Component Redundancy Versus System Redundancy in Different Stochastic Orderings

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***Abstract*—Stochastic orders are useful to compare the lifetimes of two systems. We discuss both active redundancy as well as standby redundancy. We show that redundancy at the component level is superior to that at the system level with respect to different stochastic orders, for different types of systems.**

***Index Terms*—Coherent system, k-out-of-n system, reliability, stochastic orders.**

Acronyms and Abbreviations

st	usual stochastic
sp	stochastic precedence
hr	hazard rate
rhr	reversed hazard rate
lr	likelihood ratio
hr↑	up shifted hazard rate
hr↓	down shifted hazard rate
rhr↑	up shifted reversed hazard rate
lr↑	up shifted likelihood ratio
lr ↓	down shifted likelihood ratio

Notation

X	underlying nonnegative random variable
$f_X(\cdot)$	probability density function of random variable X
$F_X(\cdot)$	cumulative distribution function of random variable X
$\bar{F}_X(\cdot)$	survival (reliability) function of random variable X
$r_X(\cdot)$	hazard rate function of random variable X
$\tilde{r}_X(\cdot)$	reversed hazard rate function of random variable X
τ	lifetime of a coherent system
$h(\cdot)$	reliability function of a coherent system having lifetime τ
$\tau_{k:n}$	lifetime of a k-out-of-n system
$\mathbf{x}(t)$	$(x_1(t), x_2(t), \ldots, x_n(t))$
$\mathbf{X}$	$(X_1, X_2, \ldots, X_n)$
$X \vee Y$	$\max\{X, Y\}$
$\mathbf{X} \vee \mathbf{Y}$	$(X_1 \vee Y_1, X_2 \vee Y_2, \ldots, X_n \vee Y_n)$
$\mathbf{X} + \mathbf{Y}$	$(X_1 + Y_1, X_2 + Y_2, \ldots, X_n + Y_n)$
$\tau(\mathbf{X} \vee \mathbf{Y})$	coherent system with active redundancy at the component level
$\tau(\mathbf{X}) \vee \tau(\mathbf{Y})$	coherent system with active redundancy at the system level
$\tau_{k:n}(\mathbf{X} \vee \mathbf{Y})$	k-out-of-n system with active redundancy at the component level
$\tau_{k:n}(\mathbf{X}) \vee \tau_{k:n}(\mathbf{Y})$	k-out-of-n system with active redundancy at the system level
$\tau_{k:n}(\mathbf{X} + \mathbf{Y})$	k-out-of-n system with standby redundancy at the component level
$\tau_{k:n}(\mathbf{X}) + \tau_{k:n}(\mathbf{Y})$	k-out-of-n system with standby redundancy at the system level
$\phi_{\tau(\mathbf{x})}$	state of $\tau(\mathbf{X})$ at time t
$\phi_{\tau_{k:n}(\mathbf{x})}$	state of $\tau_{k:n}(\mathbf{X})$ at time t
$\phi_{\tau_{k:n}(\mathbf{x} \vee \mathbf{y})}$	state of $\tau_{k:n}(\mathbf{X} \vee \mathbf{Y})$ at time t
$\phi_{\tau_{k:n}(\mathbf{x}) \vee \tau_{k:n}(\mathbf{y})}$	state of $\tau_{k:n}(\mathbf{X}) \vee \tau_{k:n}(\mathbf{Y})$ at time t
$\phi_{\tau_{k:n}(\mathbf{x} + \mathbf{y})}$	state of $\tau_{k:n}(\mathbf{X} + \mathbf{Y})$ at time t
$\phi_{\tau_{k:n}(\mathbf{x}) + \tau_{k:n}(\mathbf{y})}$	state of $\tau_{k:n}(\mathbf{X}) + \tau_{k:n}(\mathbf{Y})$ at time t
$\bar{E}_C(\cdot)$	empirical reliability function for component redundancy
$\bar{E}_S(\cdot)$	empirical reliability function for system redundancy
$GP(k, \sigma, \theta)$	Generalized Pareto distribution with reliability function given by $\bar{F}_X(t) = (1 + k(t-\theta)/(\sigma))^{-1/k}$, $t > \theta,\ k > 0,\ \sigma > 0,\ \theta > 0$.

Nomenclature

$\bar{F}_X(\cdot)$	$1 - F_X(\cdot)$
$k'(t)$	first derivative of $k(t)$ with respect to t

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$k''(t)$	second derivative of $k(t)$ with respect to t
$a \overset{\text{sign}}{=} b$	a and b have the same sign
$X =_{st} Y$	X and Y have the same distribution
$X =_{sp} Y$	$P(X > Y) = P(Y > X)$
iid	statistically independent and identically distributed

I. Introduction

We can enhance the life of a system by incorporating spares (or redundants) into the system. Then, the real problem is where to allocate spares into the system so that the lifetime of the system is maximized. Design engineers generally believe that redundancy at the component level is more effective than redundancy at the system level. Redundancy is mostly of two types: active redundancy (or parallel redundancy), and standby redundancy. In active redundancy, the original component and the redundant component work together so that the life of the system is the maximum of the lives of the original component and the redundant component. In standby redundancy, the redundant component starts to function only when the original component fails. Thus, system life is the sum of the lives of the original component and the redundant component. In this paper, we discuss both active redundancy as well as standby redundancy. Both types of redundancies are extensively studied in the literature; see, for example, Liang and Chen [29], Li and Hu [27], Misra *et al.* ([22], [23]), Li *et al.* [28], and the references there-in. In the case of active redundancy, Barlow and Proschan [5], Boland and El-Neweihi [1], Singh and Singh [19], Gupta and Nanda [3], Misra *et al.* [2], Brito *et al.* [14], Nanda and Hazra [11], and many other researchers have shown that component redundancy is superior to system redundancy in different stochastic orders. On the other hand, very few results have been developed for standby redundancy. The significant work in this direction is obtained in Boland and El-Neweihi [1].

Suppose we have two different systems, and we want to compare their reliabilities. Then, the key question is how to determine that one system is more reliable than the other. Stochastic orders are an effective solution to this problem because once the distribution functions of two lifetime random variables are known, stochastic orders use the complete information available regarding the underlying random variables through its distribution, whereas the other kind of comparison (say, in terms of means and variances) does not utilize the complete information as available with the distributions. So it is quite natural that stochastic orders will give a better method of comparison than using means or variances. In literature, many different types of stochastic orders have been defined. Each stochastic order has its own importance for comparison. For example, usual stochastic order compares two reliability functions, hazard rate order compares two failure rate functions, reversed hazard rate order compares two reversed failure rate functions, whereas likelihood ratio order is used as a sufficient condition for the above mentioned orders to hold. For modeling in terms of failure rate, one may refer to Finkelstein [21]. The following well known definitions may be obtained in Shaked and Shanthikumar [8].

Definition 1: Let X and Y be two absolutely continuous random variables with respective supports (l_X, u_X) and (l_Y, u_Y), where u_X and u_Y may be positive infinity, and l_X and l_Y may be negative infinity.

