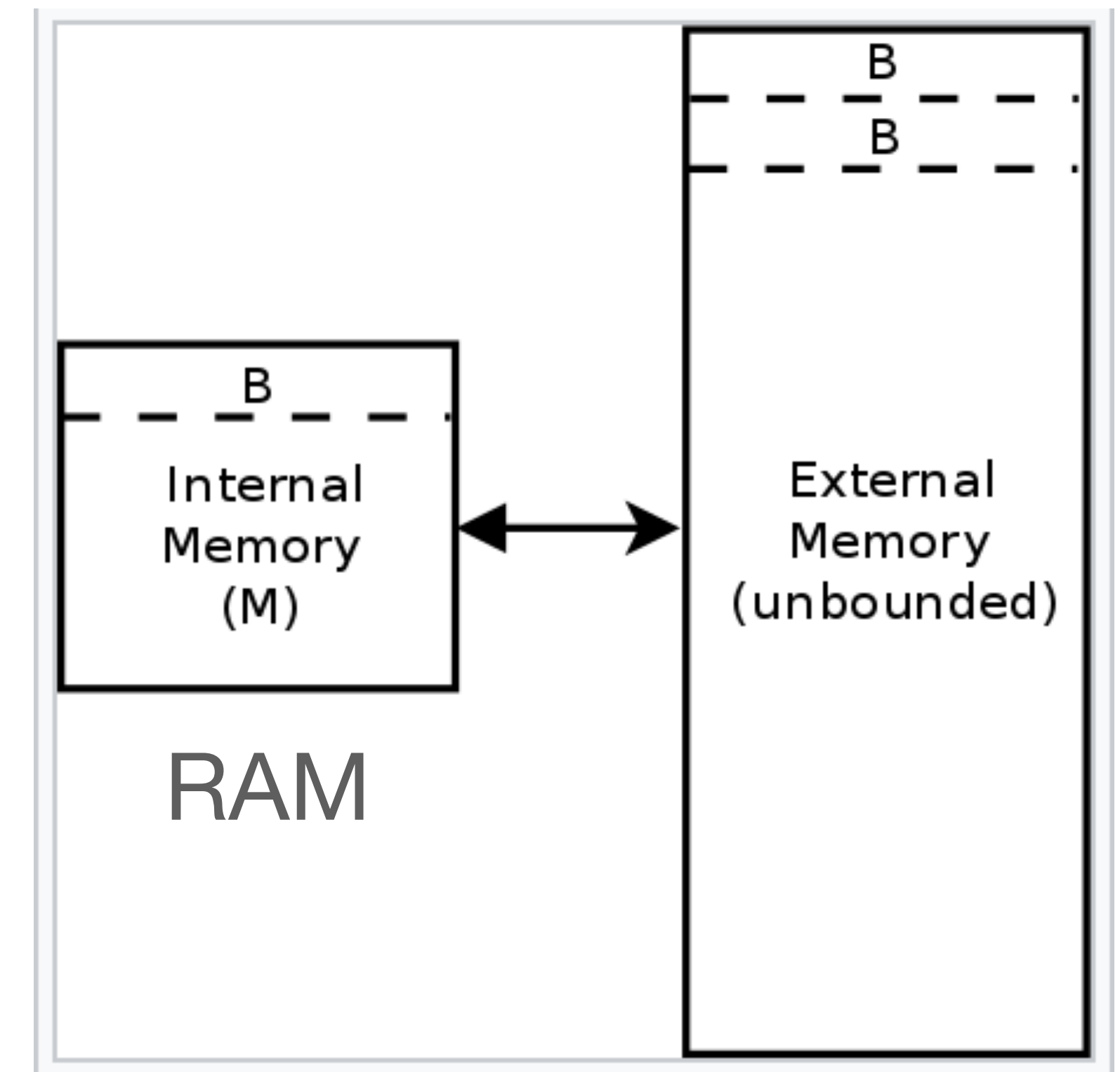
This takes 4ms time on average to access data

To make up for this latency the read/write head reads a lot of data at once (i.e. it reads the data in blocks)

Access time in Memory

Typical Image of a Hard Disk Drive
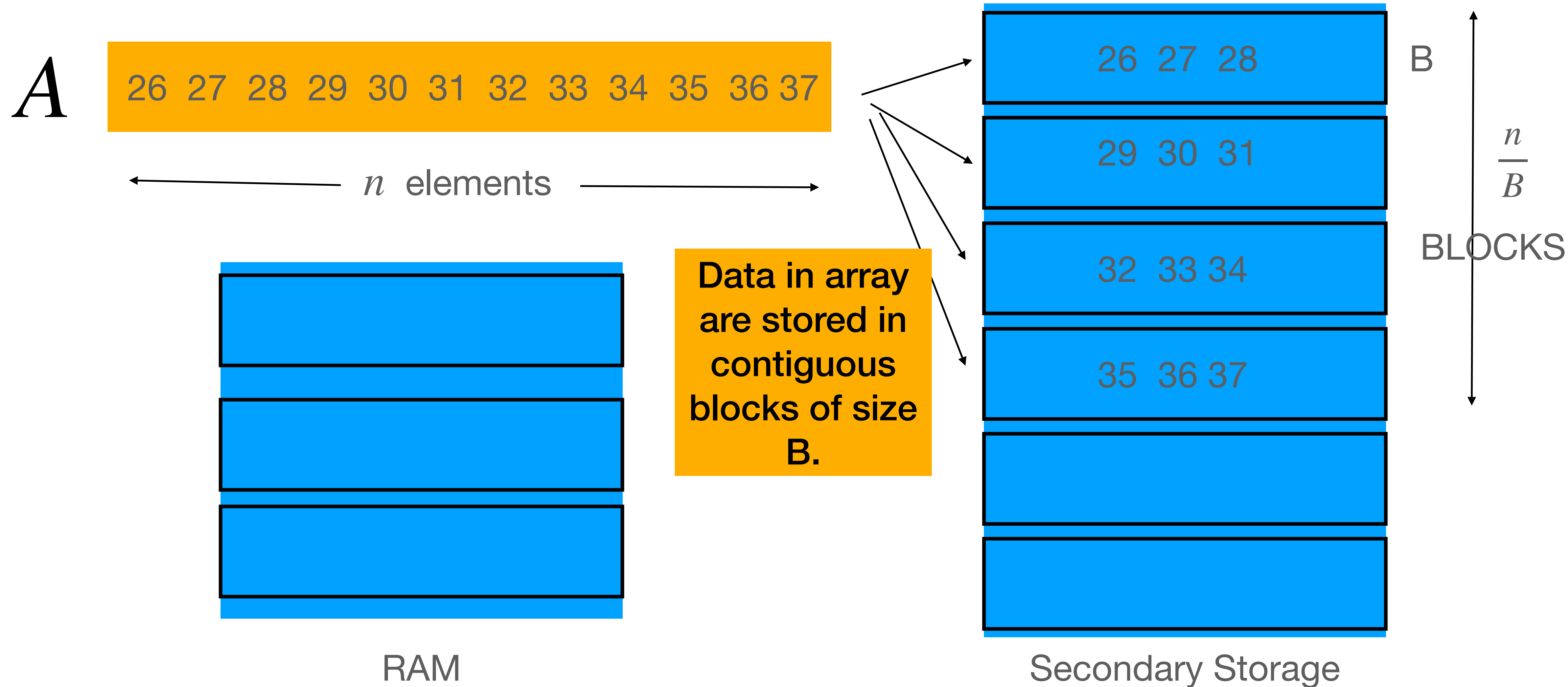

RAM

Secondary storage

Memory Hierarchy Structure

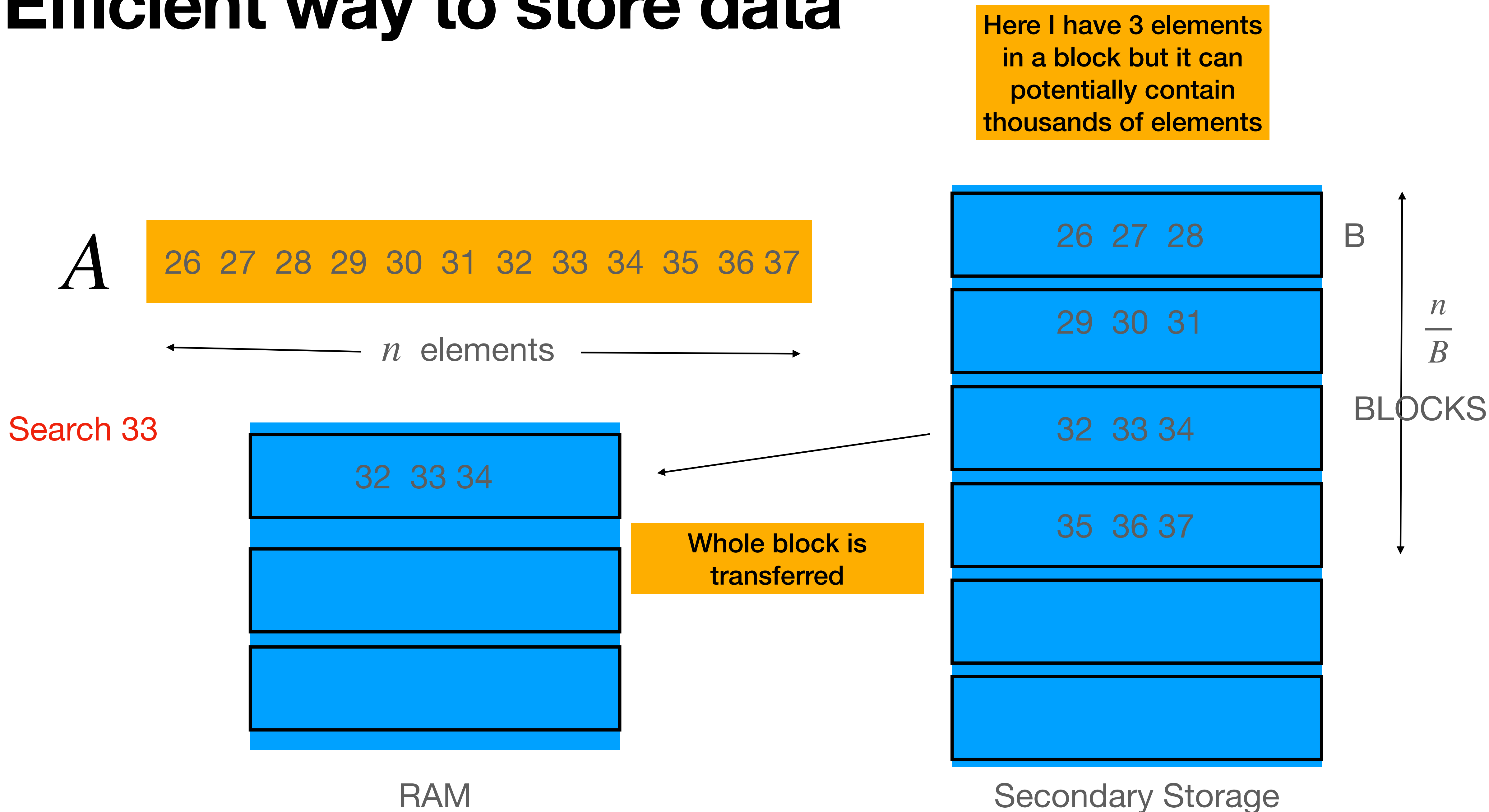| Memory unit	Size	Block size	Access time (clock cycles)
L1 cache	64KB	64B	4
L2 cache	256KB	64B	10
L3 cache	up to 40MB	64B	40–74
Main memory	128GB	16KB	200-350
(Magnetic) Disk	Arbitrarily big	16KB	20,000,000 (An SSD is only 20,000)

Efficient way to store data

In practice, searching a data in an array is much faster than searching a data in a linked list, only because of efficient way of storing data. (In theory both takes O(n) time.)

A 26 27 28 29 30 31 32 33 34 35 36 37
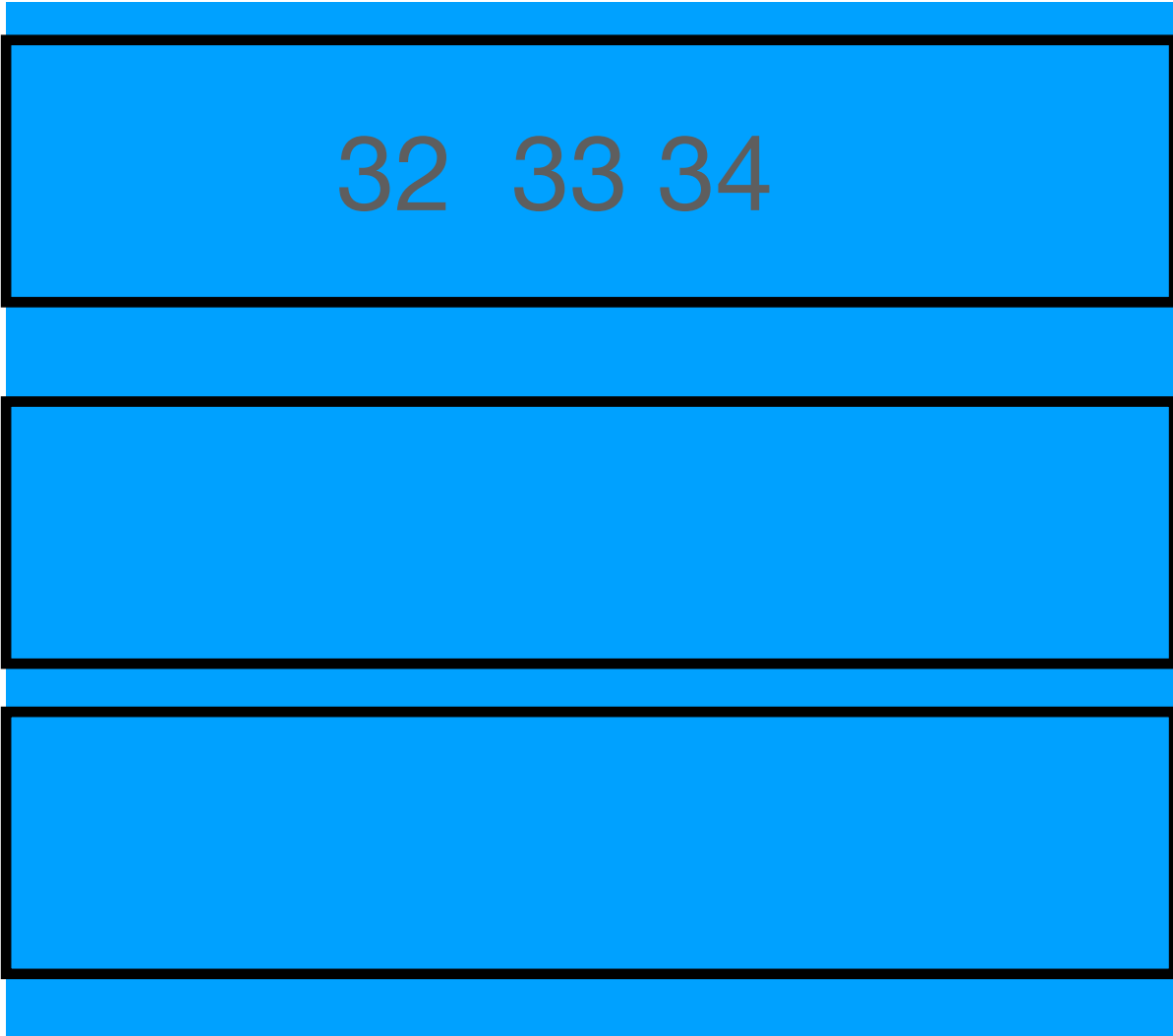
$\longleftarrow$ n elements $\longrightarrow$

Data in array are stored in contiguous blocks of size B.

