Minimal DFA's for Divisibility Testing (LSB first)

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1 Introduction

2 Non-Distinguishability Criteria

2.1 Initial ND Criteria

Lemma 2.1. *The strings s_x, s_y are non distinguishable if and only if*
$\forall d \in \mathbb{N}\ r_2(s_x)d + x \equiv 0 \iff r_2(s_y)d + y \equiv 0$

2.2 Revised ND Criteria

Lemma 2.2. *For all $\alpha \in \mathbb{Z}/p\mathbb{Z}$ there exists $\alpha^{-1} \in \mathbb{Z}/p\mathbb{Z}$ such that*
$a \times a^{-1} \equiv 1$ *(mod p) if a, p are coprime and $a \neq 0$*

Proof. If we pick a and we have that α and p are coprime we have by Bezout's identity we have that there exists integers x and y such that $\alpha x + py = 1$ which implies that

$$\alpha x + py \equiv \alpha x + 0 \equiv \alpha x \equiv 1 \pmod{p}$$

And so we take $\alpha^{-1} = x$ which fulfills our required properties □

Lemma 2.3. *The strings s_x, s_y are non distinguishable if and only if*
$(r_2(s_x))^{-1}x \equiv (r_2(s_y))^{-1}y$ *(mod p)*

3 Equivalence Relation Classes

As we have shown our distinguishability equivalence relation $_{d,p}$ is equivalent to $r_2(s_y)x \equiv r_2(s_x)y \pmod{p}$ and we want to construct our distinguishing set from this which leads us to.

Lemma 3.1. *The amount of equivalence classes under $=_{d,p}$ is exactly p*
Also said as $\Sigma^*/ND = p$

Proof. Firstly since there is only p possible values for the numbers to be congruent to mod p as they are integers we have that the amount of equivalence classes is $\leq p$.
Now all we need to do is find p possible equivalence classes of distinguishability which will force it to be p.
Consider the strings of 0 to $p-1$.
Prepend 0s to the start of these strings to make them all the same length so we have $r_2(s_x) = \alpha$ for all of them.
We then have that they are all distinct under $=_{d,p}$ as for any two such strings $s_x, s_y, x \neq y$ assuming they are non distinguishable we have by 2.3

$$(r_2(s_x))^{-1}x \equiv (r_2(s_y))^{-1}y \text{ (mod p)}$$
$$\alpha^{-1}x \equiv \alpha^{-1}y \text{ (mod p)}$$
$$x \equiv y \text{ (mod p)}$$

This forms a contradiction as we picked them to be distinct numbers between 0 and $p-1$ and so they must all be distinguishable and hence in different equivalence classes.
Thus we have found p distinct equivalence classes and so the amount of equivalence classes is exactly p. $\square$

4 Even Numbers

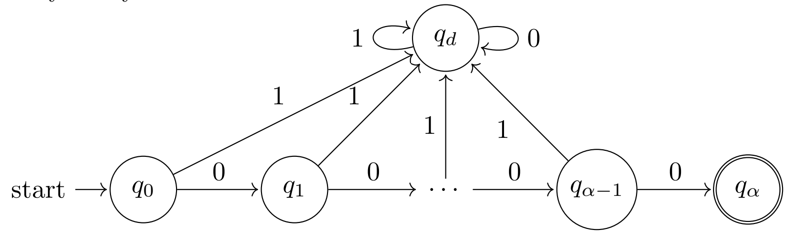
Now we aim to extend our Minimal DFA's to work with even numbers specifically numbers of the form $2^\alpha p$ where $\alpha, p \in \mathbb{N}$ and p is odd.

4.1 Construction

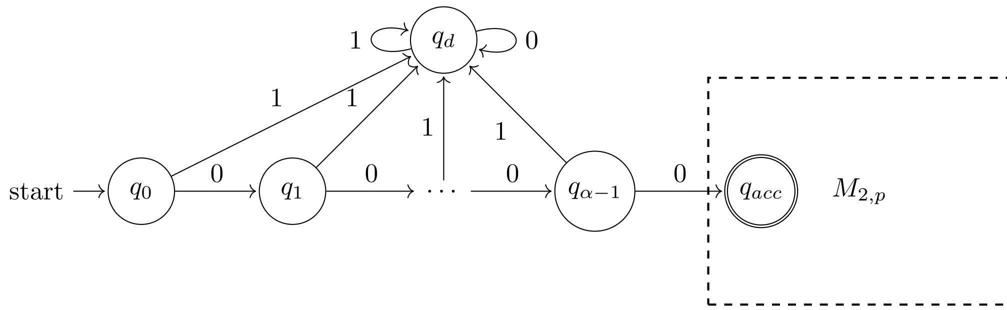
We will construct a proposed minimal DFA and prove that it works aswell as proving minimality.