1. *X is said to be smaller than Y in likelihood ratio (lr) order, denoted as $X \leq_{lr} Y$, if*
$$P(X \in B)P(Y \in A) \leq P(X \in A)P(Y \in B)$$
for all measurable sets A and B such that $A \leq B$, where $A \leq B$ means that for all $x \in A$ and $y \in B$, we have $x \leq y$. This can equivalently be written as
$$\frac{f_Y(t)}{f_X(t)} \text{ is increasing in } t \in (l_X, u_X) \cup (l_Y, u_Y).$$
2. *X is said to be smaller than Y in hazard rate (hr) order, denoted as $X \leq_{hr} Y$, if*
$$\frac{\bar{F}_Y(t)}{\bar{F}_X(t)} \text{ is increasing in } t \in (-\infty, \max(u_X, u_Y)),$$
which can equivalently be written as $r_X(t) \geq r_Y(t)$ for all t.
3. *X is said to be smaller than Y in reversed hazard rate (rhr) order, denoted as $X \leq_{rhr} Y$, if*
$$\frac{F_Y(t)}{F_X(t)} \text{ is increasing in } t \in (\min(l_X, l_Y), \infty),$$
which can equivalently be written as $\tilde{r}_X(t) \leq \tilde{r}_Y(t)$ for all t.
4. *X is said to be smaller than Y in the usual stochastic (st) order, denoted as $X \leq_{st} Y$, if $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for all $t \in (-\infty, \infty)$.* □

Many applications of convolution operation are found in different areas of mathematics and engineering. It is of interest to know whether different stochastic orders are preserved under convolution. It is well known that the likelihood ratio order is closed under convolution of independent random variables if the random variables under consideration have log-concave density functions. Shanthikumar and Yao [20] have introduced shifted likelihood ratio order which is preserved under convolution without log-concavity condition. Later, Lillo *et al.* [4] have defined some other shifted stochastic orders. These orders are frequently used to study different stochastic inequalities. Many properties of these orders are studied by different authors, viz. Nakai [26], Belzunce *et al.* [13], Lillo *et al.* [4], Hu and Zhu [9], and the references there-in. Below we give the formal definitions of shifted stochastic orders (cf. Lillo *et al.* [4], Shaked and Shanthikumar [8], and Hu and Zhu [9]).

Definition 2: Let X and Y be two random variables as defined above in Definition 1.

1. *X is said to be smaller than Y in up shifted likelihood ratio (lr ↑) order, denoted as $X \leq_{lr\uparrow} Y$, if $X - x \leq_{lr} Y$, for all $x \geq 0$. This can equivalently be written as*
$$\frac{f_Y(t)}{f_X(t+x)} \text{ is increasing in } t \in (l_X - x, u_X - x) \cup (l_Y, u_Y),$$
for all $x \geq 0$.

2. *X is said to be smaller than Y in down shifted likelihood ratio (lr $\downarrow$) order, denoted as $X \le_{lr\downarrow} Y$, if $X \le_{lr} [Y - x|Y > x]$, for all $x \ge 0$, or equivalently, if*
$$\frac{f_Y(t+x)}{f_X(t)} \text{ is increasing in } t \ge 0,$$
for all $x \ge 0$.
3. *X is said to be smaller than Y in up shifted hazard rate (hr $\uparrow$) order, denoted as $X \le_{hr\uparrow} Y$, if $X - x \le_{hr} Y$, for all $x \ge 0$, which can equivalently be written as*
$$\frac{\bar{F}_Y(t)}{\bar{F}_X(t+x)} \text{ is increasing in } t \in (-\infty, u_Y),$$
for all $x \ge 0$.
4. *X is said to be smaller than Y in down shifted hazard rate (hr $\downarrow$) order, denoted as $X \le_{hr\downarrow} Y$, if $X \le_{hr} [Y - x|Y > x]$, for all $x \ge 0$, or equivalently, if*
$$\frac{\bar{F}_Y(t+x)}{\bar{F}_X(t)} \text{ is increasing in } t \ge 0,$$
for all $x \ge 0$.
5. *X is said to be smaller than Y in up shifted reversed hazard rate (rhr $\uparrow$) order, denoted as $X \le_{rhr\uparrow} Y$, if $X - x \le_{rhr} Y$, for all $x \ge 0$, or equivalently, if*
$$\frac{F_Y(t)}{F_X(t+x)} \text{ is increasing in } t \in (l_X, \infty),$$
for all $x \ge 0$. □

In the following diagrams, we present a chain of implications of the stochastic orders (cf. Shaked and Shanthikumar [8], and Lillo *et al.* [4]). The first (resp. second) diagram shows that the up (resp. down) shifted likelihood ratio order is the most strongest order, whereas the usual stochastic order is the weakest one, and other orders lie between these two orders. There is no relation between up and down shifted orders.

$$\begin{array}{ccccc} X \le_{hr\uparrow} Y & \rightarrow & X \le_{hr} Y & & \\ \uparrow & & \uparrow & \searrow & \\ X \le_{lr\uparrow} Y & \rightarrow & X \le_{lr} Y & \rightarrow & X \le_{st} Y \\ \downarrow & & \downarrow & \nearrow & \\ X \le_{rhr\uparrow} Y & \rightarrow & X \le_{rhr} Y & & \end{array}$$

$$\begin{array}{ccccc} X \le_{hr\downarrow} Y & \rightarrow & X \le_{hr} Y & & \\ \uparrow & & \uparrow & \searrow & \\ X \le_{lr\downarrow} Y & \rightarrow & X \le_{lr} Y & \rightarrow & X \le_{st} Y \\ & & \downarrow & \nearrow & \\ & & X \le_{rhr} Y & & \end{array}.$$

Like stochastic orders, stochastic precedence order is also a very useful tool to compare the lifetimes of two systems. To know more details about stochastic precedence order, see Singh and Misra [18], Boland and Singh and Cukic [24], and Romera and Valdés and Zequeira [25]. Usual stochastic order does not always imply stochastic precedence order. If the two random variables are statistically dependent, then there is no implication between usual stochastic order and stochastic precedence order.

Definition 3: Let X and Y be two absolutely continuous random variables. Then, X is said to be greater than Y in stochastic precedence (sp) order, denoted as $X \ge_{sp} Y$, if
$$P(X > Y) \ge P(Y > X).$$

Design engineers always like to design systems which satisfy two basic requirements. First, each of the system's components should have some contribution to run the system. Second, if we replace a failed component by good one, then the system life must increase. On the basis of these two fundamental considerations, design engineers created a term to specify such a system, which is coherent system. Another well known system is the k-out-of-n:G system, which is a special type of coherent system. If there is no ambiguity, then we simply write it as a k-out-of-n system. Two extreme cases of the k-out-of-n system are the n-out-of-n system, known as a series system, and the 1-out-of-n system, known as a parallel system. For definitions of coherent systems and k-out-of-n systems, one may refer to Barlow and Proschan [5], and Samaniego [17].