26 27 28   B

29 30 31

32 33 34

35 36 37

$\dfrac{n}{B}$

BLOCKS

RAM

Secondary Storage

Efficient way to store data

A 26 27 28 29 30 31 32 33 34 35 36 37

$\longleftarrow$ n elements $\longrightarrow$

Search 33

32 33 34

Whole block is transferred

26 27 28   B

29 30 31 $\dfrac{n}{B}$

32 33 34   BLOCKS

35 36 37

RAM

Secondary Storage

Efficient way to store data

A | 26 27 28 29 30 31 32 33 34 35 36 37

$\longleftarrow \quad n \text{ elements} \quad \longrightarrow$
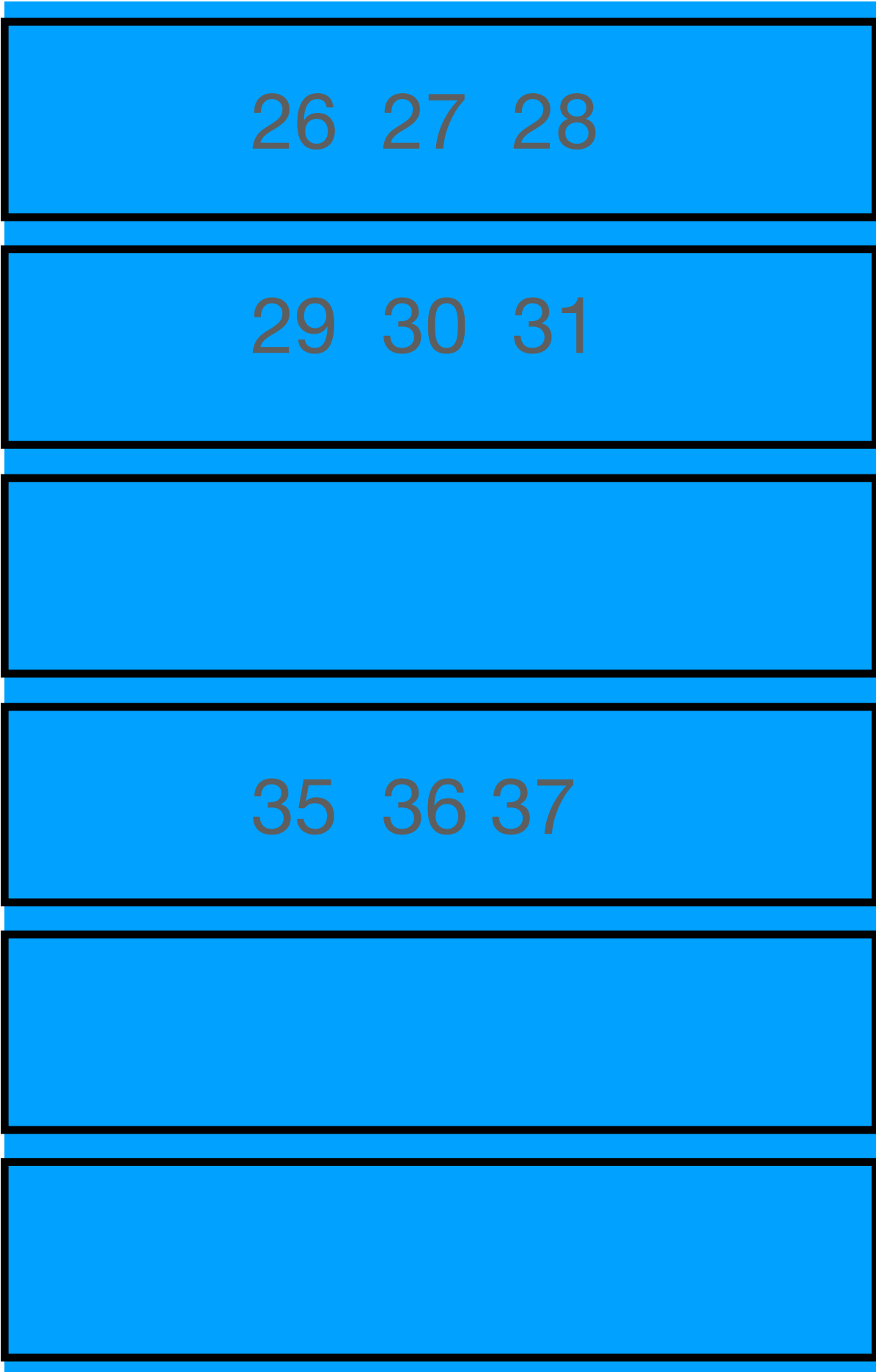
Now, if we call search 34 now. Since 34 is in the RAM now, it takes so less time to search 34 in the RAM that we don't care about the running time to search 34.

| 32 33 34 |

RAM

| 26 27 28 |
| 29 30 31 |
| 32 33 34 |
| 35 36 37 |

B

$\dfrac{n}{B}$

BLOCKS

Secondary Storage

Efficient way to store data

A 26 27 28 29 30 31 32 33 34 35 36 37

$\longleftarrow$ n elements $\longrightarrow$
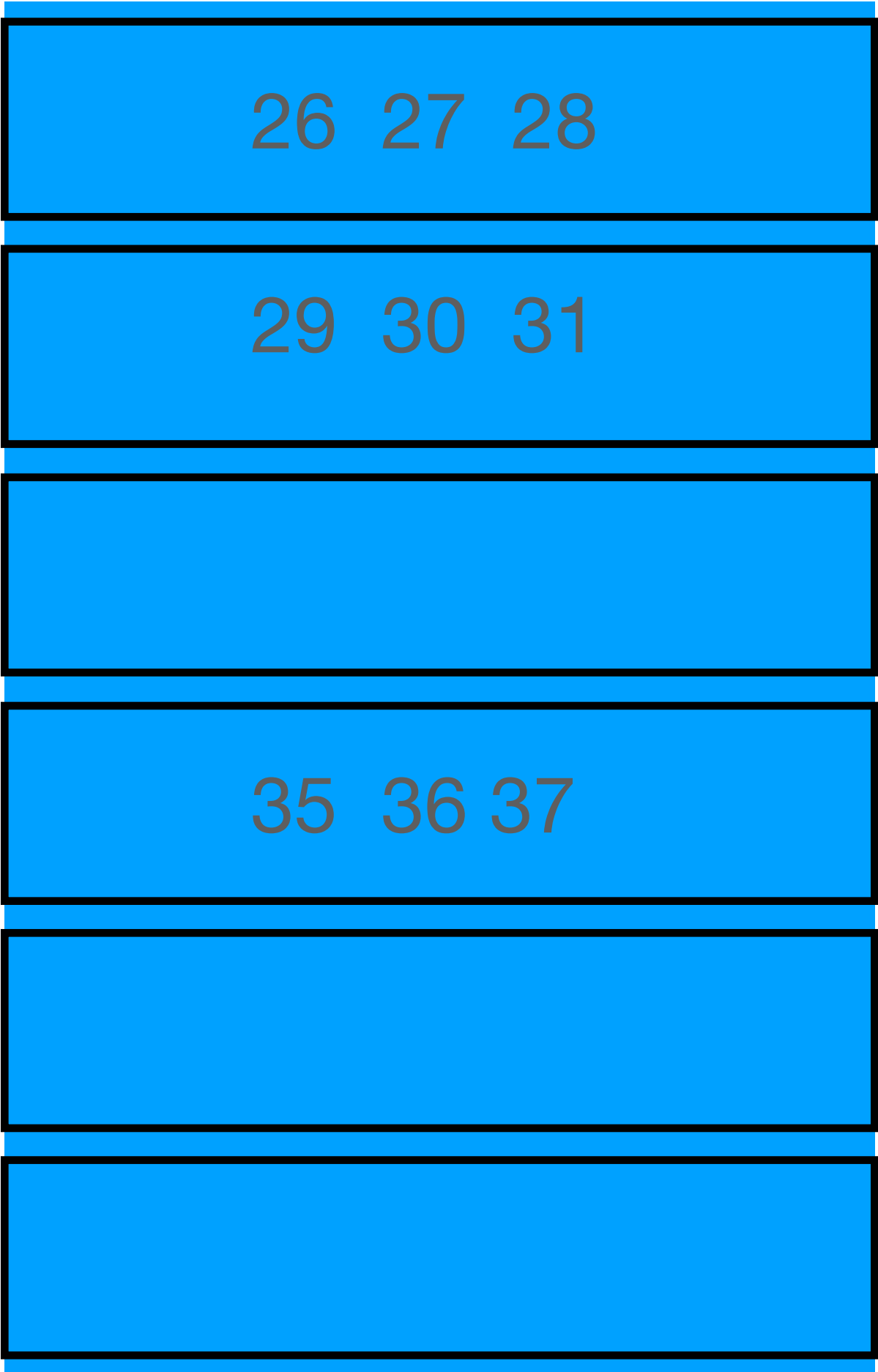
So, what we care about is "How many block transfers from Secondary storage need to be made for accessing a particular element?"
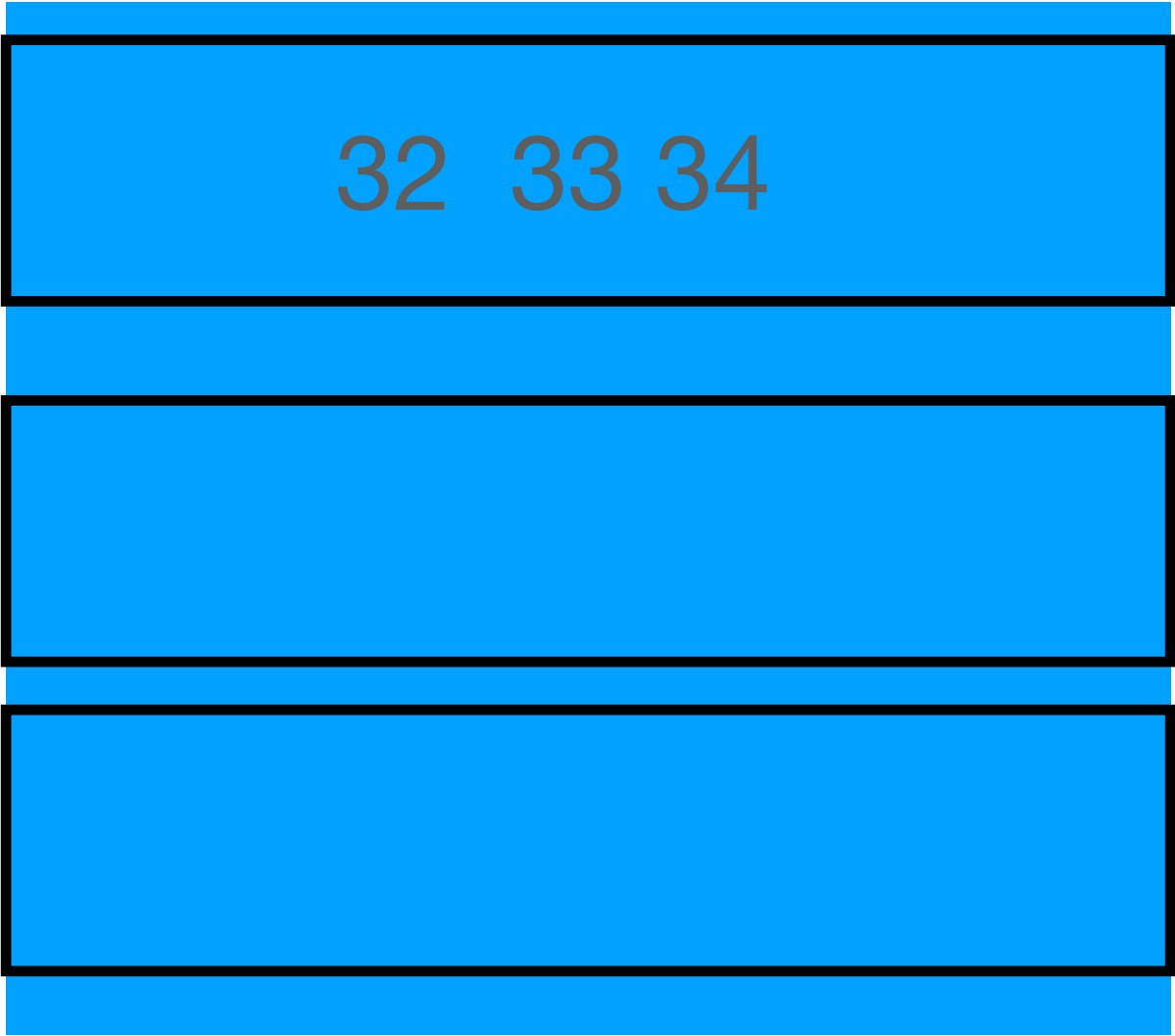
26 27 28

29 30 31

B

$\dfrac{n}{B}$

BLOCKS

32 33 34

35 36 37

RAM

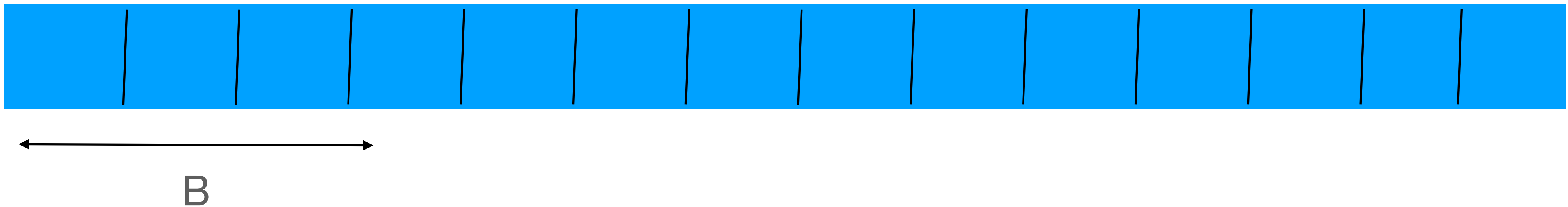Secondary Storage

Efficient way to store data (Problem 2)

(a) Assume your data is stored on disk. Your data is a sorted array of size n, and spans many blocks. What is the number of block transfers needed (i.e. cost) of doing a linear search for an item? Leave your answer in terms of n and B.

(b) Assume your data is stored on disk. Your data is a sorted array of size n, and spans many blocks. What is the number of block transfers needed (i.e. cost) of doing a binary search for an item? Leave your answer in terms of n and B.

Efficient way to store data (Problem 2)

(a) Assume your data is stored on disk. Your data is a sorted array of size n, and spans many blocks. What is the number of block transfers needed (i.e. cost) of doing a linear search for an item? Leave your answer in terms of n and B.

(b) Assume your data is stored on disk. Your data is a sorted array of size n, and spans many blocks. What is the number of block transfers needed (i.e. cost) of doing a binary search for an item? Leave your answer in terms of n and B.

B

Since after transferring one block of data from secondary storage to RAM, we don't care about the searching time in that block so, we can consider the block of B elements as equivalent to 1 element in the array.

Efficient way to store data (Problem 2)

(a) Assume your data is stored on disk. Your data is a sorted array of size n, and spans many blocks. What is the number of block transfers needed (i.e. cost) of doing a linear search for an item? Leave your answer in terms of n and B.

$O(n/B)$. (Answer)

(b) Assume your data is stored on disk. Your data is a sorted array of size n, and spans many blocks. What is the number of block transfers needed (i.e. cost) of doing a binary search for an item? Leave your answer in terms of n and B.

$O(log(n/B))$. (Answer)
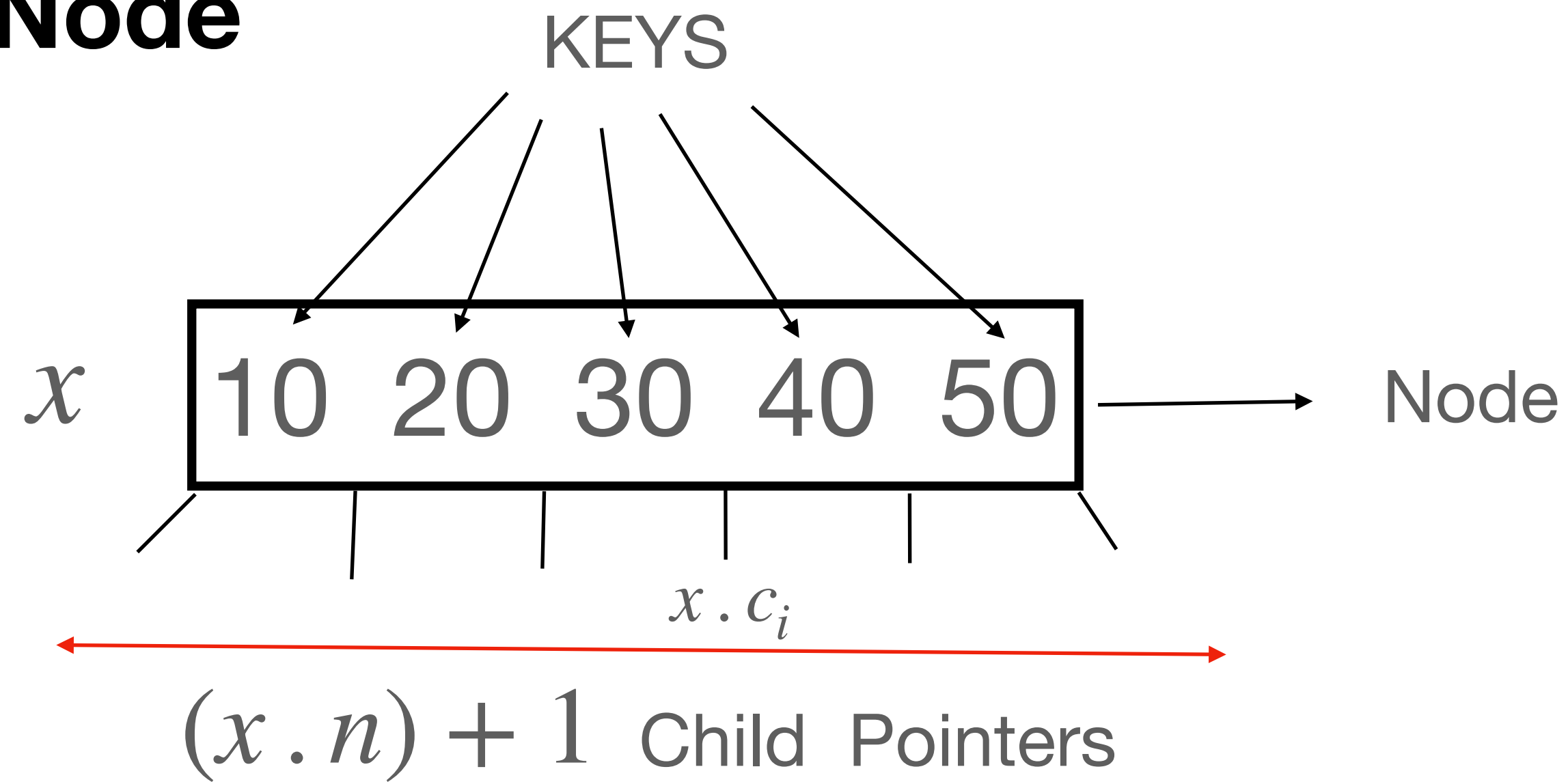
Efficient way to store data
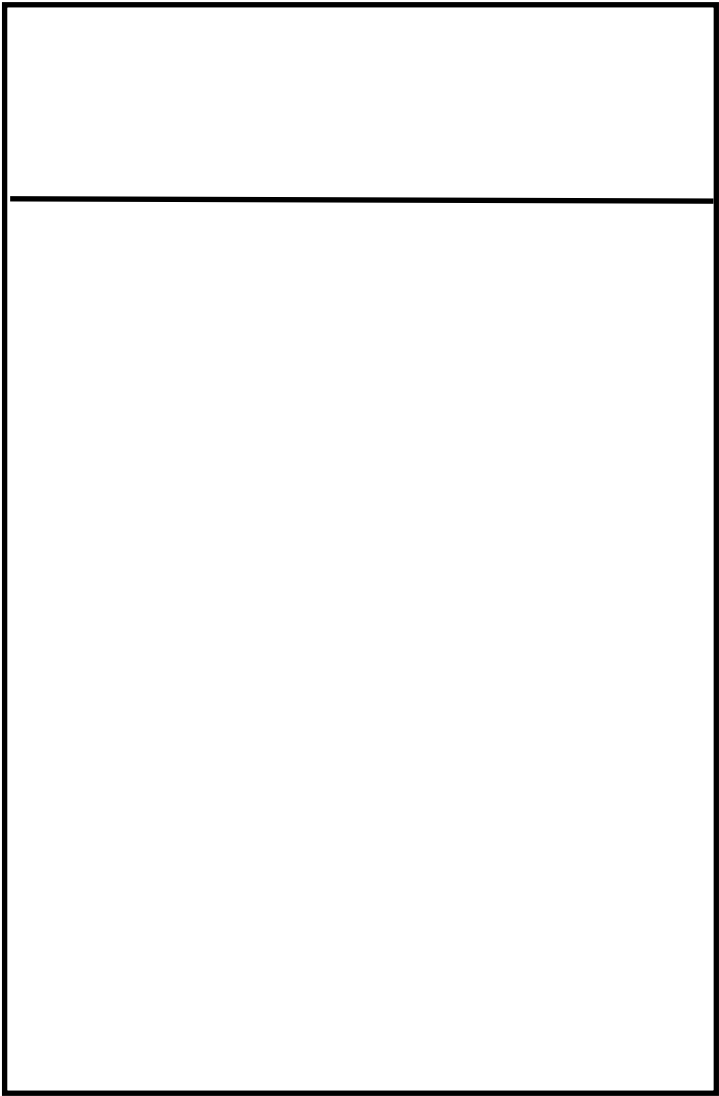
B

Memory

10 20 30 40 50

Create a node which has a
lot of keys (data) in it.