4.1.1 Checking for Divisibility by 2^α

Firstly it is easy to see that for checking if a binary string is divisible by 2^α you only need to check the first α digits and so we can construct a DFA for that fairly easily.



Next we can construct a DFA as a joining of those two DFAs because if the first α digits are 0s that doesnt affect the divisibility by p.



4.2 Proof of it working

Lemma 4.1. *For any binary string s_w the run of s_w on $M_{2,2^\alpha p}$ is accepting if and only if s_w is divisible by $2^\alpha p$*

Proof. We know from our previous work that $M_{2,p}$ accepts a string s_d if and only if the string is divisible by p.
All strings that are divisible by $2^\alpha p$ have the value of $2^\alpha d$ for any s_d divisible by p and because of this we can write any string divisible by $2^\alpha p$ as $s_w = 0^\alpha \circ s_d$ for some s_d divisible by p. If we partially calculate where the run on $M_{2,2^\alpha p}$ will end up at after processing 0^α we get that it ends up at the start state (also the accepting state) of $M_{2,p}$ and then since s_d is divisible by p the run from the start state of $M_{2,p}$ will be accepting and so we have proven the machine for all s_w

$\square$

4.3 Reachability

Lemma 4.2. *For all q_m in the states of $M_2, 2^\alpha p$ there exists a string s_w such that the run of s_w ends at q_m*

Proof. We split q_m into cases based on which part of the DFA it is in, we consider 3 cases.
Case 1: q_m is in the 2^α part of the DFA $0 \leq m \leq \alpha - 1$ then the string is just $s_w = 0^m$ and we know the run of this ends at q_m by definition of the 2^α part of the DFA.
Case 2: $q_m = q_d$ in which case the string is $s_w = 1$ and we can see the run of this ends at q_d
Case 3: q_m is a state in $M_{2,p}$ in which case we can construct s_w by first taking the string 0^α which will take us to the accepting state (also the start state of $M_{2,p}$) and then concatenating it with a string from the equivalence class of q_m and so we can the run of this ends at q_m by definition of $M_{2,p}$ $\square$

4.4 Distinguishability

Lemma 4.3. *For all q_m in the states of $M_{2,2^\alpha p}$ where $q_m \neq q_d$ we have that there exists a string s_w such that running the string w from q_m we get to the accepting state.*

Proof. We split q_m into cases based on which part of the DFA it is in, we consider 2 cases.
Case 1: q_m is in the 2^α part of the dfa $0 \leq m \leq \alpha - 1$, if this is the case we have the trivial string of $s_w = 0^{\alpha - m}$ which we can see will go to the accepting state.
Case 2: q_m is a state in $M_{2,p}$ if this is the case we know that after running the string s_w at the state the value will be $r_2(s_x)w + x$ and so all we need to do is find a string s_w such that $w \equiv -(r_2(s_x))^{-1}x$ which we know we can find as $r_2(s_x)^{-1}$ must exist by 2.2 □

Lemma 4.4. *For all q_m q_n in the states of $M_{2,2^\alpha p}$ $m \neq n$ we have that q_m is distinguishable from Sqn*

Proof. We split q_m and q_n into cases based on which part of the DFA they are in we consider 3 cases for each of them being the dead state q_d, being any other state in the divisibility by 2^α part $q_l, 0 \leq l \leq \alpha - 1$ or being any state in $M_{2,p}$.
Case 1: $q_m \neq q_d$ and $q_n = q_d$. In this case we have that the distinguishing string is the accepting string of q_m which we know exists from 4.3 with this q_m will go to the accepting state and q_n will stay at the dead state and so they are distinguishable.
Case 2: q_m is in the 2^α part of the dfa $0 \leq m \leq \alpha - 1$ and q_n in the states of $M_{2,p}$, in this case for our distinguishing string we first take 1 which will take q_m to q_d and q_n to another state in $M_{2,p}$ as no transitions leave the machine. We then take the rest of the distinguishing string to be the accepting string of the state q_n goes to and then we have that the run on q_n will be accepting and the run on q_m will go to the dead state and stay there and so they are distinguishable.
Case 3: q_m, q_n are both in the 2^α part of the dfa $0 \leq m, n \leq \alpha - 1$, in this case the distinguishing string is $0^{\alpha - \max m,n}$ which will take the state the closest to the accepting state to the accepting state and leave the other state still in the 2^α part of the DFA and so these states are distinguishable.
Case 4: q_m, q_n are both in $M_{2,p}$ we know these states are all distinguishable from our construction of the machine using equivalence clases.

□