Let us consider a coherent system τ formed by components $\mathbf{X}$. Further, let $\mathbf{x}(t) \in \{0,1\}^n$ be the state vector of $\mathbf{X}$, where $x_i(t) = 1$ if the ith component is working, and $x_i(t) = 0$ if it is not working, at time t. Without any loss of generality, we write $\mathbf{x}$ in place of $\mathbf{x}(t)$, for mathematical simplicity, when there is no ambiguity. Then, the state of $\tau(\mathbf{X})$ at time t, is defined as
$$\phi_{\tau(\mathbf{x})} = \begin{cases} 1, & \text{if the system is functioning} \\ 0, & \text{if the system is failed,} \end{cases}$$
and its reliability function is defined as the probability that it is working at time t. Thus,
$$P(\tau(\mathbf{X}) > t) = P(\phi_{\tau(\mathbf{X})} = 1).$$
If the components are statistically independent, then the system reliability can be written as a function of component reliabilities, and hence
$$\begin{aligned} P(\tau(\mathbf{X}) > t) &= h\left(\bar{F}_{X_1}(t), \bar{F}_{X_2}(t), \ldots, \bar{F}_{X_n}(t)\right) \\ &= h(p_1, p_2, \ldots, p_n) \\ &= h(\mathbf{p}), \end{aligned}$$
where $p_i = \bar{F}_{X_i}(t)$, $i = 1, 2, \ldots, n$, and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. We write $h(p)$ in place of $h(\mathbf{p})$ whenever components are statistically identical. In contrast to the dichotomous state system as defined above, a multistate system is also possible; see, for instance, El-Neweihi *et al.* [15], and Zaitseva and Levashenko [16]. Further, let $Y_1, Y_2, \ldots, Y_n$ be the random variables representing the lifetimes of n spares. Clearly, $\boldsymbol{\tau}(\mathbf{X} \vee \mathbf{Y})$ (resp. $\tau(\mathbf{X} + \mathbf{Y})$) represents the life of a coherent system with active (resp. standby) redundancy at the component level, whereas $\tau(\mathbf{X}) \vee \tau(\mathbf{Y})$ (resp. $\tau(\mathbf{X}) + \tau(\mathbf{Y})$) represents that of a coherent system with active (resp. standby) redundancy at the system level.

It is natural that the components and the spares may not be identical. By matching spares, we mean that the distribution of the lifetime of a spare is identical with that of the corresponding original component; and if the spare and its corresponding original component do not have the same distribution, they are called non-matching spares. It is to be mentioned here that most of the results discussed in this paper can be generalized to any random variable with obvious modification.

The rest of the paper is organized as follows. In Section II, we discuss active redundancy. In Section II.A, all the results are proved for matching spares of iid components and iid spares. We prove that, under some sufficient conditions, redundancy at the component level is superior to that at the system level with respect to the up shifted hazard rate, the up shifted reversed hazard rate, and the up shifted likelihood ratio orders, for any coherent system. In Section II.B, we talk about matching spares of not necessarily iid components and spares. We prove that component redundancy is more reliable than system redundancy in up shifted hazard rate order, for series system. In Section II.C, we discuss non-matching spares of iid components and iid spares. We give sufficient conditions under which component redundancy offers better reliability than system redundancy with respect to the up shifted reversed hazard rate order, for any coherent system. In Section II.D, we discuss non-matching spares of not necessarily iid components and iid spares. We prove that redundancy at the component level is better than redundancy at the system level with respect to the stochastic precedence order, for k-out-of-n systems. We also show that, under some sufficient conditions, this principle holds in the hazard and the reversed hazard rate orders, for 2-out-of-2 systems. In Section III, we discuss standby redundancy. We show that component redundancy is more reliable than system redundancy in stochastic precedence order, for k-out-of-n systems. In Section IV, we discuss some simulation results. The work is summarized in Section V.

Throughout the manuscript, when we write a function to be increasing it means that the function may be constant in some parts of the domain and strictly increasing in other parts. Similar convention is followed for a decreasing function.

II. Active Redundancy

In this section, we discuss active redundancy. We show that redundancy at the component level is more reliable than that at the system level with respect to different stochastic orderings.

A. Matching Spares of IID Components and IID Spares

Gupta and Nanda [3] proved that, for a coherent system, component redundancy offers better reliability than system redundancy in reversed hazard rate order. In Theorem 1, we extent this result for up shifted reversed hazard rate order. Below we give two lemmas to be used in proving the theorem. The first lemma is analogous to Theorem 6.19 of Lillo *et al.* [4]. The proof is omitted.

Lemma 1: Let X and Y be two absolutely continuous random variables with interval supports. If X or Y have log-concave distribution functions, and if $X \geq_{rhr} Y$, then $X \geq_{rhr\uparrow} Y$. □

Lemma 2: Let $h(\cdot)$ be the reliability function of a coherent system of n iid components. Further, let components have log-concave distribution functions. If $(1-p)h'(p)/(1-h(p))$ is increasing in p, then the lifetime of the coherent system has a log-concave distribution function.

Proof: Let the reliability function, and the probability density function of a component having lifetime X be $\bar{F}_X(\cdot)$, and $f_X(\cdot)$, respectively. Note that the reversed hazard rate of the coherent system is given by $f_X(t)h'(\bar{F}_X(t))/(1-h(\bar{F}_X(t)))$. Again, we know that the lifetime of a system has a log-concave distribution function iff its reversed hazard rate function is decreasing. Therefore, to prove the result, it suffices to show that $f_X(t)h'(\bar{F}_X(t))/\left(1-h(\bar{F}_X(t))\right)$ is decreasing in t. We see that

$$\frac{f_X(t)h'(\bar{F}_X(t))}{\left(1-h(\bar{F}_X(t))\right)} = \frac{(1-\bar{F}_X(t))h'(\bar{F}_X(t))}{\left(1-h(\bar{F}_X(t))\right)}\frac{f_X(t)}{1-\bar{F}_X(t)}.$$

Because $(1-p)h'(p)/(1-h(p))$ is increasing in p, it follows that $(1-\bar{F}_X(t))h'(\bar{F}_X(t))/(1-h(\bar{F}_X(t)))$ is decreasing in t. Also, from the hypothesis, we have $f_X(t)/(1-\bar{F}_X(t))$ is decreasing in t. Hence, the result follows. □

Theorem 1 below shows that, under some sufficient conditions, $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{rhr\uparrow} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$ holds.

Theorem 1: Let $X_1, X_2, \ldots, X_n$ be iid component lifetimes, and $Y_1, Y_2, \ldots, Y_n$ be those of iid spares, all having log-concave distribution functions. Further, let X_i and Y_i be statistically independent, and $X_i =_{st} Y_i$, $i = 1, 2, \ldots, n$. Suppose that either or both of (a) or (b), and (c) hold.

(a) $h(p) \leq p$ *for all* $p \in [0,1]$.
(b) $(1-h(p))\,h'(p)/\left((1-p)h'\left(2p-p^2\right)\right)$ *is increasing in* $p \in (0,1)$.
(c) $(1-p)h'(p)/(1-h(p))$ *is increasing in* $p \in (0,1)$.

Then, $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{rhr\uparrow} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$.