(a, b) **Trees**
Attributes of a Node

KEYS

$$x \quad \boxed{10 \quad 20 \quad 30 \quad 40 \quad 50} \longrightarrow \text{Node}$$

$x . c_i$

$(x . n) + 1$ Child Pointers

$x . n$	Number of keys in the node x
$x . leaf$	Returns a boolean value TRUE if x is a leaf node, FALSE otherwise
$x . key_i$	Returns the value of the ith key
$x . c_i$	Pointer to the ith child node

(a, b) **Trees**
Rules

Rule 1: (a, b) child Policy

$(a - 1) \leq x.n \leq (b - 1)$ where $2 \leq a \leq (b + 1)/2$

Root node has a special treatment

$1 \leq root.n \leq (b - 1)$

All nodes except the leaf nodes has $((x.n) + 1))$ child nodes.

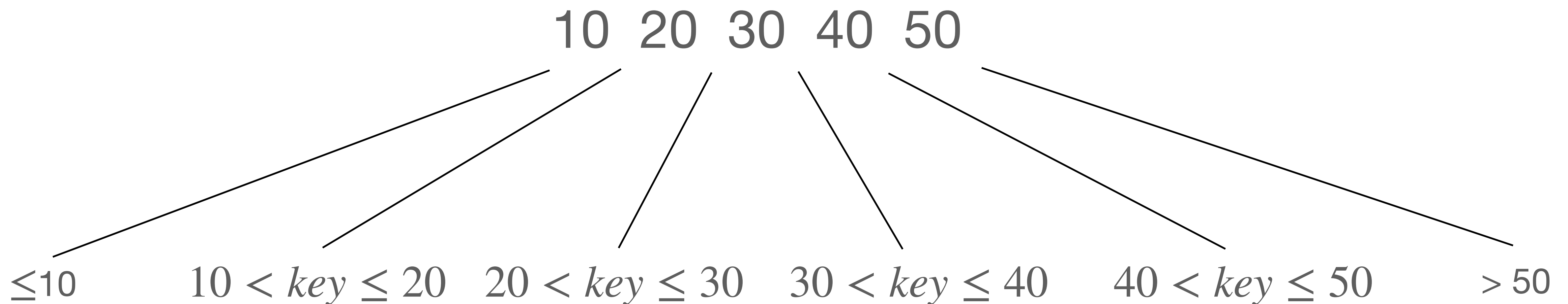# (a, b) **Trees**
Rules

Rule 2: Key ranges

$$x . key_{i-1} < (x . c_i) . key \leq x . key_i$$

All keys are stored in sorted order in a node.

10 20 30 40 50

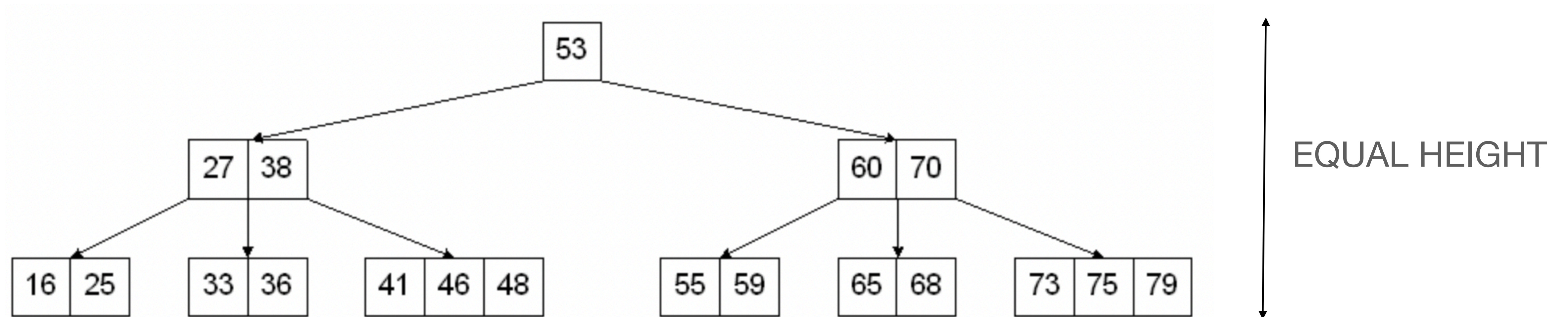≤ 10　　$10 < key \leq 20$　$20 < key \leq 30$　$30 < key \leq 40$　$40 < key \leq 50$　　> 50

(a, b) **Trees**
Rules

Rule 3: Leaf nodes are at the same height

Balancing Rules (Problem 1(a))

Is (a, b) tree balanced ? If it is, which rules ensures that? If not why?

Balancing Rules (Problem 1(a))

Is (a, b) tree balanced ? If it is, which rules ensures that? If not why?

Both Rule 1 "(a, b) child Policy", and Rule 3 " All leaves are at the same height " together makes it balanced.

Insertion

Insert the keys 6, 19, 17, 11 in a (2, 4) tree.

Insertion

Insert the keys 6, 19, 17, 11 in a (2, 4) tree.

6

Insertion

Insert the keys 6, 19, 17, 11 in a (2, 4) tree.

| 6 |

| 6 | 19 |

Insertion

Insert the keys 6, 19, 17, 11 in a (2, 4) tree.
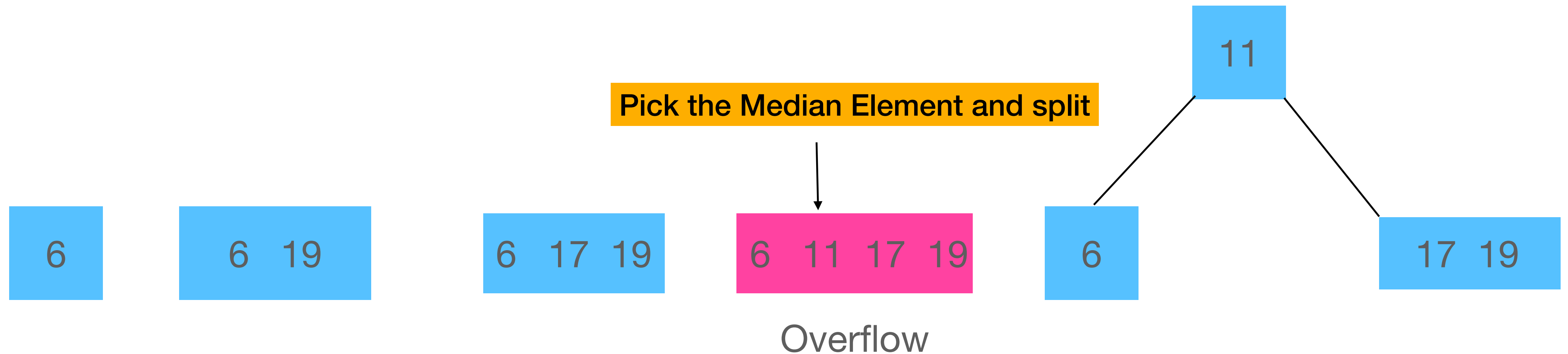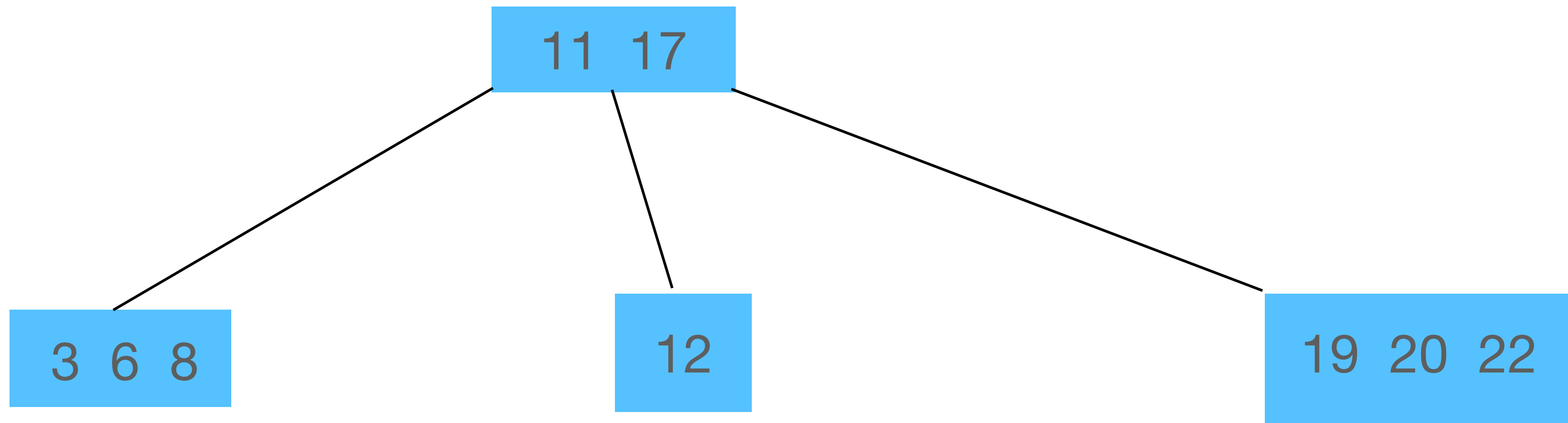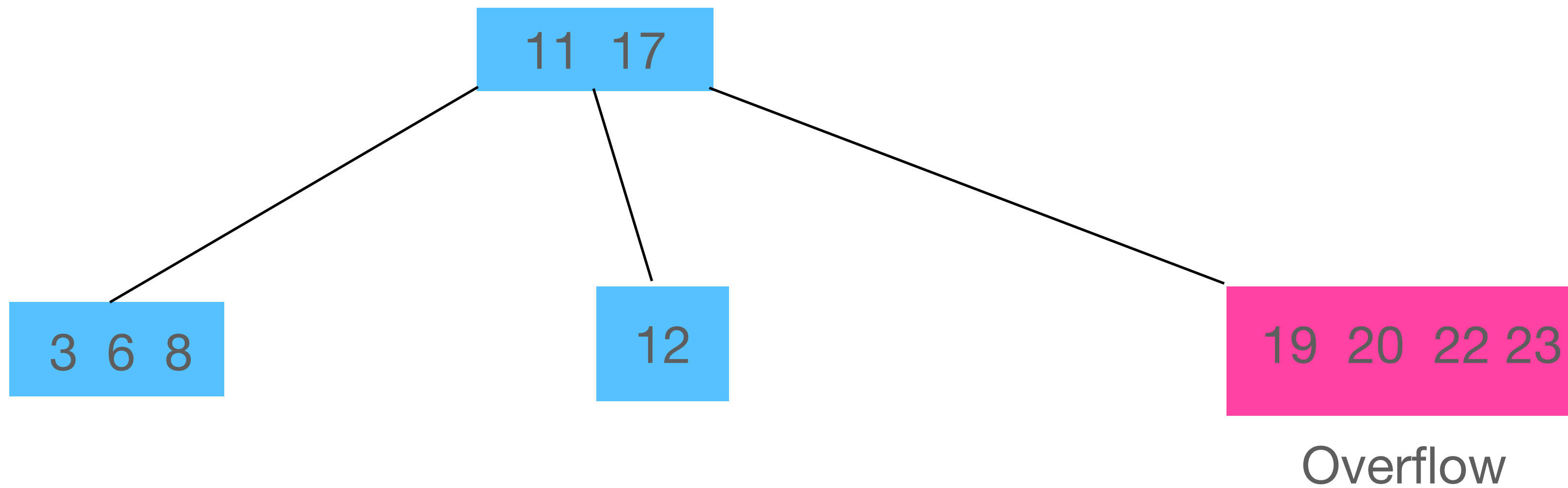
| 6 |

| 6 | 19 |

| 6 | 17 | 19 |

Insertion

Insert the keys 6, 19, 17, 11 in a (2, 4) tree.

| 6 |

| 6 | 19 |

| 6 | 17 | 19 |

| 6 | 11 | 17 | 19 |

Overflow

Insertion

Insert the keys 6, 19, 17, 11 in a (2, 4) tree.

Insertion

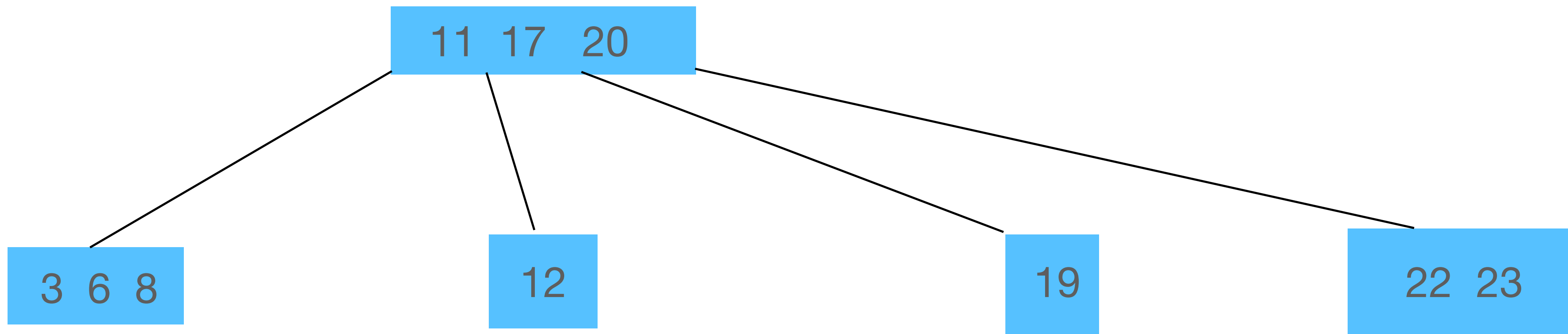After inserting the keys 3, 12, 8, 20, 22 in the previous tree.


```
                    ┌─────────┐
                    │  11  17 │
                    └─────────┘
           ┌────────────┼────────────────┐
    ┌──────────┐   ┌──────┐         ┌──────────────┐
    │  3  6  8 │   │  12  │         │  19  20  22  │
    └──────────┘   └──────┘         └──────────────┘
```

Now, insert 23

Insertion

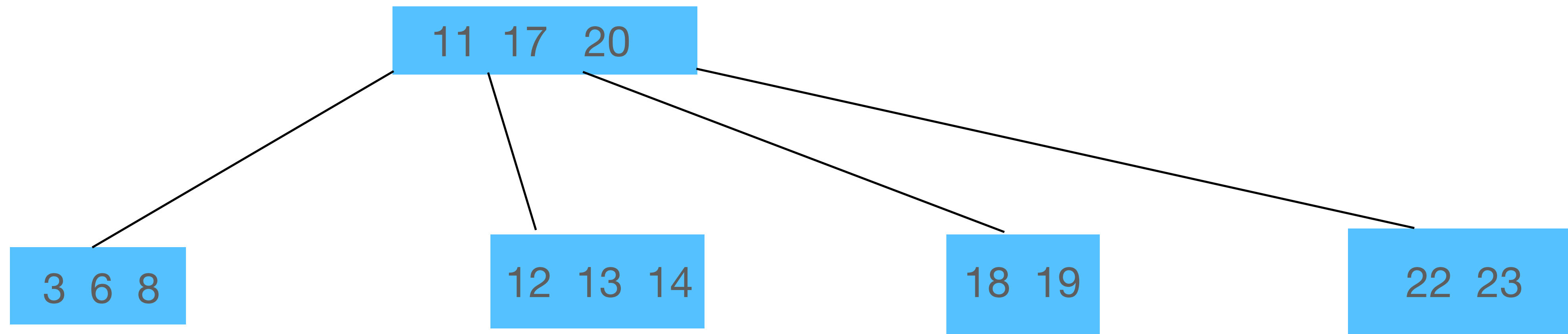After inserting the keys 3, 12, 8, 20, 22 in the previous tree.