Proof: Because the conditions (*a*) and (*c*) hold, by Theorem 3.2 of Gupta and Nanda [3] we have $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{rhr} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$. If (*b*) holds, then also $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{rhr} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$ (see, Theorem 3.2 of Misra *et al.* [2]). Further, $\tau(\mathbf{X} \vee \mathbf{Y})$ has a log-concave distribution function because of Lemma 2. Hence, from Lemma 1, it follows that $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{rhr\uparrow} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$. □

Remark 1: Note that condition (a) holds for a series system where each component is a k_j-out-of-n_j subsystem, and at least one subsystem consists of a single component (see, Remark 3.2 of Gupta and Nanda [3]), whereas conditions (b) and (c) are satisfied for any k-out-of-n system (see Corollary 3.2 of Misra et al. [2], and Corollary 2.1 of Nanda et al. [7]). Thus, the above result is valid for k-out-of-n systems. □

The following counterexample shows that the log-concavity condition given in Theorem 1 cannot be relaxed.

Counterexample 1: Let X_1, X_2, Y_1, and Y_2 be iid random variables with cumulative distribution function given by

$$F_{X_1}(t) = \begin{cases} \frac{t^3}{26} & \text{for } 0 \leq t \leq 1 \\ \frac{t^5-2t^2+2}{26} & \text{for } 1 \leq t \leq 2 \\ 1 & \text{for } t \geq 2, \end{cases}$$

which is not log-concave. Note that, for $1 \leq t \leq 2$,

$$F_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t) = \left(\frac{t^5-2t^2+2}{26}\right)^2\left(2-\left(\frac{t^5-2t^2+2}{26}\right)^2\right),$$

and

$$F_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t) = \left(1-\left(\frac{24+2t^2-t^5}{26}\right)^2\right)^2.$$

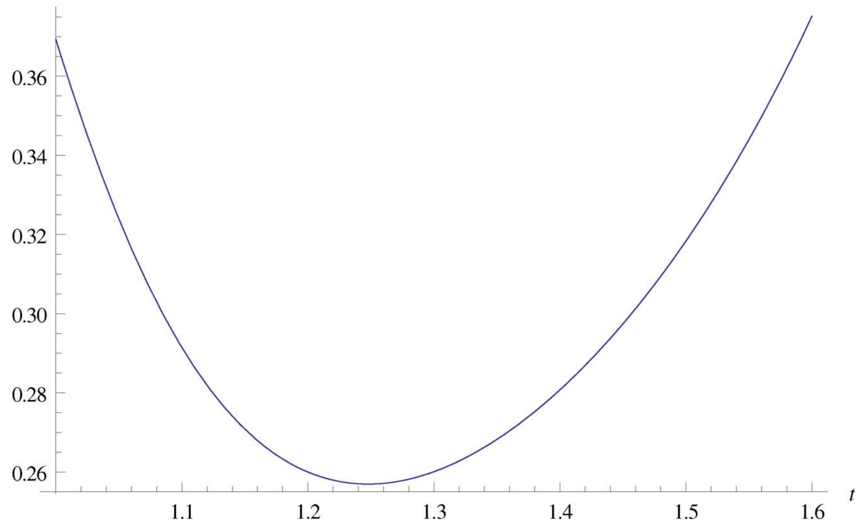

Fig. 1. Plot of $k_1(t)$ against $t \in [1, 1.6]$ (Counterexample 1).

Writing $k_1(t) = F_{\tau_{2:2}(\mathbf{X} \vee \mathbf{Y})}(t)/F_{\tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})}(t+0.1)$, *we get*

$$k_1(t) = \frac{(t^5 - 2t^2 + 2)^2 \left(1352 - (t^5 - 2t^2 + 2)^2\right)}{\left(676 - (24 + 2(t+0.1)^2 - (t+0.1)^5)^2\right)^2}.$$

From Fig. 1, it is clear that $k_1(t)$ *is not monotone. Hence,* $\tau_{2:2}(\mathbf{X} \vee \mathbf{Y}) \ngeq_{rhr\uparrow} \tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})$. □

That component redundancy for a coherent system is superior to system redundancy in hazard rate order is shown by Boland and El-Neweihi [1]. In the following theorem, we generalize this result for the up shifted hazard rate order. Below we give two lemmas to be used in proving the upcoming theorem. The first lemma is taken from Lillo *et al.* [4].

Lemma 3: Let X *and* Y *be two absolutely continuous random variables with interval supports. If* X *or* Y *have log-concave survival functions, and if* $X \geq_{hr} Y$, *then* $X \geq_{hr\uparrow} Y$. □

Lemma 4: Let $h(\cdot)$ *be the reliability function of a coherent system of* n *iid components. Further, let components have log-concave survival functions. If* $ph'(p)/h(p)$ *is decreasing in* p, *then the lifetime of the coherent system has a log-concave survival function.*

Proof: Let the reliability function of a component having lifetime X be $\bar{F}_X(\cdot)$. Write $\alpha_1(u) = \log h(e^u)$ for $u \in (-\infty, 0]$. Because $ph'(p)/h(p)$ is positive and decreasing in p, it follows that $\alpha_1(u)$ is increasing and concave in u. Further, write $\xi(t) = \log \bar{F}_X(t)$, so that $\alpha_1(\xi(t)) = \log h(\bar{F}_X(t))$, which is increasing and concave in ξ. Now, differentiating $\alpha_1(\xi(t))$ twice with respect to t, we get

$$\frac{d^2\alpha_1(\xi(t))}{dt^2} = \left(\frac{d^2\alpha_1(\xi)}{d\xi^2}\right)\left(\frac{d\xi(t)}{dt}\right)^2 + \left(\frac{d\alpha_1(\xi)}{d\xi}\right)\left(\frac{d^2\xi(t)}{dt^2}\right) \leq 0.$$

The inequality follows because $\alpha_1(\cdot)$ is increasing and concave as shown above, and $\xi(\cdot)$ is concave by the hypothesis. Thus, $h(\bar{F}_X(t))$ is log-concave in t. □

In the following theorem, we prove that, under some sufficient conditions, $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{hr\uparrow} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$.

Theorem 2: Let $X_1, X_2, \ldots, X_n$ *be iid component lifetimes, and* $Y_1, Y_2, \ldots, Y_n$ *be those of iid spares, all having log-concave survival functions. Further, let* X_i *and* Y_i *be statistically independent, and* $X_i =_{st} Y_i, i = 1, 2, \ldots, n$. *Suppose that* ($a$) *or* ($b$) *or both, and* ($c$) *hold.*

(a) $h(p) \leq p$ *for all* $p \in [0, 1]$.

(b) $(1 - h(p))\, h'(p)/\left((1-p)h'\left(2p - p^2\right)\right)$ *is increasing in* $p \in (0, 1)$.

(c) $ph'(p)/h(p)$ *is decreasing in* $p \in (0, 1)$.

Then $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{hr\uparrow} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$.

Proof: Because the conditions (a) and (c) hold, by Theorem 2 of Boland and El-Neweihi [1], we have $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{hr} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$. If ($b$) holds, then also $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{hr} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$ (see, Theorem 3.2 of Misra *et al.* [2]). Further, $\tau(\mathbf{X} \vee \mathbf{Y})$ has a log-concave survival function because of Lemma 4. Hence, the result follows from Lemma 3. □

Remark 2: In Remark 1, it is mentioned that condition (a) *holds for some special type of series system. The conditions* (b) *and* (c) *hold for any* k*-out-of-*n *system (see, Corollary* 3.2 *of Misra et al. [2], and Barlow and Proschan [5, p. 109], respectively). Thus, the above result is valid for* k*-out-of-*n *systems.* □

That Theorem 2 does not hold without the log-concavity condition is verified by the following counterexample.

Counterexample 2: Let X_1, X_2, Y_1, *and* Y_2 *be iid random variables with survival functions given by* $\bar{F}_{X_1}(t) = e^{-t^{0.8}}$, $t > 0$, *which is not log-concave. Now, we have*

$$\bar{F}_{\tau_{2:2}(\mathbf{X} \vee \mathbf{Y})}(t) = e^{-2t^{0.8}}\left(2 - e^{-t^{0.8}}\right)^2,$$

and

$$\bar{F}_{\tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})}(t) = 2e^{-2t^{0.8}} - e^{-4t^{0.8}}.$$

Writing $k_2(t) = \bar{F}_{\tau_{2:2}(\mathbf{X} \vee \mathbf{Y})}(t)/\bar{F}_{\tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})}(t+1)$, *we get*

$$k_2(t) = \frac{e^{-2t^{0.8}}\left(2 - e^{-t^{0.8}}\right)^2}{2e^{-2(t+1)^{0.8}} - e^{-4(t+1)^{0.8}}}.$$

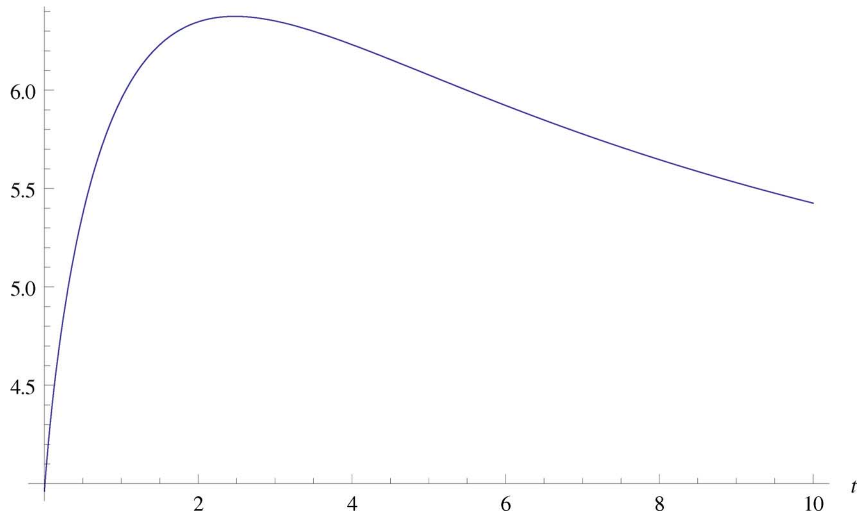

Fig. 2. Plot of $k_2(t)$ against $t \in [0, 10]$ (Counterexample 2).

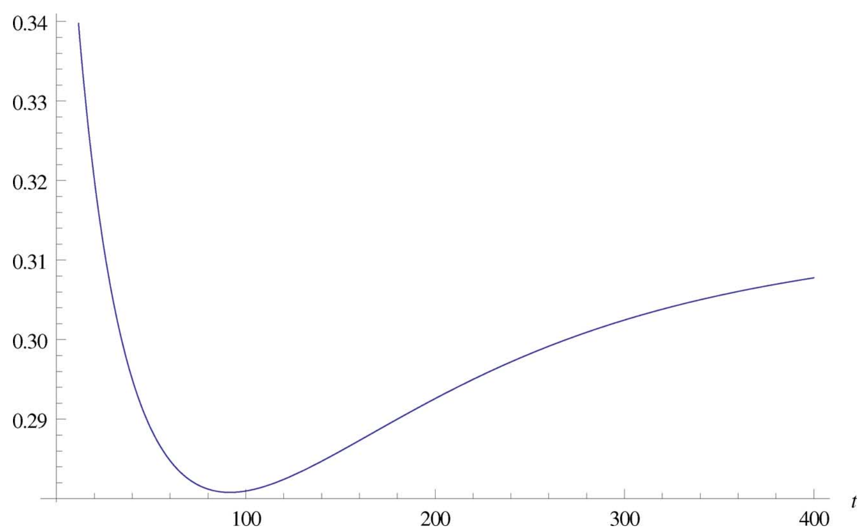

Fig. 3. Plot of $k_3(t)$ against $t \in [0, 400]$ (Counterexample 3).

From Fig. 2, see that $k_2(t)$ is not monotone in $t > 0$. Thus, $\tau_{2:2}(\mathbf{X} \vee \mathbf{Y}) \ngeq_{hr\uparrow} \tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})$. □

The next counterexample shows that a result similar to Theorem 2 does not hold for down shifted hazard rate order even if components have log-convex survival functions.

Counterexample 3: Let X_1, X_2, Y_1, and Y_2 be iid random variables with survival functions given by $\bar{F}_{X_1}(t) = e^{-0.01t^{0.99}}$, $t > 0$, which is log-convex. We see that

$$\bar{F}_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t) = e^{-0.02t^{0.99}} \left(2 - e^{-0.01t^{0.99}}\right)^2,$$

and

$$\bar{F}_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t) = 2e^{-0.02t^{0.99}} - e^{-0.04t^{0.99}}.$$

Writing $k_3(t) = \bar{F}_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t+100)/\bar{F}_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t)$, we get

$$k_3(t) = \frac{e^{-0.02(t+100)^{0.99}} \left(2 - e^{-0.01(t+100)^{0.99}}\right)^2}{2e^{-0.02t^{0.99}} - e^{-0.04t^{0.99}}}.$$

Fig. 3 shows that $k_3(t)$ is not monotone in $t > 0$. Hence, $\tau_{2:2}(\mathbf{X} \vee \mathbf{Y}) \ngeq_{hr\downarrow} \tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})$. □

It is well known that the likelihood ratio order is stronger than both hazard and reversed hazard rate orders. Again, up shifted likelihood ratio order implies likelihood ratio order. Misra *et al.* [2] proved that for a coherent system, component redundancy is more reliable than system redundancy in likelihood ratio order under certain conditions. In the following theorem we show that this principle holds for up shifted likelihood ratio order. Before going to the details of the next theorem, we give two lemmas. The first lemma is taken from Lillo *et al.* [4]. The proof of the second lemma, although done by us, comes from an idea due to Belzunce *et al.* ([12], Theorem 6.4).