Insertion

After inserting the keys 3, 12, 8, 20, 22 in the previous tree.
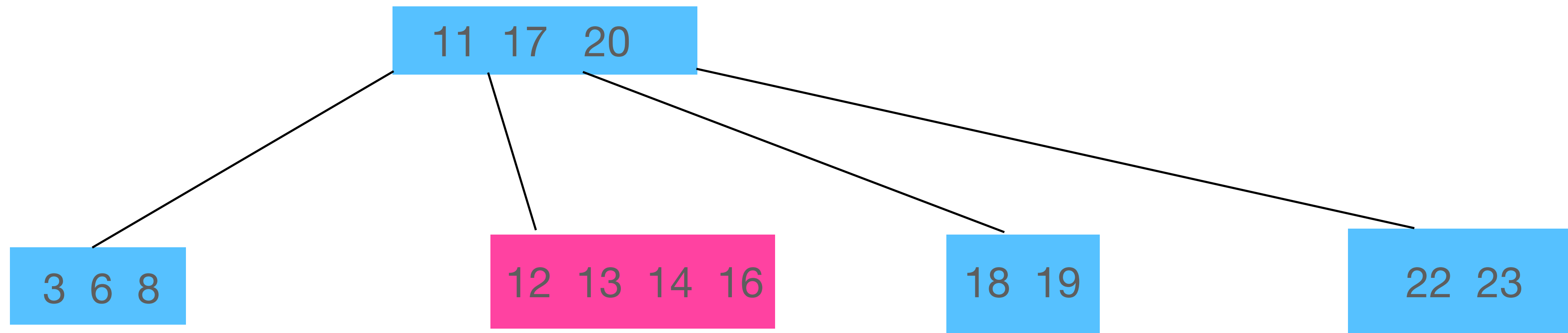

After inserting 23

Insertion

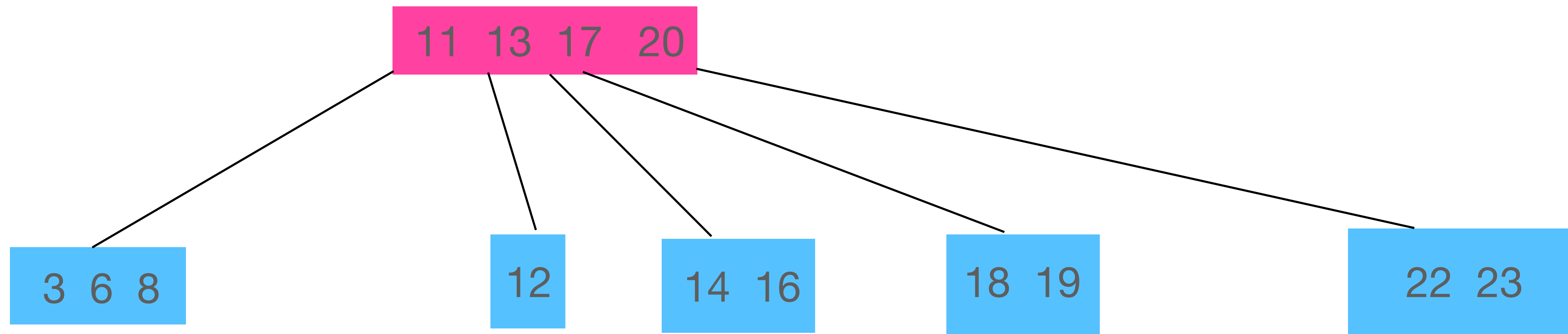After inserting the keys 13, 14, 18 in the previous tree.


```
              ┌─────────────┐
              │ 11  17   20 │
              └─────────────┘
          ╱       │      ╲         ╲
  ┌───────┐  ┌───────────┐  ┌─────────┐  ┌─────────┐
  │ 3 6 8 │  │ 12  13  14│  │ 18  19  │  │ 22  23  │
  └───────┘  └───────────┘  └─────────┘  └─────────┘
```

Now insert 16

Insertion

After inserting the keys 13, 14, 18 in the previous tree.


```
                    ┌──────────────┐
                    │ 11  17   20  │
                    └──────────────┘
        ┌──────────────┼──────────┬──────────────────┐
┌───────────┐   ┌──────────────────┐  ┌─────────┐  ┌──────────┐
│  3  6  8  │   │ 12  13  14  16   │  │ 18  19  │  │  22  23  │
└───────────┘   └──────────────────┘  └─────────┘  └──────────┘
```

Now insert 16

Insertion

After inserting the keys 13, 14, 18 in the previous tree.

11 13 17 20

3 6 8   12   14 16   18 19   22 23

Now insert 16

Insertion

After inserting the keys 13, 14, 18 in the previous tree.
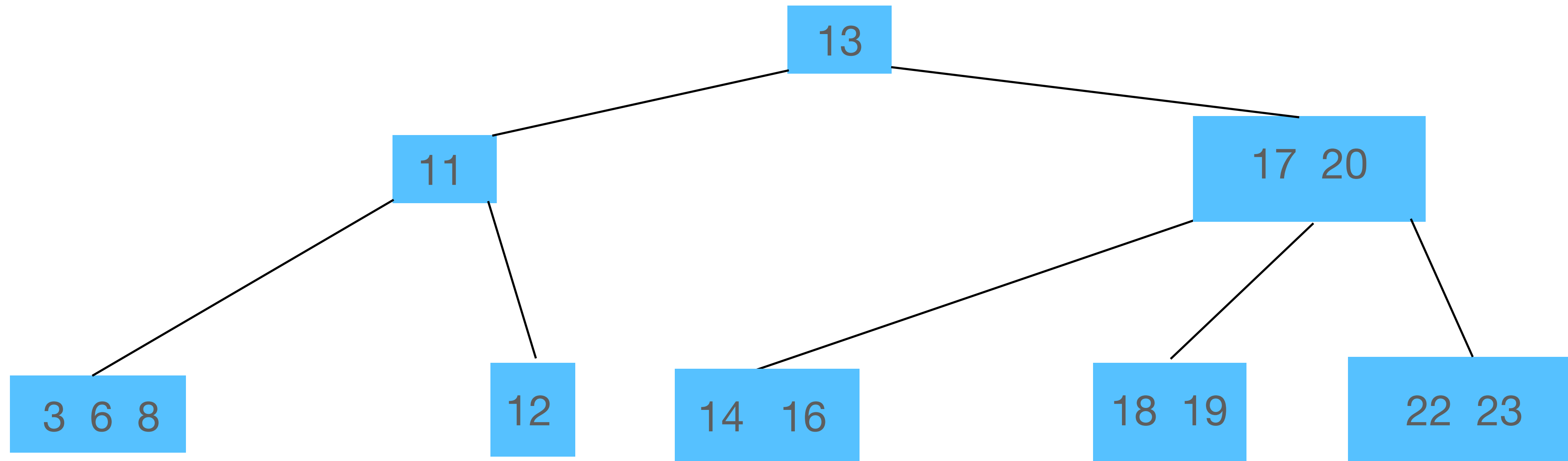
After inserting 16

Strategy for Insertion

(1) At first put the element in its correct place in the node (which is in sorted order)

(2) If the node overflows, then find the median ($= \dfrac{b}{2}$ if b is even and $\dfrac{b+1}{2}$ if b is odd) key in that node.

(3) Put the median key in the parent and split the current node.

Searching

Search 18
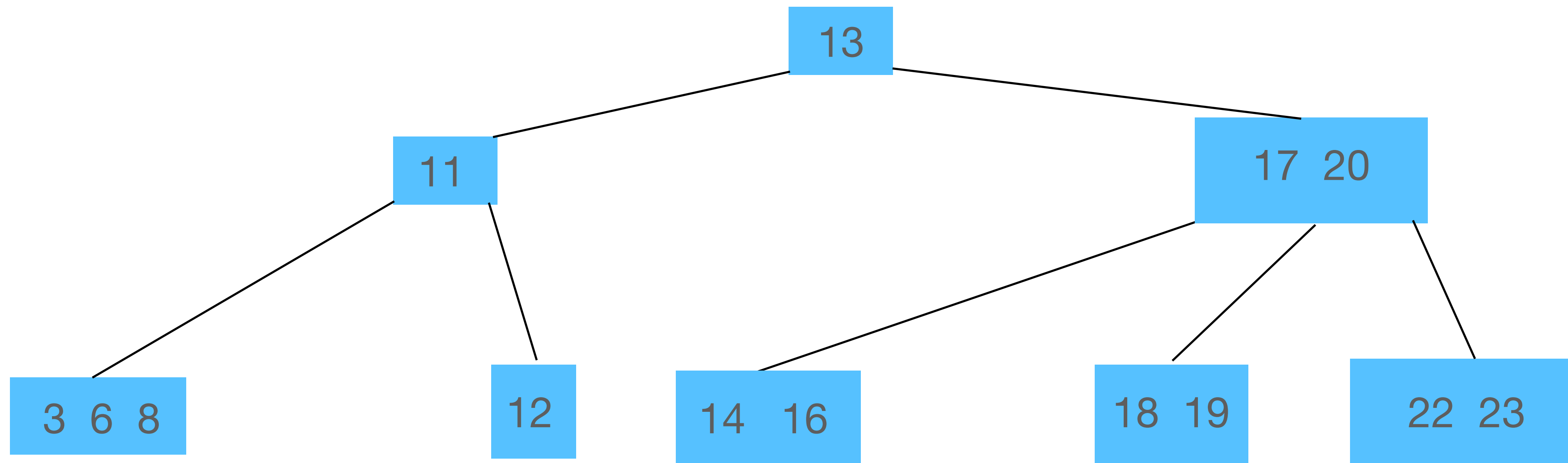
Searching

Search 18

Search like we do in BST.

Only difference is we have to do either a linear search or binary search in each node to find the correct child pointer to traverse.
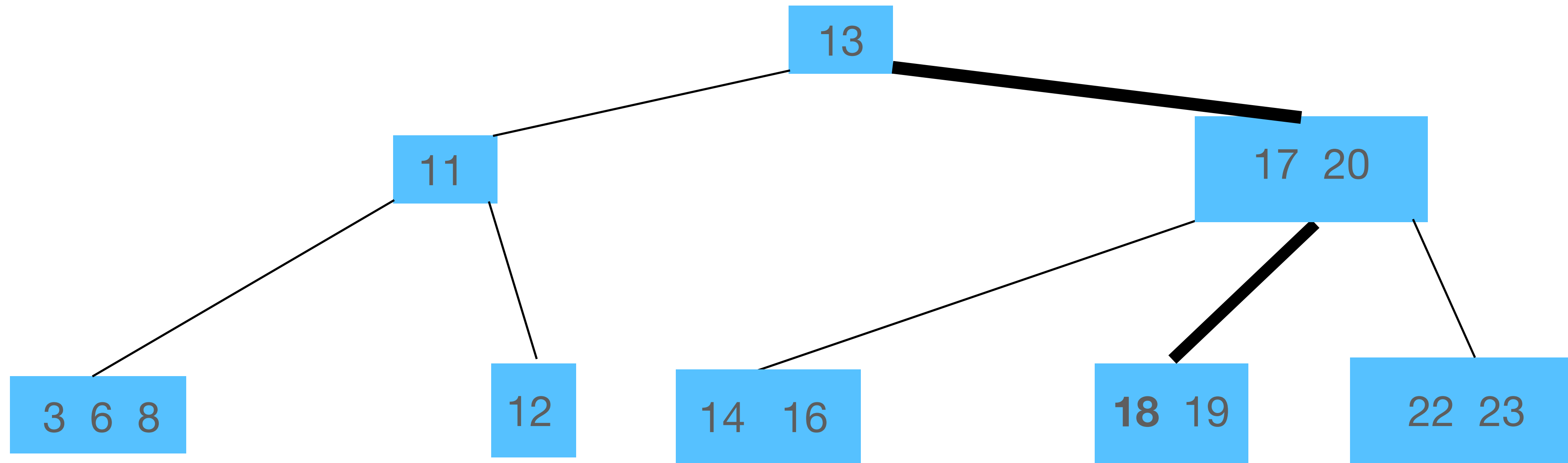
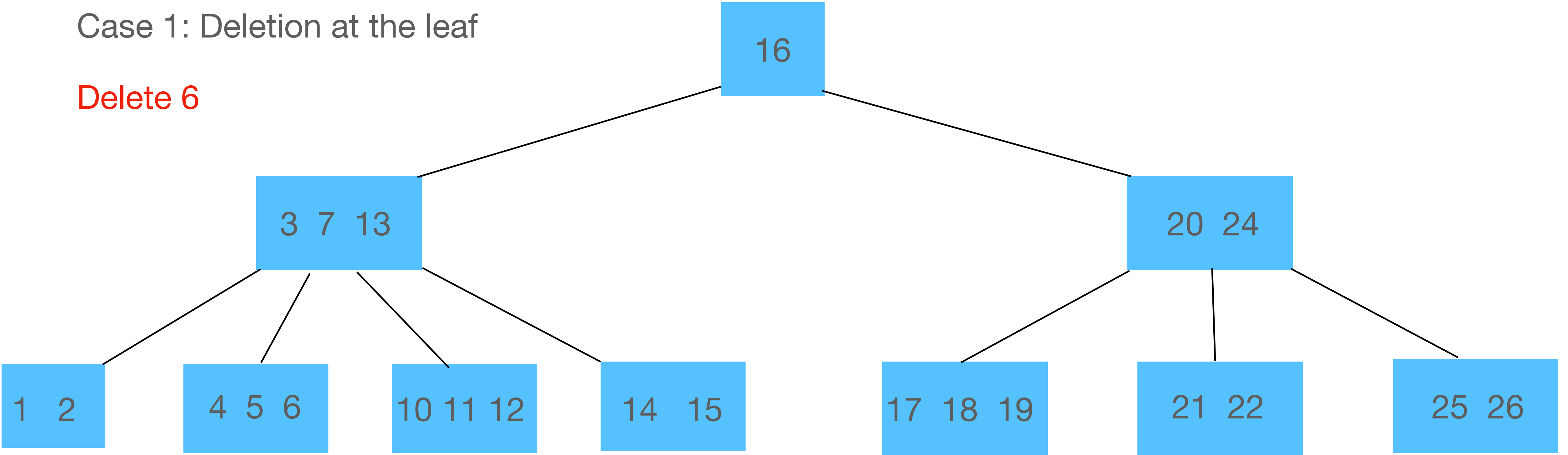# Pseudocode of Searching (Problem 1(c))

B-TREE-SEARCH(x, k)

1   $i = 1$
2   **while** $i \leq x.n$ and $k > x.key_i$
3       $i = i + 1$
4   **if** $i \leq x.n$ and $k == x.key_i$
5   **return** (x, i)
6   **elseif** $x.leaf$
7   **return** NIL
8   **else** DISK-READ$(x.c_i)$
9   **return** B-TREE-SEARCH$(x.c_i, k)$