Lemma 5: Let X and Y be two absolutely continuous random variables with interval supports. If X or Y or both have log-concave densities, and if $X \geq_{lr} Y$, then $X \geq_{lr\uparrow} Y$. □

Lemma 6: Let $h(\cdot)$ be the reliability function of a coherent system of n iid components. Further, let components have log-concave densities. If there exists a point $\mu \in [0, 1]$ such that

(a) *$ph''(p)/h'(p)$ is decreasing and positive for all $p \in [0, \mu)$, and*
(b) *$(1-p)h''(p)/h'(p)$ is decreasing and negative for all $p \in (\mu, 1]$,*

then the lifetime of the coherent system has log-concave density.

Proof: Let the reliability function, the probability density function, the hazard rate function, and the reversed hazard rate function of a component having lifetime X be $\bar{F}_X(\cdot)$, $f_X(\cdot)$, $r_X(\cdot)$, and $\tilde{r}_X(\cdot)$, respectively. Note that the lifetime of the coherent system has log-concave density iff

$$\frac{1}{h'\left(\bar{F}_X(t)\right) f_X(t)} \frac{d}{dt}\left(h'\left(\bar{F}_X(t)\right) f_X(t)\right) \text{ is decreasing in } t,$$

or equivalently,

$$\frac{f'_X(t)}{f_X(t)} - \frac{f_X(t)h''(\bar{F}_X(t))}{h'(\bar{F}_X(t))} \text{ is decreasing in } t.$$

Now, we see that

$$\begin{aligned}\frac{f'_X(t)}{f_X(t)} &- \frac{f_X(t)h''(\bar{F}_X(t))}{h'(\bar{F}_X(t))} \\ &= \frac{f'_X(t)}{f_X(t)} - r_X(t)\left(\frac{\bar{F}_X(t)h''\left(\bar{F}_X(t)\right)}{h'\left(\bar{F}_X(t)\right)}\right) \quad (1)\\ &= \frac{f'_X(t)}{f_X(t)} - \tilde{r}_X(t)\left(\frac{\left(1-\bar{F}_X(t)\right)h''\left(\bar{F}_X(t)\right)}{h'\left(\bar{F}_X(t)\right)}\right). \quad (2)\end{aligned}$$

Because $\log f_X(t)$ is concave, it implies that

$$\frac{f'_X(t)}{f_X(t)} \text{ is decreasing in } t. \quad (3)$$

Again, $r_X(t)$ is increasing in t, and $\tilde{r}_X(t)$ is decreasing in t because $\log f_X(t)$ is concave in t (see Lemma 5.8 of Barlow and Proschan [5], and Proposition 1(e) of Sengupta and Nanda [6]). Let us consider the following two cases.

Case I: Let $p \in [0, \mu)$. Because condition (a) holds, it follows that $\bar{F}_X(t)h''\left(\bar{F}_X(t)\right)/h'(\bar{F}_X(t))$ is increasing and positive for all t. Therefore,

$$r_X(t)\left(\frac{\bar{F}_X(t)h''\left(\bar{F}_X(t)\right)}{h'\left(\bar{F}_X(t)\right)}\right) \text{ is increasing in } t. \quad (4)$$

Hence, from (3) and (4), we get that

$$\frac{f'_X(t)}{f_X(t)} - r_X(t)\left(\frac{\bar{F}_X(t)h''\left(\bar{F}_X(t)\right)}{h'\left(\bar{F}_X(t)\right)}\right) \text{ is decreasing in } t.$$

Thus, the result follows from (1).

Case II: Let $p \in (\mu, 1]$. Then, $\left(1-\bar{F}_X(t)\right)h''\left(\bar{F}_X(t)\right)/h'(\bar{F}_X(t))$ is increasing and negative for all t because condition (b) holds. Therefore,

$$\tilde{r}_X(t)\left(-\frac{\left(1-\bar{F}_X(t)\right)h''\left(\bar{F}_X(t)\right)}{h'\left(\bar{F}_X(t)\right)}\right) \text{ is decreasing in } t. \quad (5)$$

Hence, from (3) and (5), we have that

$$\frac{f'_X(t)}{f_X(t)} - \tilde{r}_X(t)\left(\frac{\left(1-\bar{F}_X(t)\right)h''\left(\bar{F}_X(t)\right)}{h'\left(\bar{F}_X(t)\right)}\right) \text{ is decreasing in } t.$$

Thus, the result follows from (2). □

Theorem 3 below shows that, under some sufficient conditions, $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{lr\uparrow} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$.

Theorem 3: Let $X_1, X_2, \ldots, X_n$ be iid component lifetimes, and $Y_1, Y_2, \ldots, Y_n$ be those of iid spares, all having log-concave densities. Further, let X_i and Y_i be statistically independent, and $X_i =_{st} Y_i, i = 1, 2, \ldots, n$. Suppose three conditions hold.

(a) *$(1-h(p))\,h'(p)/\left((1-p)h'\left(2p-p^2\right)\right)$ is increasing in $p \in (0,1)$, and for some point $\mu \in [0,1]$,*
(b) *$ph''(p)/h'(p)$ is decreasing and positive for all $p \in [0, \mu)$, and*
(c) *$(1-p)h''(p)/h'(p)$ is decreasing and negative for all $p \in (\mu, 1]$.*

Then $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{lr\uparrow} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$.

Proof: Because $(1-h(p))\,h'(p)/\left((1-p)h'\left(2p-p^2\right)\right)$ is increasing in p, by Theorem 3.2 of Misra *et al.* [2] we have $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{lr} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$. Again, $\tau(\mathbf{X} \vee \mathbf{Y})$ has a log-concave density because of Lemma 6. Hence, from Lemma 5, it follows that $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{lr\uparrow} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$. □

Remark 3: It is to be noted that all the conditions (a), (b), and (c) given in Theorem 3 are satisfied by a k-out-of-n system (cf. Corollary 3.2 of Misra et al. [2], and proof of Corollary 6.6 of Belzunce et al. [12]).

In the following counterexample, we show that Theorem 3 does not hold without the log-concavity condition.

Counterexample 4: Let X_1, X_2, Y_1, and Y_2 be iid random variables with probability density $f_{X_1}(t) = 0.9t^{-0.10}e^{-t^{0.90}}$, $t > 0$, which is not log-concave. Note that, for $t > 0$,

$$f_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t) = 3.6t^{-0.10}e^{-2t^{0.90}}\left(1-e^{-t^{0.90}}\right)\left(2-e^{-t^{0.90}}\right),$$

and

$$f_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t) = 3.6t^{-0.10}e^{-2t^{0.90}}\left(1-e^{-2t^{0.90}}\right).$$

Writing $k_4(t) = f_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t)/f_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t+30)$, we have

$$k_4(t) = \frac{t^{-0.10}e^{-2t^{0.90}}\left(1-e^{-t^{0.90}}\right)\left(2-e^{-t^{0.90}}\right)}{(t+30)^{-0.10}e^{-2(t+30)^{0.90}}\left(1-e^{-2(t+30)^{0.90}}\right)}.$$

From Fig. 4, it is clear that $k_4(t)$ is not monotone in $t > 0$. Hence, $\tau_{2:2}(\mathbf{X} \vee \mathbf{Y}) \ngeq_{lr\uparrow} \tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})$. □

One may wonder whether a result analogous to Theorem 3 holds for down shifted likelihood ratio order. We answer this question in negative through the following counterexample.