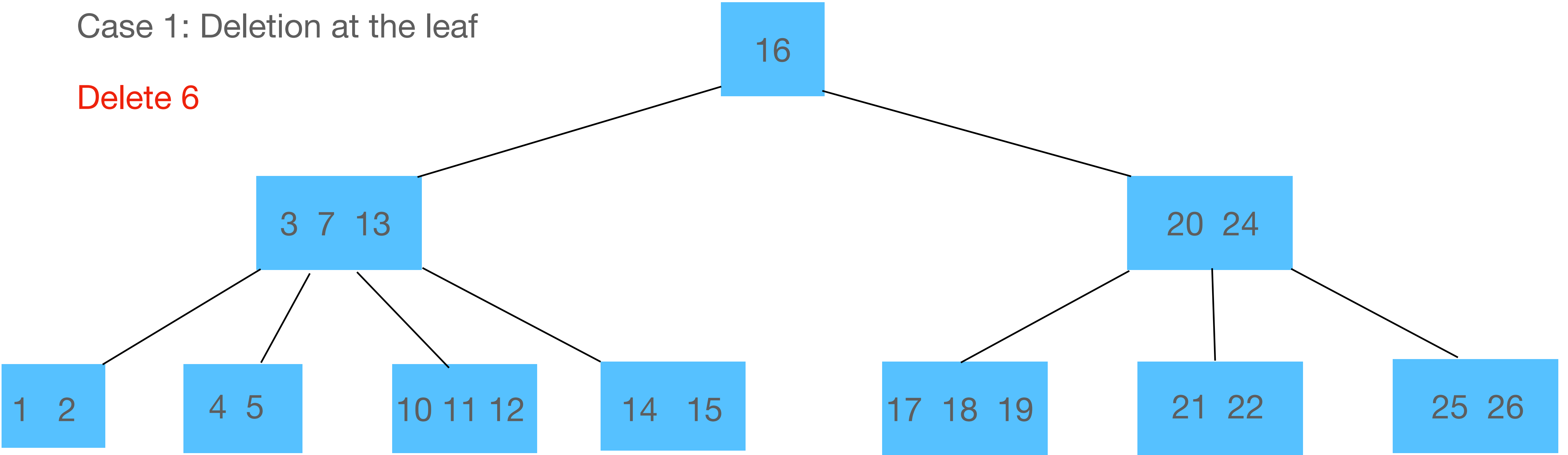# Deletion in (3,6) tree

Case 1: Deletion at the leaf

Delete 6

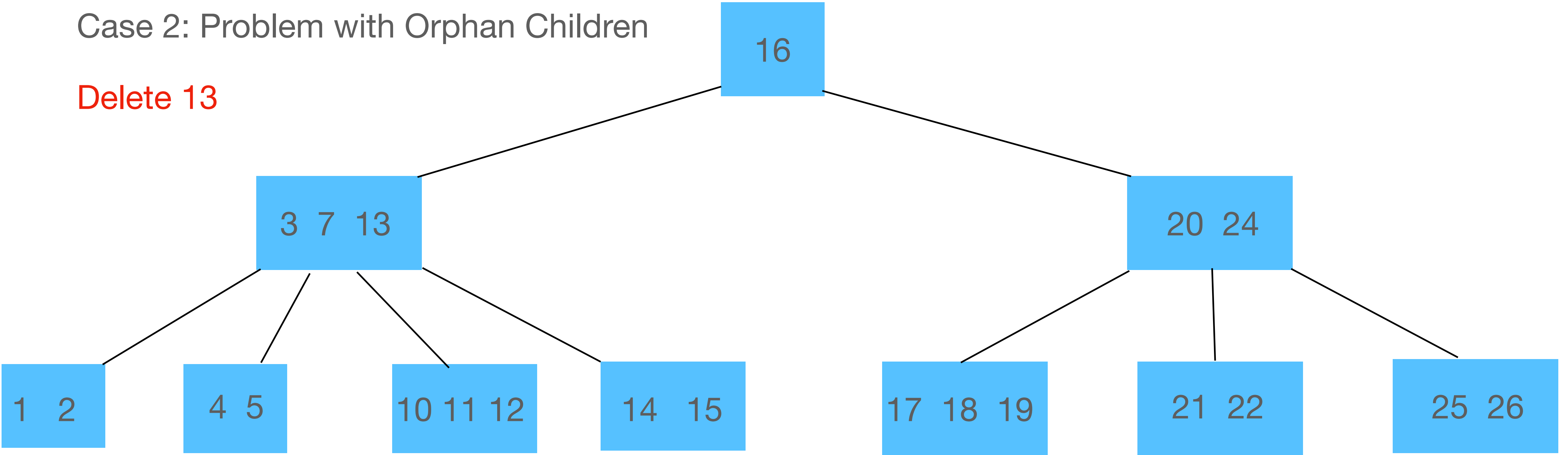# Deletion in (3,6) tree

Case 1: Deletion at the leaf

Delete 6

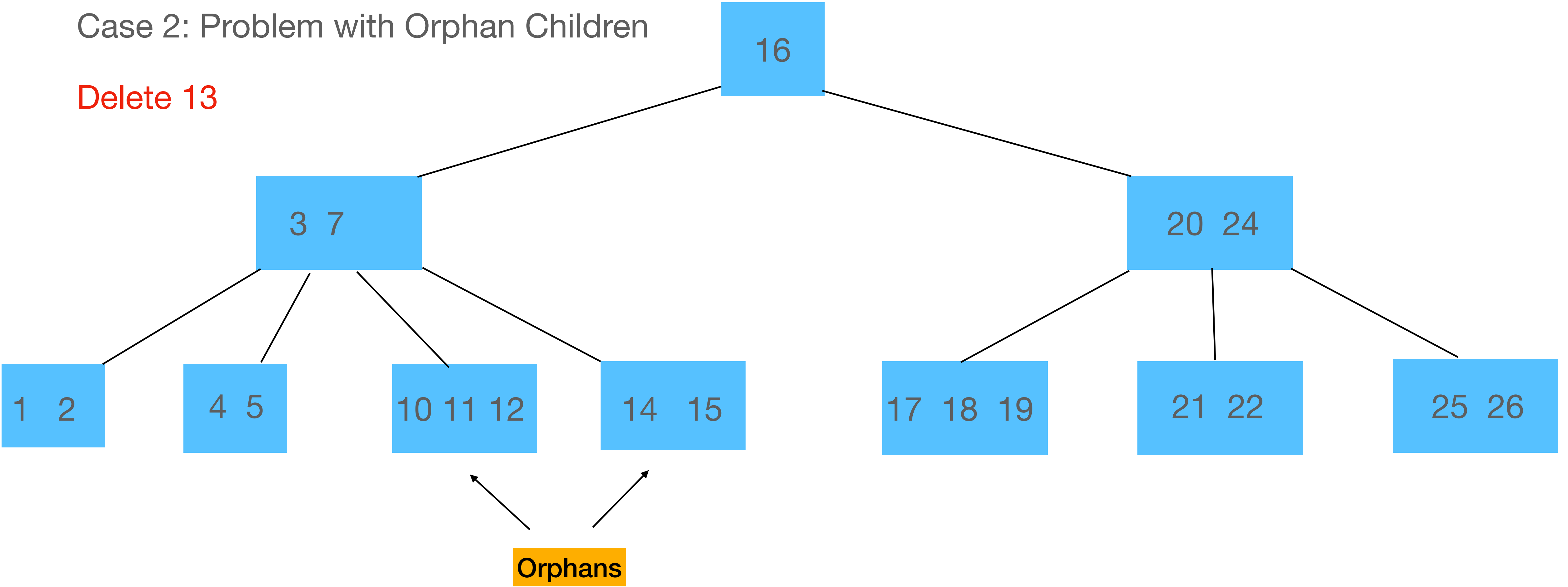# Deletion in (3,6) tree

Case 2: Problem with Orphan Children

Delete 13

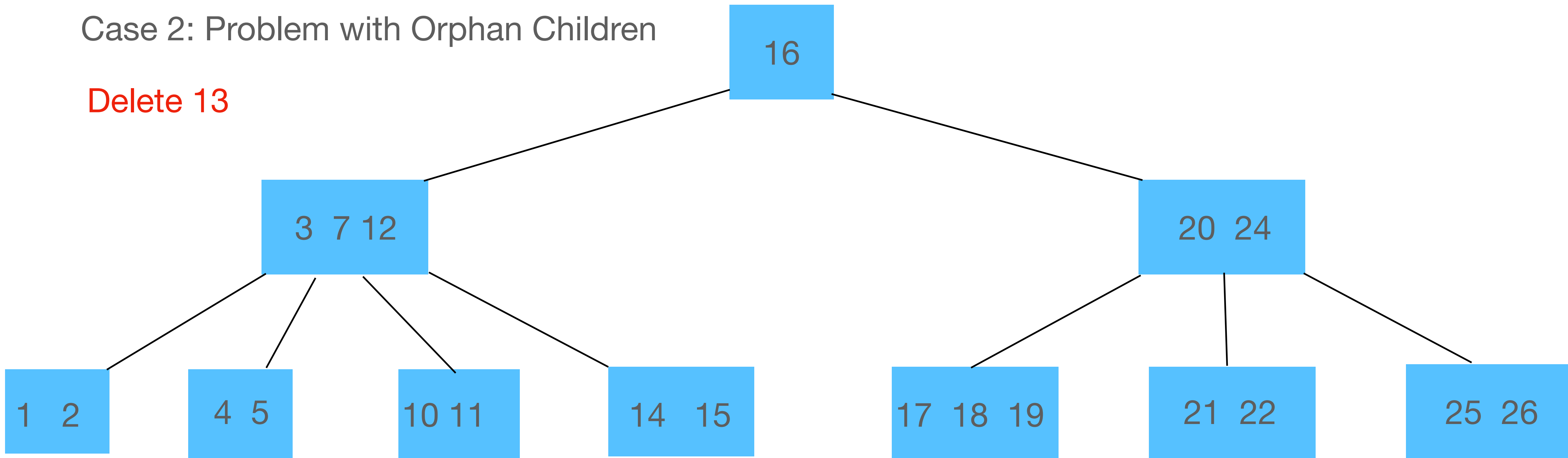# Deletion in (3,6) tree

Case 2: Problem with Orphan Children

Delete 13

16

3 7

20 24

1 2

4 5

10 11 12

14 15

17 18 19

21 22

25 26

Orphans

Deletion in (3,6) tree

Case 2: Problem with Orphan Children

Delete 13

```
                              16

        3  7  12                          20   24

1  2      4  5     10 11    14   15    17 18  19    21  22    25  26
```
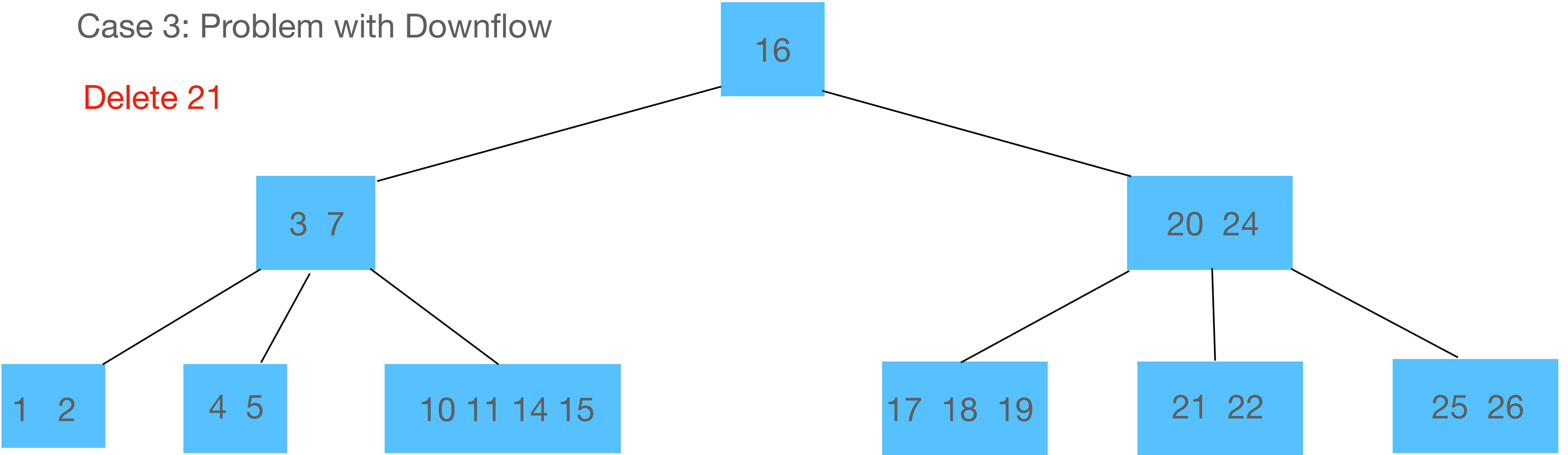
Replace the parent key with the predecessor key which is the rightmost key in the left subtree or the successor key which is the leftmost key in the right subtree

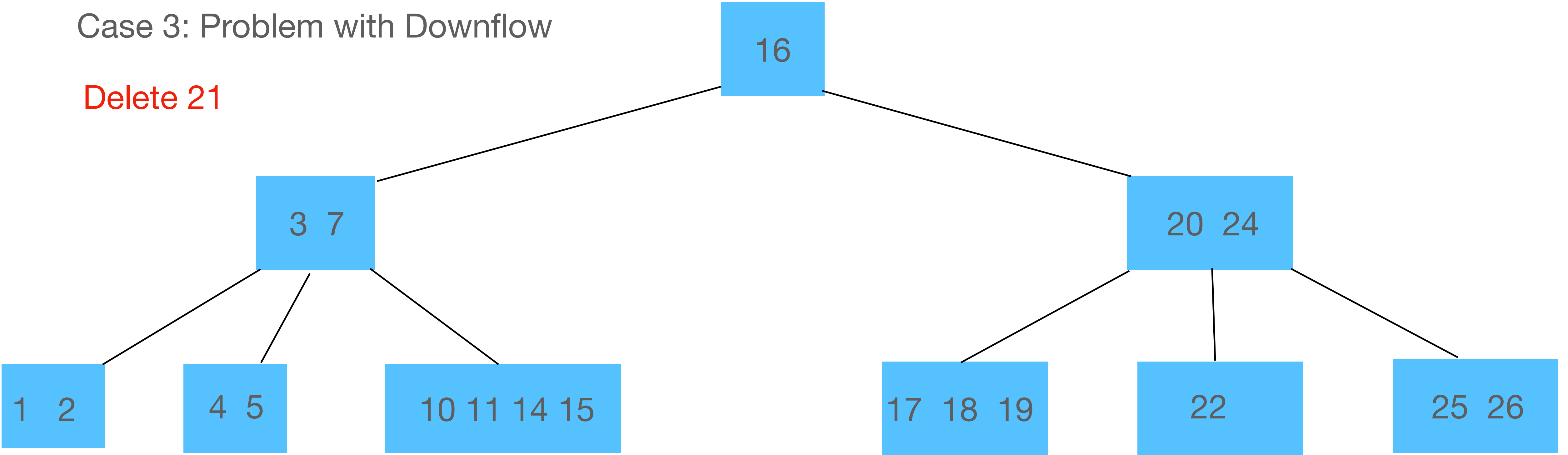# Deletion in (3,6) tree

Case 3: Problem with Downflow

Delete 21

Deletion in (3,6) tree
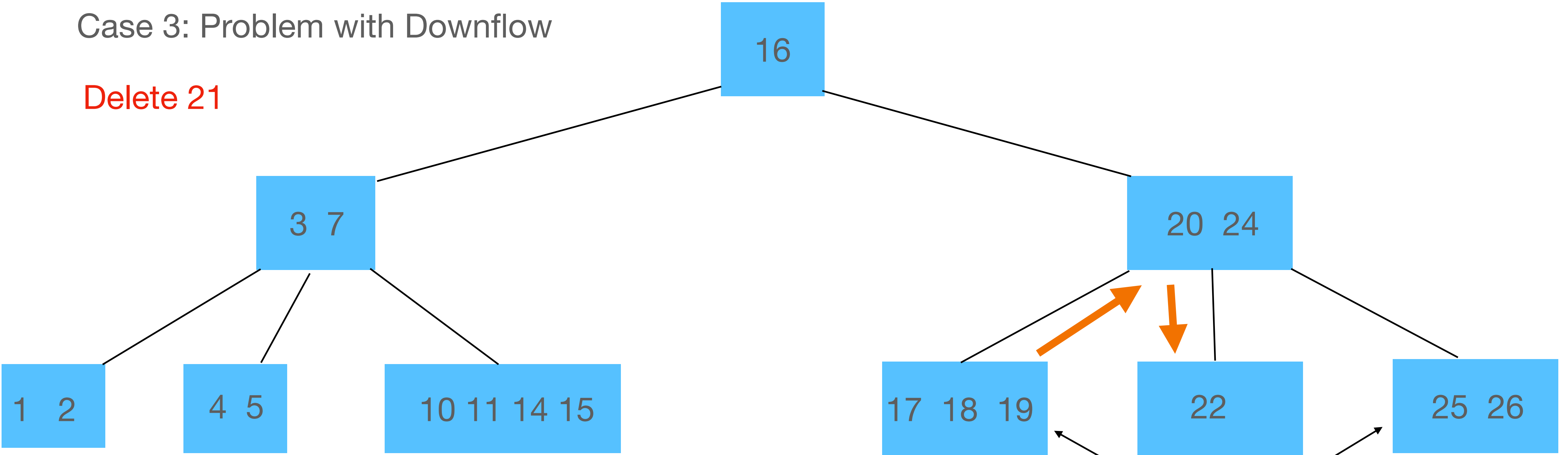
Case 3: Problem with Downflow

Delete 21

Deletion in (3,6) tree

Case 3: Problem with Downflow

Delete 21

16

3 7

20 24
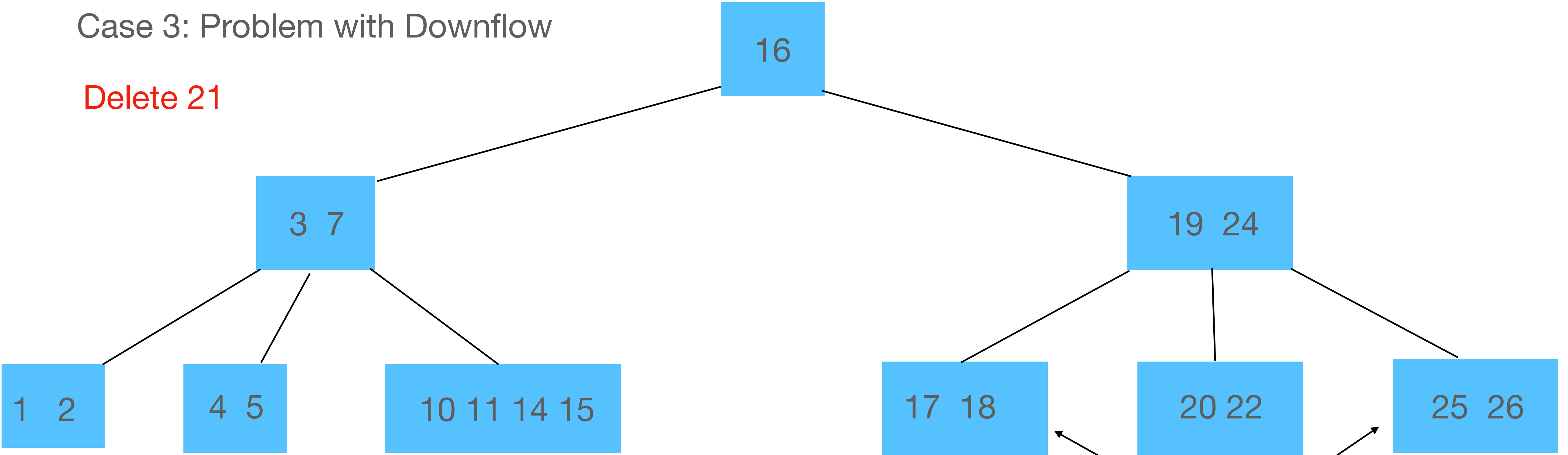
1 2

4 5

10 11 14 15

17 18 19

22

25 26

If at least one sibling has enough keys (at least 'a' keys) then it can donate its key to the other sibling

Deletion in (3,6) tree

Case 3: Problem with Downflow

Delete 21

16

3 7

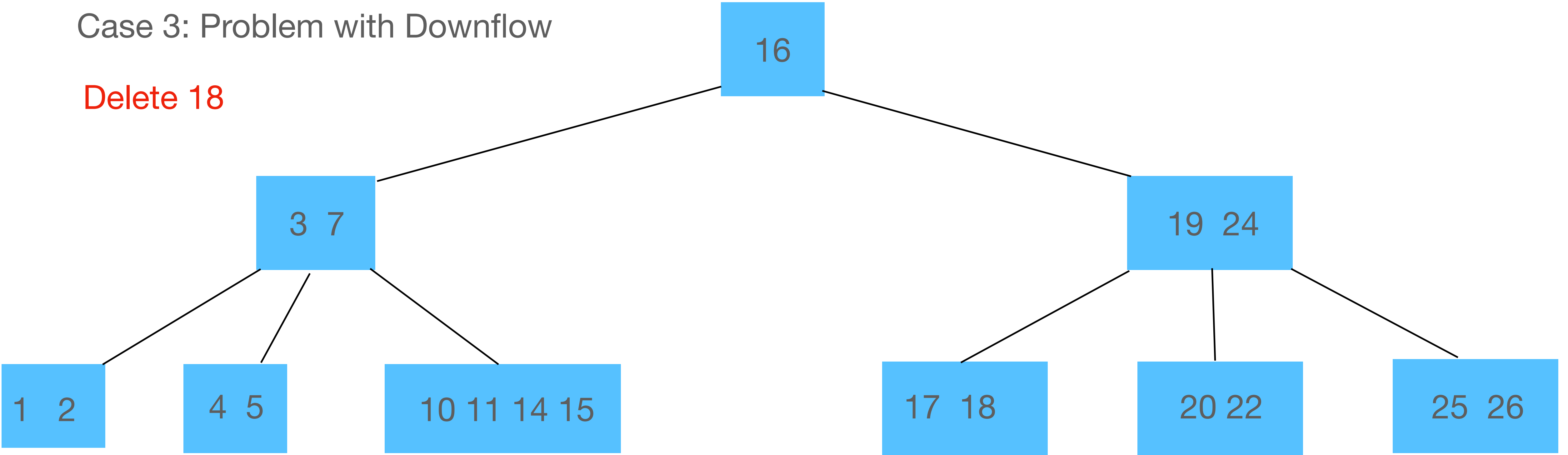19 24

1 2

4 5

10 11 14 15

17 18

20 22

25 26

If at least one sibling has enough keys (at least 'a' keys) then it can donate its key to the other sibling

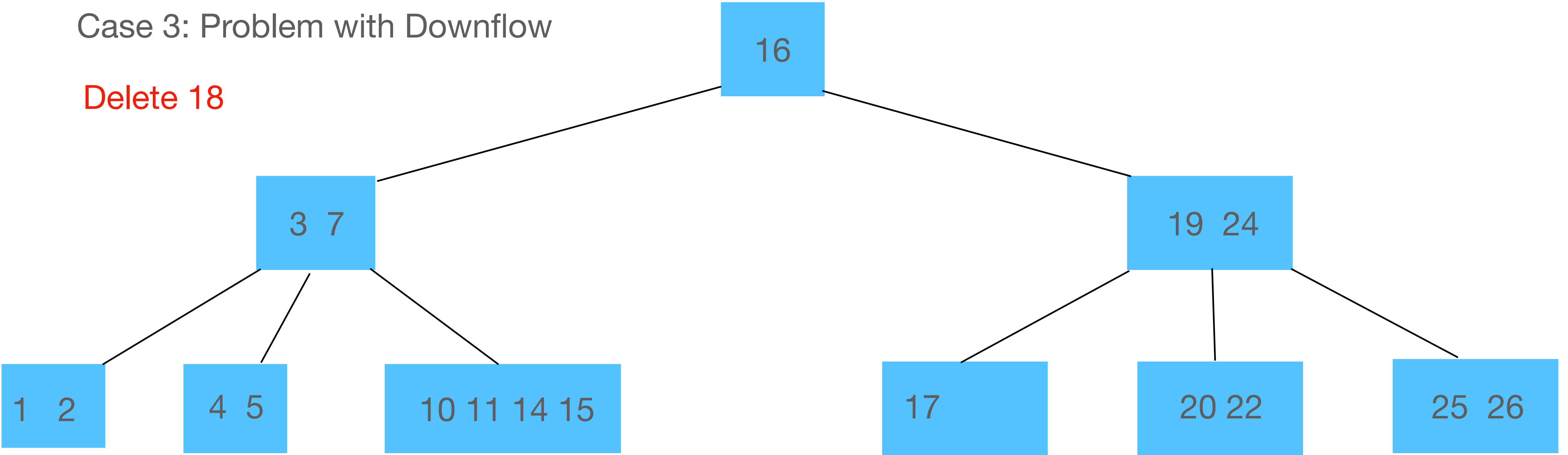# Deletion in (3,6) tree

Case 3: Problem with Downflow

Delete 18
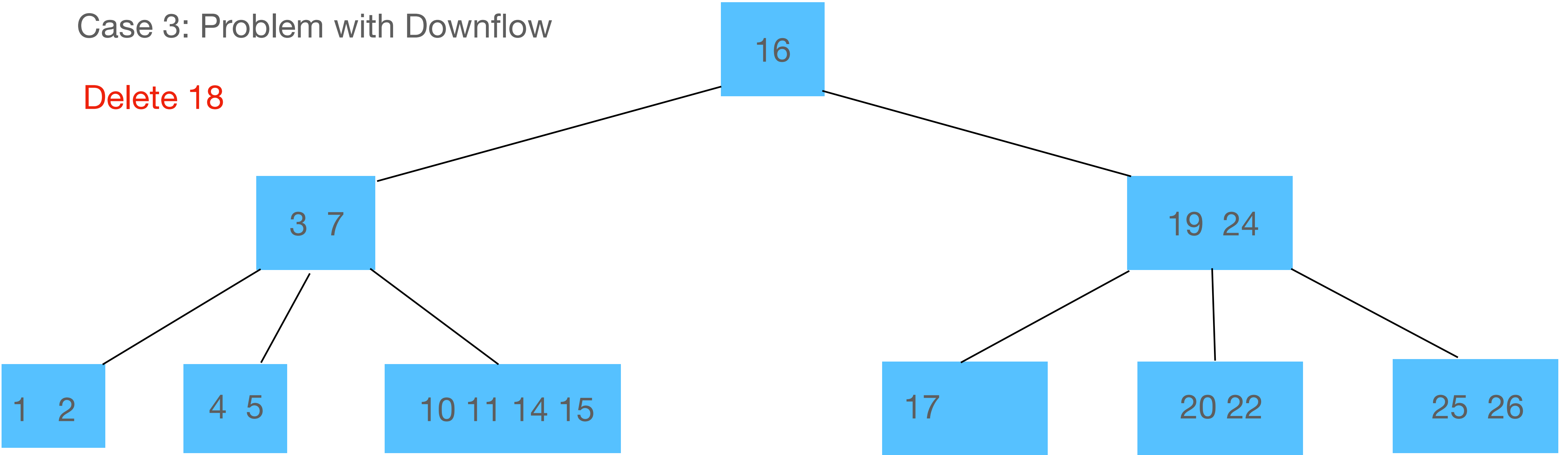
Deletion in (3,6) tree

Case 3: Problem with Downflow

Delete 18

Deletion in (3,6) tree

Case 3: Problem with Downflow

Delete 18


```
                              16
                 ┌────────────┴────────────┐
               3  7                      19  24
          ┌─────┼─────┐            ┌───────┼───────┐
        1  2   4  5  10 11 14 15  17    20 22    25 26
```
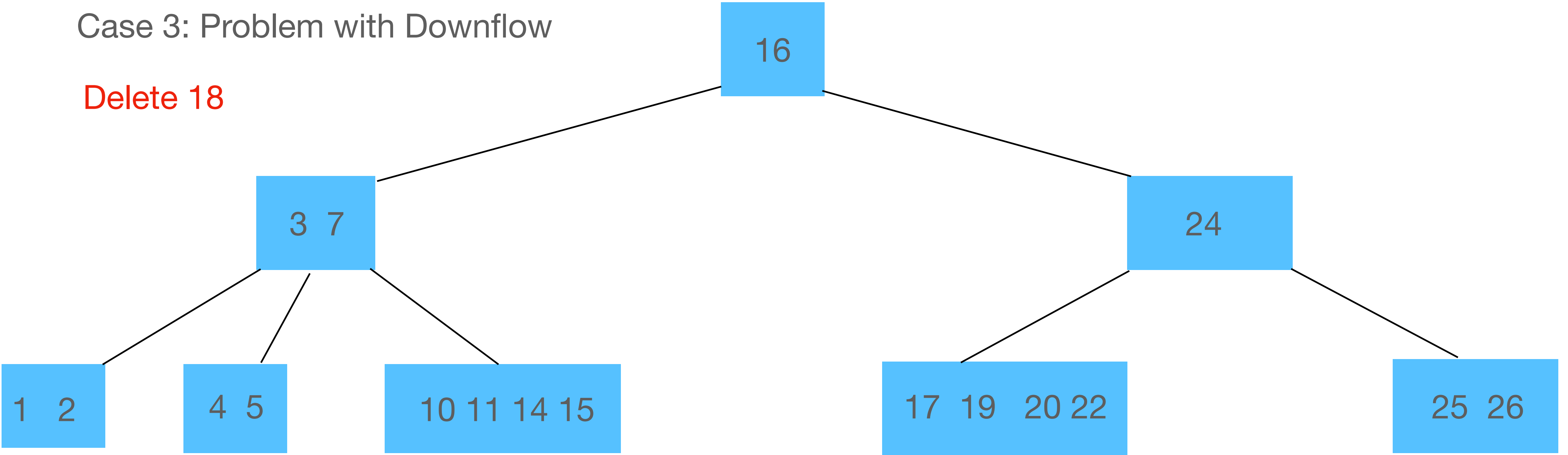
If the sibling does not have enough keys (at least 'a' keys) then merge the two siblings and the parent together

Deletion in (3,6) tree

Case 3: Problem with Downflow

Delete 18

16

3 7

24

1 2

4 5

10 11 14 15

17 19   20 22

25 26

If the sibling does not have enough keys (at least 'a' keys) then merge the two siblings and the parent together

Deletion in (3,6) tree

Case 3: Problem with Downflow

Delete 18

Downflow Again

16

3 7

24

1 2
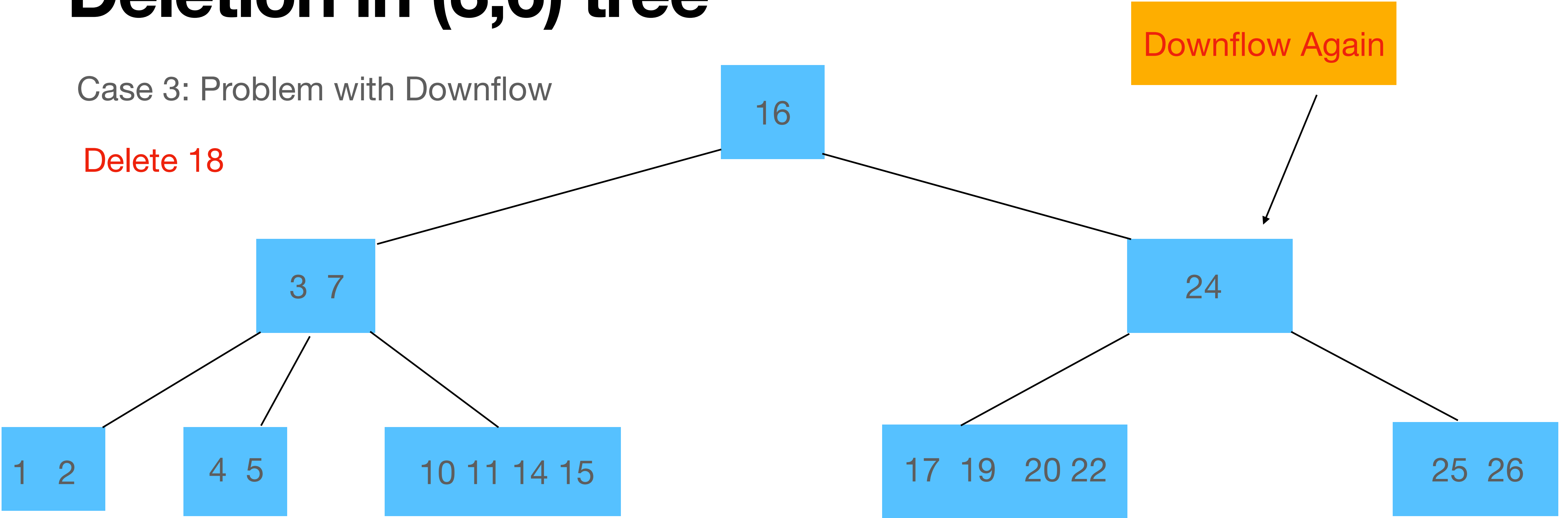
4 5

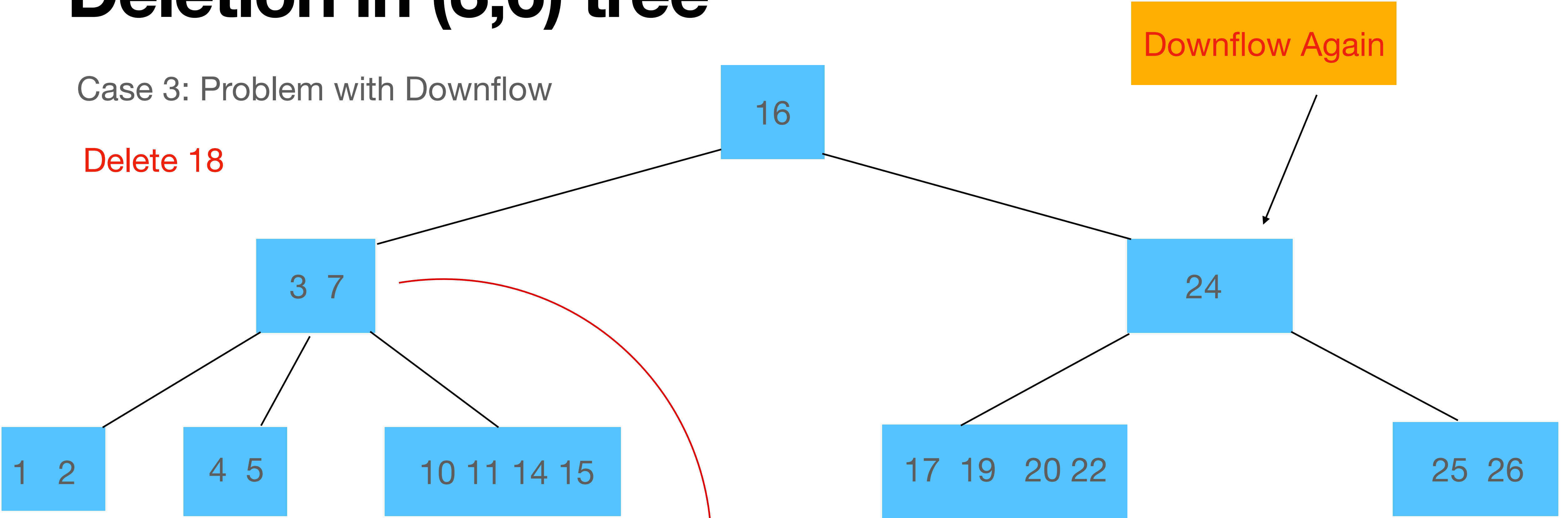10 11 14 15

17 19   20 22

25 26

If the sibling does not have enough keys (at least 'a' keys) then merge the two siblings and the parent together

Deletion in (3,6) tree

Case 3: Problem with Downflow

Delete 18

16

3 7

24
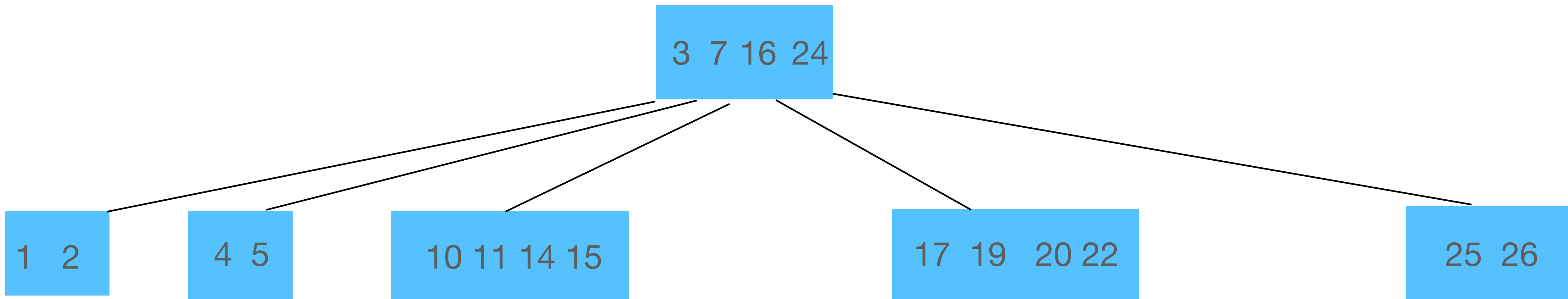
1 2

4 5

10 11 14 15

17 19   20 22

25 26

Since the sibling does not have enough keys here. So we merge.