Counterexample 5: Let X_1, X_2, Y_1, and Y_2 be iid random variables with survival functions given by $\bar{F}_{X_1}(t) = e^{-0.01t^{0.99}}$, $t > 0$. Clearly, X_i, Y_i, $i = 1, 2$, have log-convex densities. Note that, for $t > 0$,

$$f_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t) = 0.0396t^{-0.01}e^{-0.02t^{0.99}}\left(1-e^{-0.01t^{0.99}}\right)\left(2-e^{-0.01t^{0.99}}\right),$$

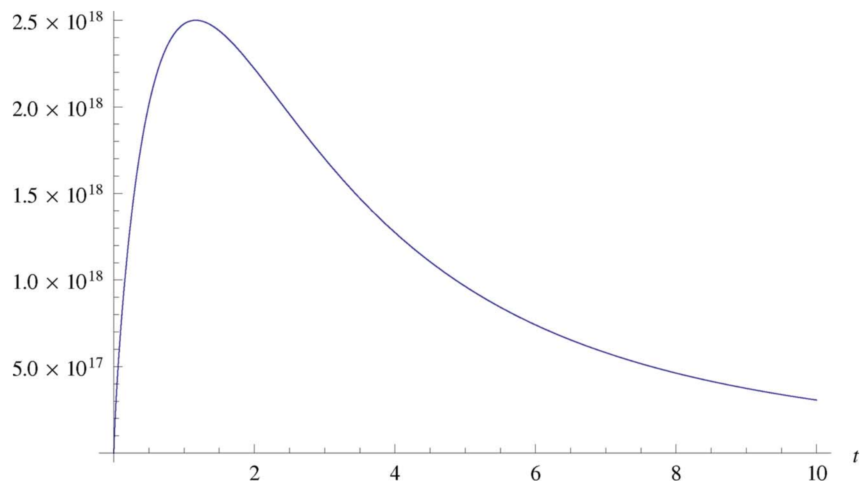

Fig. 4. Plot of $k_4(t)$ against $t \in [0, 10]$ (Counterexample 4).

and

$$f_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t) = 0.0396t^{-0.01}e^{-0.02t^{0.99}}\left(1 - e^{-0.02t^{0.99}}\right).$$

Writing $k_5(t) = f_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t+20)/f_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t)$, *we get,* see the equation at the bottom of the page. *From Fig. 5, it is clear that* $k_5(t)$ *is not monotone in* $t > 0$. *Hence,* $\tau_{2:2}(\mathbf{X} \vee \mathbf{Y}) \ngeq_{lr\downarrow} \tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})$. □

B. Matching Spares of Not Necessarily IID Components and IID Spares

Boland and El-Neweihi [1] have shown that, in the case of a series system, component redundancy is better than system redundancy in hazard rate order. In the next theorem, we generalize this result for up shifted hazard rate order. Below we give a well known lemma without proof to be used in proving the next theorem.

Lemma 7: If the components of a series system have statistically independent lifetimes with log-concave survival functions, then the system lifetime will have a log-concave survival function. □

Theorem 4: Let $X_1, X_2, \ldots, X_n$ *be statistically independent component lifetimes, and* $Y_1, Y_2, \ldots, Y_n$ *be those of statistically independent spares, all having log-concave survival functions. Further, let* X_i *and* Y_i *be statistically independent, and* $X_i =_{st} Y_i, i = 1, 2, \ldots, n$. *Then* $\tau_{n:n}(\mathbf{X}\vee\mathbf{Y}) \geq_{hr\uparrow} \tau_{n:n}(\mathbf{X})\vee\tau_{n:n}(\mathbf{Y})$.

Proof: Note that, for each $i = 1, 2, \ldots, n$, $\max\{X_i, Y_i\}$ has a log-concave survival function (cf. Theorem 4.1(a) of Nanda and Shaked [10]). Then, from Lemma 7, it follows that $\tau_{n:n}(\mathbf{X} \vee \mathbf{Y})$ has a log-concave survival function. Further, we have $\tau_{n:n}(\mathbf{X} \vee \mathbf{Y}) \geq_{hr} \tau_{n:n}(\mathbf{X}) \vee \tau_{n:n}(\mathbf{Y})$ (cf. Theorem 1 of Boland and El-Neweihi [1]). Therefore, $\tau_{n:n}(\mathbf{X} \vee \mathbf{Y}) \geq_{hr\uparrow} \tau_{n:n}(\mathbf{X}) \vee \tau_{n:n}(\mathbf{Y})$ which follows from Lemma 3. □

Remark 4: From Counterexample 2, it is clear that the log-concavity condition given in Theorem 4 cannot be relaxed, whereas Counterexample 3 shows that a result similar to Theorem 4 does not hold for down shifted hazard rate order. □

C. Non-Matching Spares of IID Components and IID Spares

Gupta and Nanda [3] proved that, for 2-component non-matching spares of iid components and iid spares, component redundancy is more reliable than system redundancy in reversed hazard rate order. Later, Misra *et al.* [2] generalize this result for a coherent system. We know that up shifted reversed hazard rate order is stronger than reversed hazard rate order. Thus, a natural question raises, which is whether the result holds for up shifted reversed hazard rate order. We answer this question in affirmative in the following theorem. But before stating the next theorem, we give a lemma which is taken from Sengupta and Nanda [6] to be used in proving the next theorem.

Lemma 8: Let $h(\cdot)$ *be the reliability function of a coherent system of* n *iid components. Further, let components have log-concave distribution functions. If* $ph'(p)/h(p)$ *is decreasing in* p*, then the lifetime of the coherent system has a log-concave distribution function.* □

Theorem 5: Let $X_1, X_2, \ldots, X_n$ *be iid component lifetimes, and* $Y_1, Y_2, \ldots, Y_n$ *be those of iid spares, all having log-concave distribution functions. Further, let* X_i *and* Y_i *be statistically independent,* $i = 1, 2, \ldots, n$. *Suppose that conditions* (a) *or* (b)*, and* (c) *hold.*

(a) $ph'(p)/h(p)$ *is decreasing in* $p \in (0, 1)$,

$$k_5(t) = \frac{(t+20)^{-0.01}e^{-0.02(t+20)^{0.99}}\left(1 - e^{-0.01(t+20)^{0.99}}\right)\left(2 - e^{-0.01(t+20)^{0.99}}\right)}{t^{-0.01}e^{-0.02t^{0.99}}\left(1 - e^{-0.02t^{0.99}}\right)}.$$