If the sibling does not have enough keys (at least 'a' keys) then merge the two siblings and the parent together

Deletion in (3,6) tree

Case 3: Problem with Downflow

Delete 18


```
                        3  7  16  24


1  2        4  5      10 11 14 15            17  19  20 22              25  26
```

If the sibling does not have enough keys (at least 'a' keys) then merge the two siblings and the parent together

Strategy for Deletion

(1) At first delete the key at the node it is present currently.

(2) Now, detect does it have "the problem of orphan children" or "the problem of downflow"

(3) If it is "the problem of orphan children" then replace it with either the successor or predecessor key.

(4) if it is "the problem of down flow" then there are again two cases

(a) Its sibling (note that there can be two siblings left and right sibling) has enough (at least "a" keys) keys, then SHARE (look at the process of sharing) its key with the other sibling.

(b) Its sibling does not have enough keys then we MERGE (look how we merge).
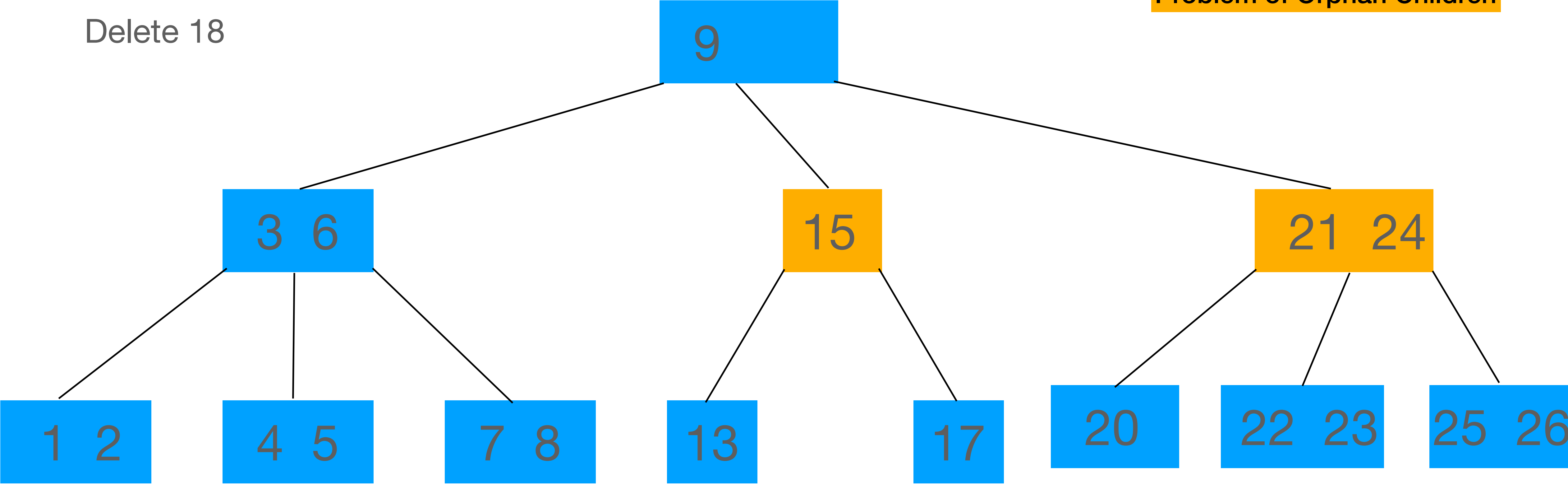
Another Example of Deletion

Delete 18

Another Example of Deletion
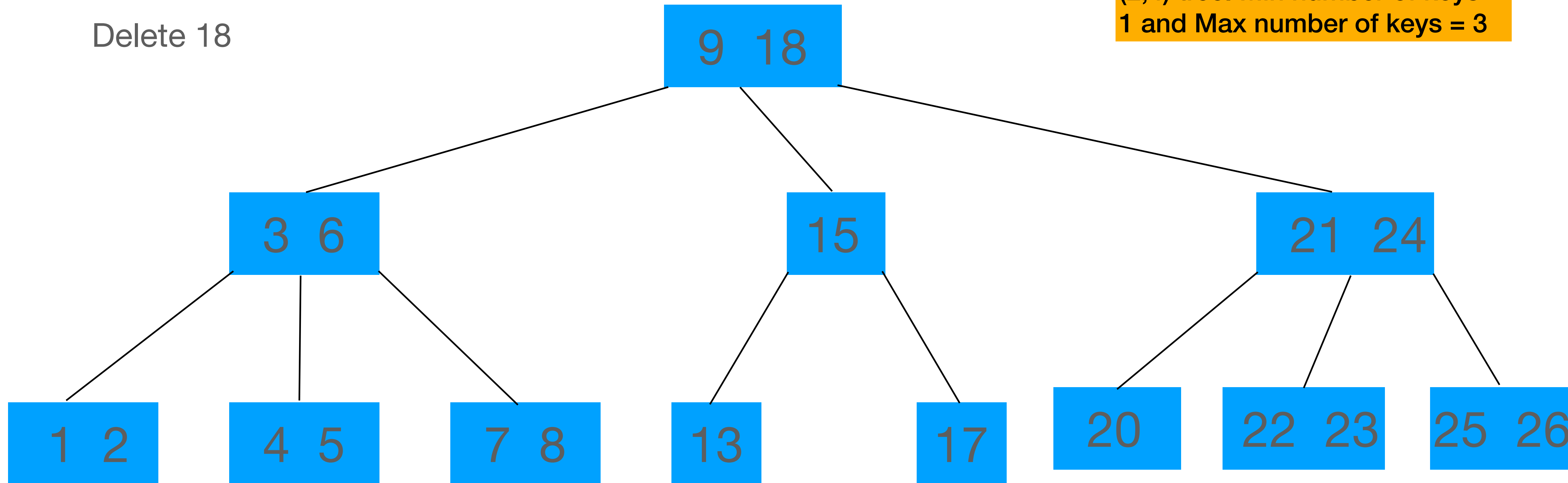
Delete 18

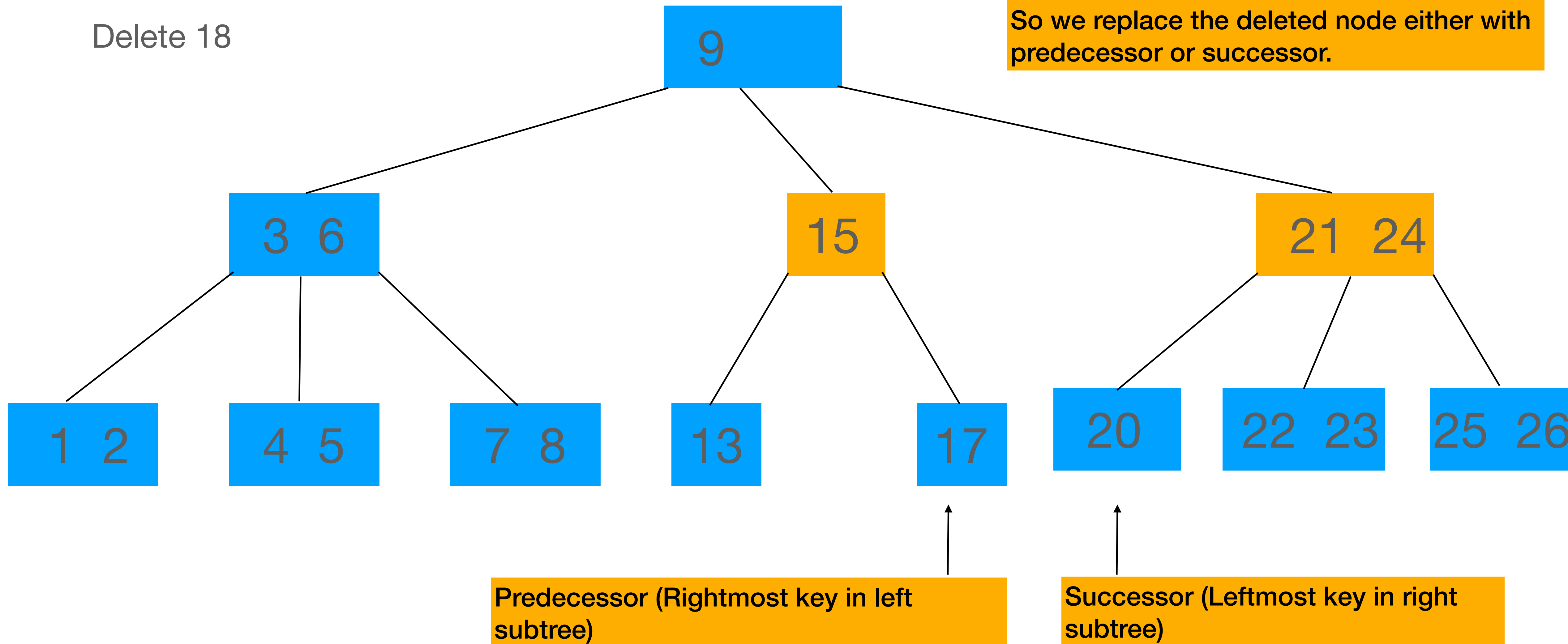# Another Example of Deletion

Problem of Orphan Children

Delete 18

So we replace the deleted node either with predecessor or successor.

Predecessor (Rightmost key in left subtree)

Successor (Leftmost key in right subtree)

Another Example of Deletion
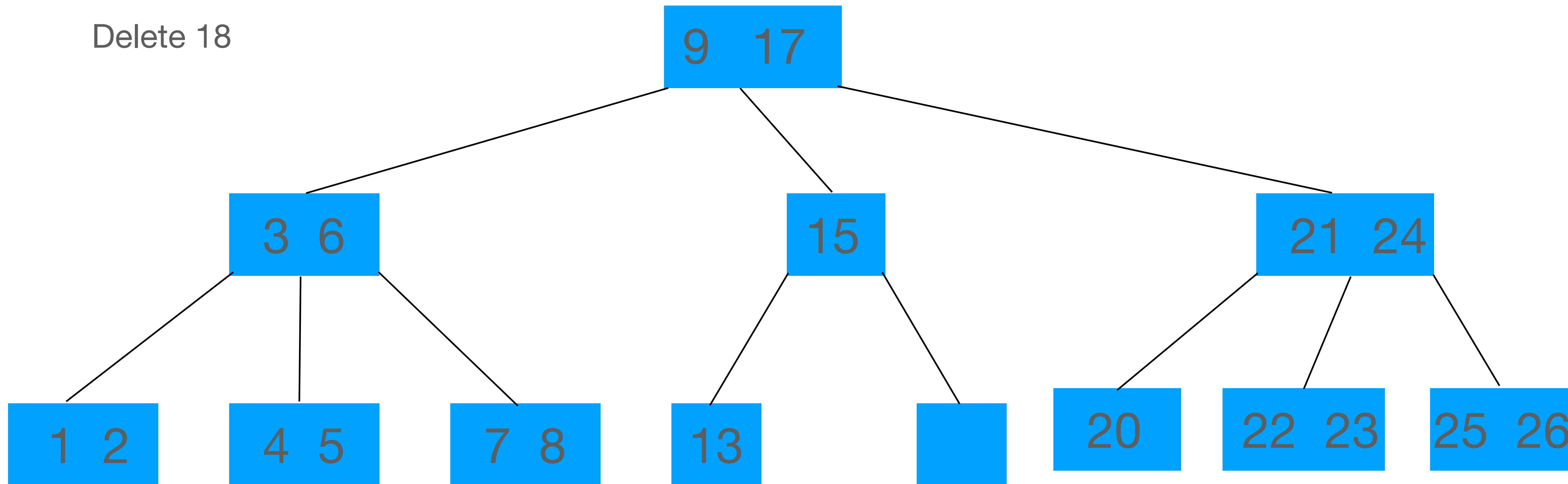
Delete 18

Downflow

Another Example of Deletion

Delete 18

Sibling does not have enough (at least 2) keys. So, we MERGE two siblings and the parent together.

Downflow

Another Example of Deletion

Delete 18

Downflow
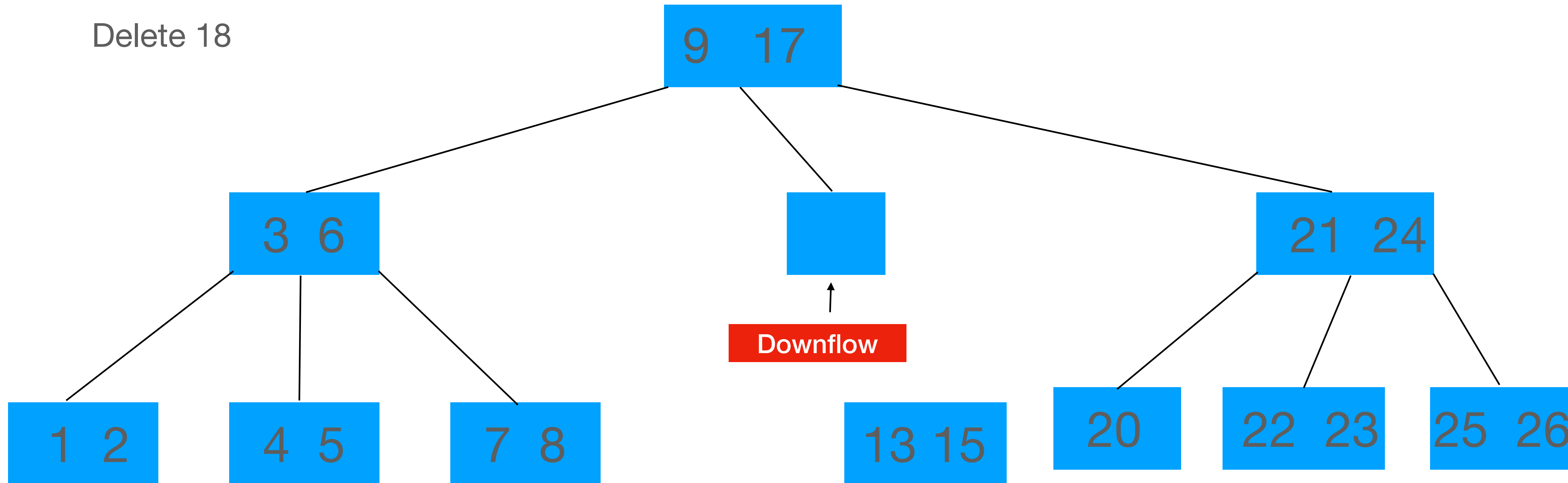
Sibling does not have enough (at least 2) keys. So, we MERGE two siblings and the parent together.

Another Example of Deletion

Delete 18

Downflow

Now the sibling has enough (at least 2) keys. So, this sibling SHARE its key with other sibling.

Another Example of Deletion

Delete 18

9 17

Process of Sharing

3 6

Downflow

21 24

1 2   4 5   7 8

13 15

20   22 23   25 26
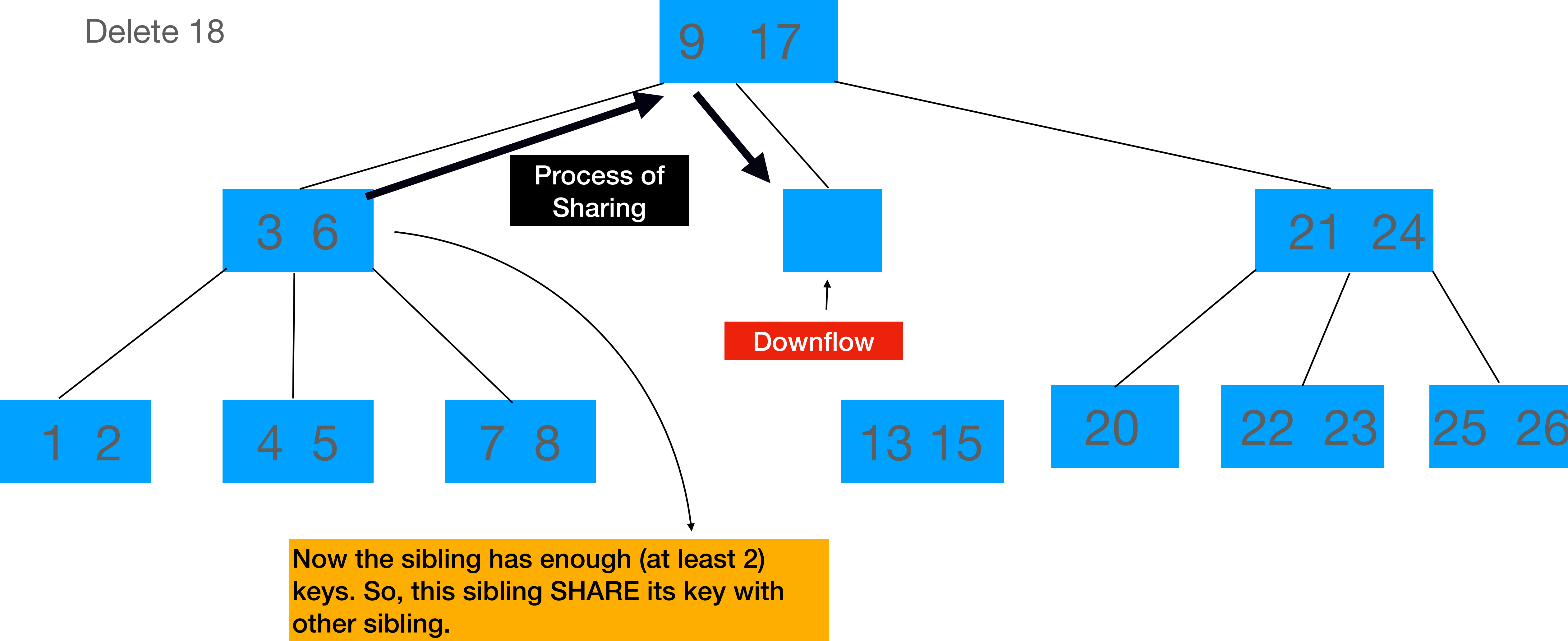
Now the sibling has enough (at least 2) keys. So, this sibling SHARE its key with other sibling.

Another Example of Deletion

Delete 18

Height of a B-tree (Problem 1(b))

What is the minimum and maximum height of a B-Tree with n keys?

Height of a B-tree

What is the minimum and maximum height of a B-Tree with n keys?

Maximum height when,

- Minimum branching factor of internal nodes (a children each)
- Equivalently, each non-root node has $a-1$ keys
- Root node has **1** key and **2** children

Minimum height when,

- Maximum branching factor of internal nodes (b children each)
- Equivalently, each non-root node has $b-1$ keys

Height of a B-tree

Here, I am calculating the maximum height when the root node has two children

Number of Keys
1
$2(a-1)$

Height of a B-tree

Root

Number of Keys at level 0

$(a-1)$ keys

$(a-1)$ keys

at level 1

a child nodes

a child nodes

at level 2

Each child node has $(a-1)$ keys
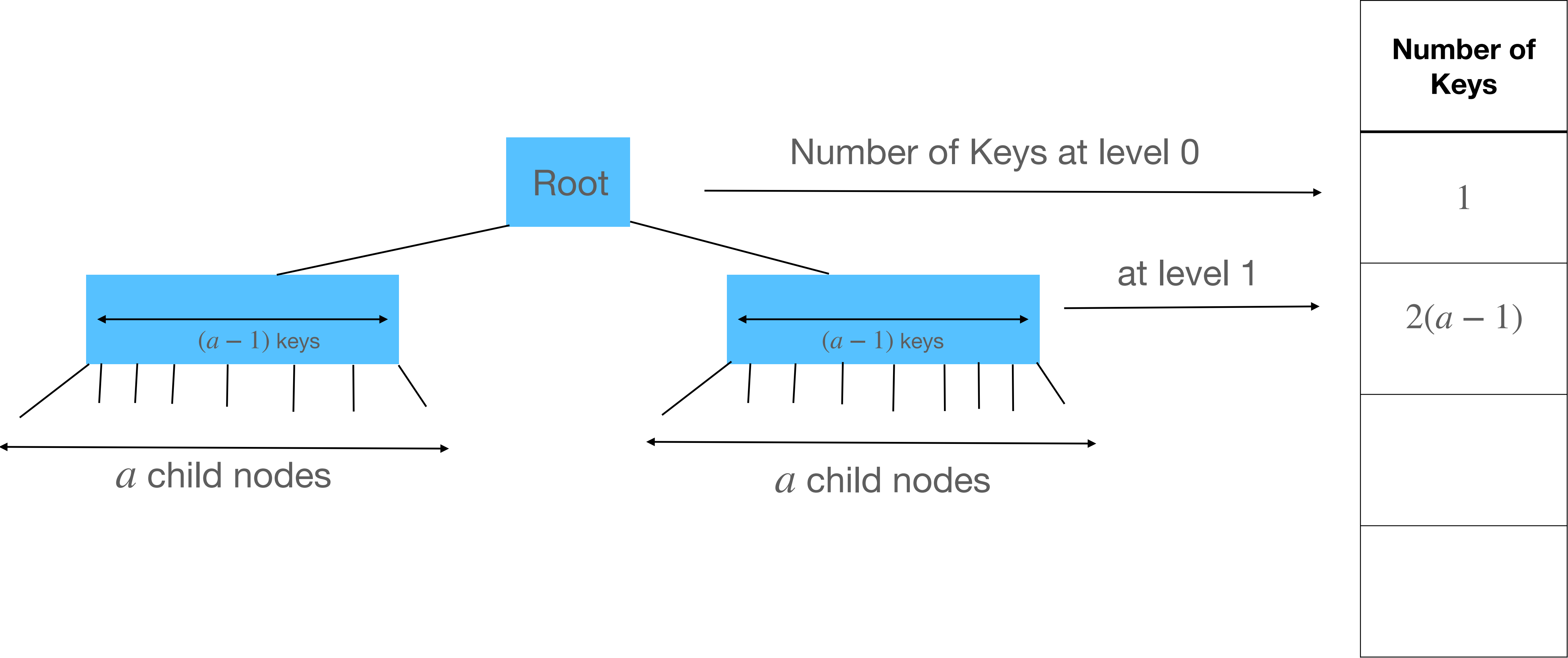
Number of Keys
1
$2(a-1)$
$2a \times (a-1)$
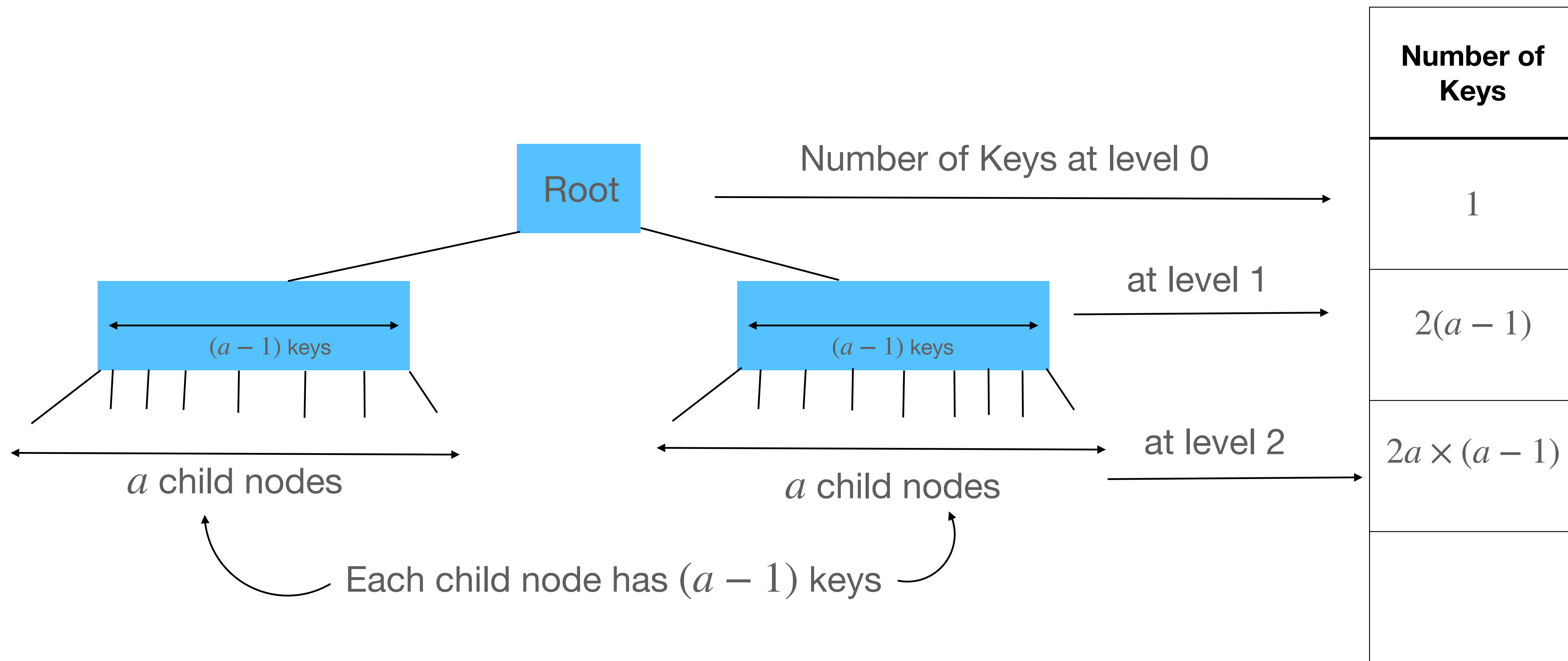
Height of a B-tree

At height h we will have the number of keys $= 2a^{h-1} \times (a-1)$

Root

Number of Keys at level 0

$(a-1)$ keys

$(a-1)$ keys

at level 1

a child nodes

a child nodes

at level 2

Each child node has $(a-1)$ keys

at level 3

Number of Keys
1
$2(a-1)$
$2a \times (a-1)$
$2a^2 \times (a-1)$
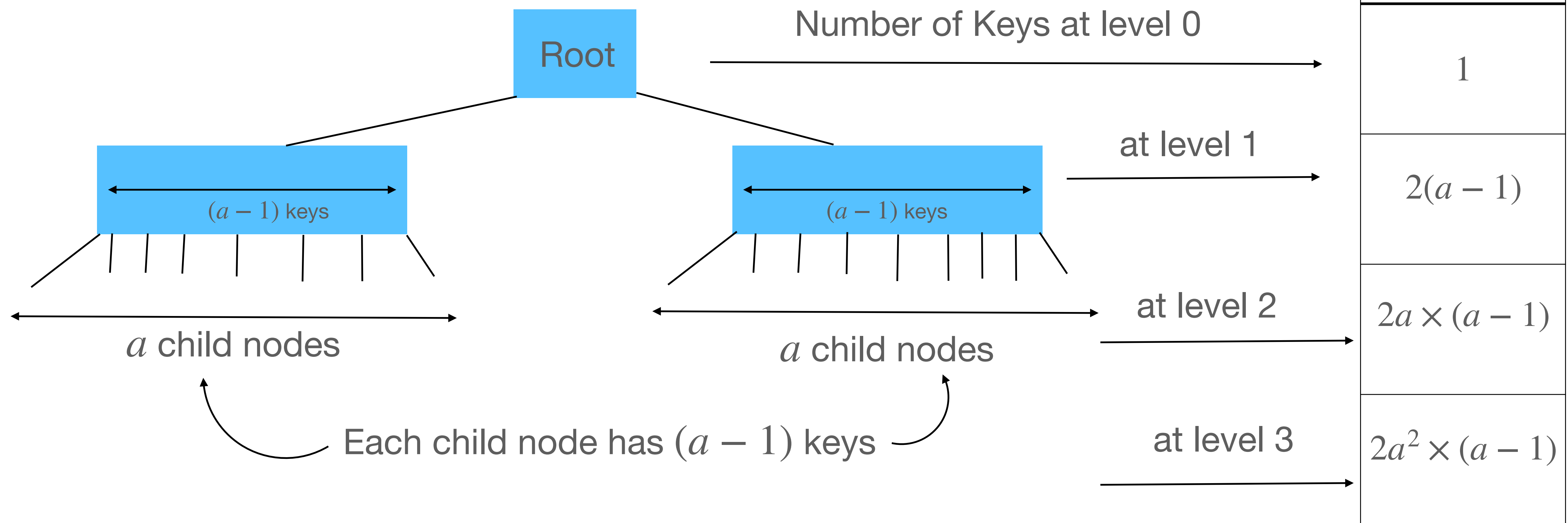
Height of a B-Tree

So, the total number of keys = $1 + (a - 1) \times (2 + 2a + 2a^2 + 2a^3 + \ldots + 2a^{h-1}) = n$

Height of a B-Tree

So, the total number of keys $= 1 + (a - 1) \times (2 + 2a + 2a^2 + 2a^3 + \ldots + 2a^{h-1}) = n$

$$\implies 1 + 2(a - 1) \times (1 + a + a^2 + a^3 + \ldots + a^{h-1}) = n$$

$$\implies 1 + 2(a - 1) \times \frac{a^h - 1}{a - 1} = n$$

$$\implies 1 + 2 \times (a^h - 1) = n$$

$$\implies 2a^h - 1 = n$$

$$h = log_a(n)$$

Using similar calculations we can calculate the minimum height which is $log_b(n)$

Cost of searching the keylist, splitting, merging, sharing

Problem 2(c)

What is the cost of searching a keylist in a B-tree node?

What is the cost of splitting a B-tree node?

What is the cost of merging or sharing B-tree nodes?

Cost of searching the keylist, splitting, merging, sharing
Problem 2(c)

What is the cost of searching a keylist in a B-tree node?

What is the cost of splitting a B-tree node?

What is the cost of merging or sharing B-tree nodes?

Answer: They are all $O(1)$ since they require some constant number of block transfers followed by operation on the block in-memory.

Cost of Searching, Insertion, Deletion
Problem 2(d)

What is the cost of searching a B-tree? What is the cost of inserting or deleting in a B-tree?

These are all $O(log_a n)$ because

1. Each node is contained in a block. i.e., the cost only depends on the height of the tree.

2. Cost of the in-memory operations on a single node is only $O(1)$.

Cost of Searching, Insertion, Deletion
Problem 1(d), 1(g), 1(h)

What is the cost of searching a B-tree? What is the cost of inserting or deleting in a B-tree?

Normally, we do not care about the time taken to search an element when the block is in the RAM but suppose now we care about that time also, then the time taken would be $O(log_a n) \times O(log_2 b)$ because we are doing a binary search within the block to search an element, insert in the correct position and delete the correct element.

Space Complexity of B-Trees

Since we are storing n keys in total in a B-tree, the space complexity would be $O(n)$.

Note that the difference in between storing datas in an array and in a B-tree is the way we are storing our data which does not change the space complexity.

1 Billion Data in only 3 levels!!!
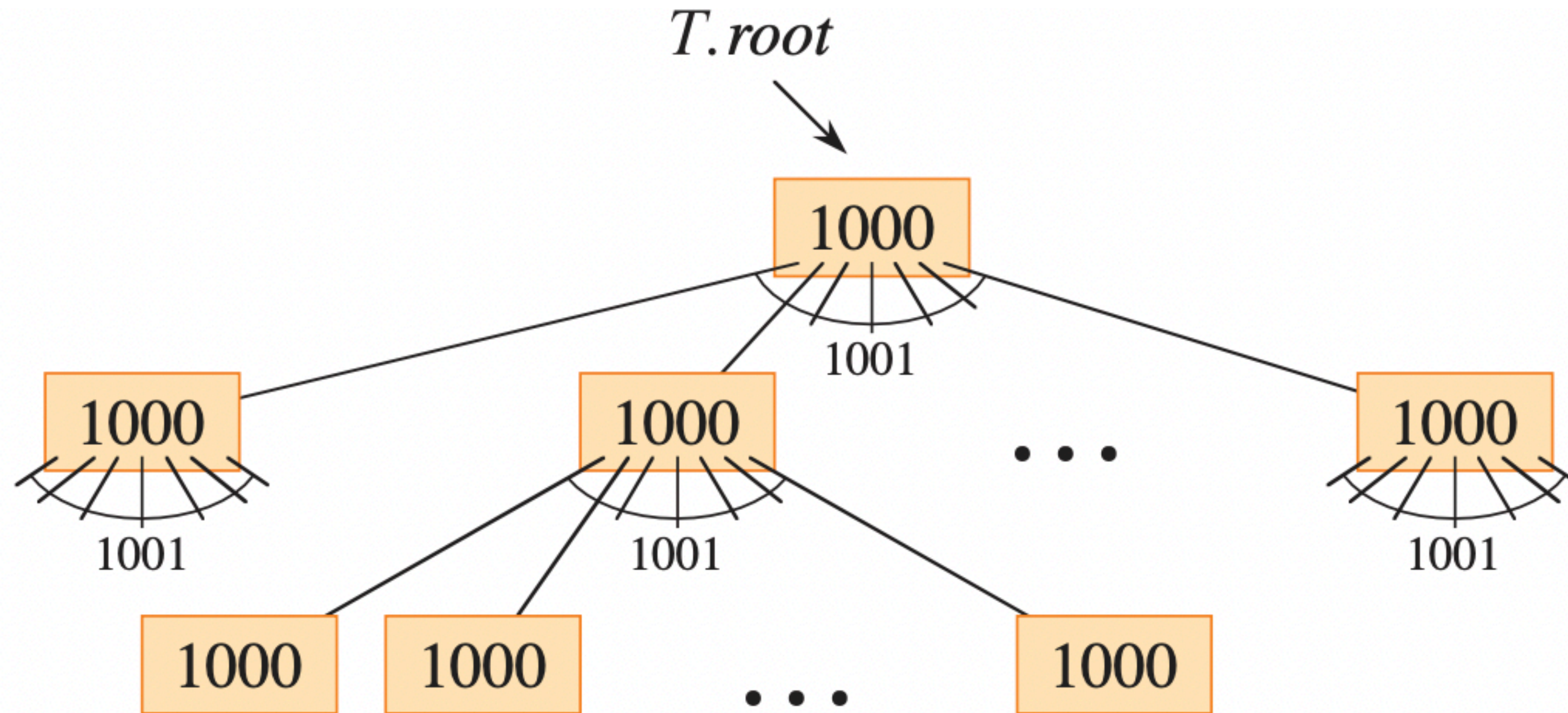
1 node,
  1000 keys

1001 nodes,
  1,001,000 keys

1,002,001 nodes,
  1,002,001,000 keys

We can access any data among these 1 billion data in only 3 disk accesses

Thank You