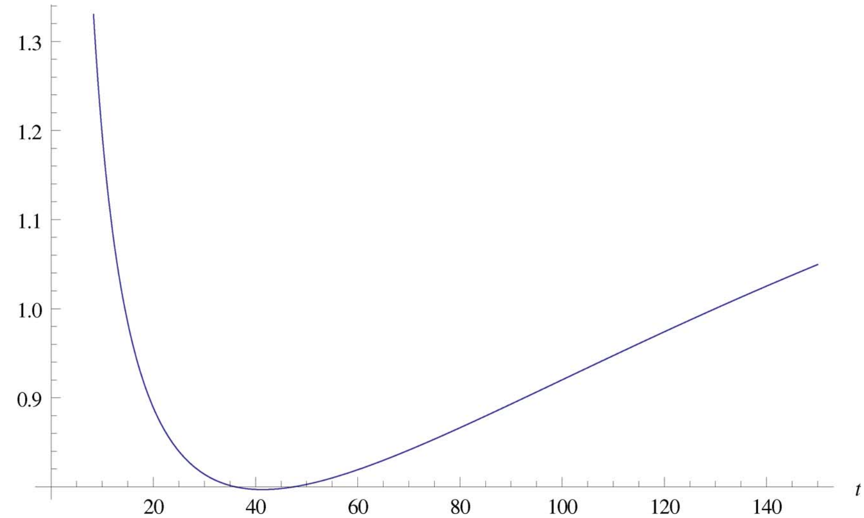

Fig. 5. Plot of $k_5(t)$ against $t \in [0, 150]$ (Counterexample 5).

(b) $(1-p)h'(p)/(1-h(p))$ *is increasing in* $p \in (0,1)$,
(c) *for any fixed* $q_2 \in (0,1)$, $h'(1-(1-q_1)q_2)/h'(q_1)$ *is decreasing in* $q_1 \in (0,1)$.

Then $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{rhr\uparrow} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$.

Proof: Because, for any $q_2 \in (0,1)$, $h'(1-(1-q_1)q_2)/h'(q_1)$ is decreasing in $q_1 \in (0,1)$, by Theorem 3.1 of Misra *et al.* [2], we have $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{rhr} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$. Again, both $\tau(\mathbf{X})$ and $\tau(\mathbf{Y})$ have log-concave distribution functions because of Lemma 8 (or Lemma 2). Consequently, $\tau(\mathbf{X}) \vee \tau(\mathbf{Y})$ has a log-concave distribution function (see Theorem $2(e)$ of Sengupta and Nanda [6]). Therefore, from Lemma 1, it folows that $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{rhr\uparrow} \tau(\mathbf{X}) \vee \tau(\mathbf{Y})$. ☐

Remark 5: Note that, for any k-out-of-n system, all the conditions (a), (b), *and* (c) *of Theorem 5 are validated (cf. Barlow and Proschan [5] on p.109, Corollary 2.1 of Nanda et al. [7], and Corollary 3.1 of Misra et al. [2]).* ☐

Remark 6: From Counterexample 1, it is seen that the log-concavity condition given in Theorem 5 cannot be removed. ☐

D. Non-Matching Spares of Not Necessarily IID Components and IID Spares

We are well aware that redundancy at the component level is always superior to that at the system level in usual stochastic order, for coherent systems (cf. Barlow and Proschan [5, p. 23]). In Theorem 6, we show that the same principle holds in stochastic precedence order, for k-out-of-n systems.

Theorem 6: Let $X_1, X_2, \ldots, X_n$ *be component lifetimes, and* $Y_1, Y_2, \ldots, Y_n$ *be those of spares. Further, let* X_i *and* Y_i, $i = 1, 2, \ldots, n$, *be statistically independent. Then*

$$\tau_{k:n}(\mathbf{X} \vee \mathbf{Y}) \geq_{sp} \tau_{k:n}(\mathbf{X}) \vee \tau_{k:n}(\mathbf{Y}) \text{ for } k = 2, 3, \ldots, n,$$

and

$$\tau_{1:n}(\mathbf{X} \vee \mathbf{Y}) =_{sp} \tau_{1:n}(\mathbf{X}) \vee \tau_{1:n}(\mathbf{Y}).$$

Proof: Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be the state vectors of $\mathbf{X}$, and $\mathbf{Y}$, respectively. We show that

$$\phi_{\tau_{k:n}(\mathbf{x}) \vee \tau_{k:n}(\mathbf{y})} > \phi_{\tau_{k:n}(\mathbf{x} \vee \mathbf{y})} \tag{6}$$

is never possible. To see this result, note that (6) holds iff

$$\phi_{\tau_{k:n}(\mathbf{x}) \vee \tau_{k:n}(\mathbf{y})} = 1, \quad \text{and} \quad \phi_{\tau_{k:n}(\mathbf{x} \vee \mathbf{y})} = 0.$$

Or equivalently, one of the following three systems of equations holds.

$$\begin{cases} \phi_{\tau_{k:n}(\mathbf{x})} = 1 \\ \phi_{\tau_{k:n}(\mathbf{y})} = 0 \\ \phi_{\tau_{k:n}(\mathbf{x} \vee \mathbf{y})} = 0 \end{cases}$$
$$\begin{cases} \phi_{\tau_{k:n}(\mathbf{x})} = 0 \\ \phi_{\tau_{k:n}(\mathbf{y})} = 1 \\ \phi_{\tau_{k:n}(\mathbf{x} \vee \mathbf{y})} = 0 \end{cases}$$
$$\begin{cases} \phi_{\tau_{k:n}(\mathbf{x})} = 1 \\ \phi_{\tau_{k:n}(\mathbf{y})} = 1 \\ \phi_{\tau_{k:n}(\mathbf{x} \vee \mathbf{y})} = 0. \end{cases}$$

Or equivalently, one of the following three systems of inequalities is satisfied.

$$\begin{cases} x_1 + x_2 + \cdots + x_n \geq k \\ y_1 + y_2 + \cdots + y_n \leq k-1 \\ \vee(x_1, y_1) + \vee(x_2, y_2) + \cdots + \vee(x_n, y_n) \leq k-1 \end{cases}$$
$$\begin{cases} x_1 + x_2 + \cdots + x_n \leq k-1 \\ y_1 + y_2 + \cdots + y_n \geq k \\ \vee(x_1, y_1) + \vee(x_2, y_2) + \cdots + \vee(x_n, y_n) \leq k-1 \end{cases}$$
$$\begin{cases} x_1 + x_2 + \cdots + x_n \geq k \\ y_1 + y_2 + \cdots + y_n \geq k \\ \vee(x_1, y_1) + \vee(x_2, y_2) + \cdots + \vee(x_n, y_n) \leq k-1, \end{cases}$$

where $x_i, y_i \in \{0,1\}$ for all $i = 1,2,\ldots,n$. It is easy to see that none of the above three systems has any solution. Thus,

$$P\left[\tau_{k:n}(\mathbf{X}) \vee \tau_{k:n}(\mathbf{Y}) > \tau_{k:n}(\mathbf{X} \vee \mathbf{Y})\right] = 0. \tag{7}$$

One may wonder whether $P\left[\tau_{k:n}(\mathbf{X} \vee \mathbf{Y}) > \tau_{k:n}(\mathbf{X}) \vee \tau_{k:n}(\mathbf{Y})\right]$ is also zero. Below we show that this is not the case always. Note that the inequality

$$\phi_{\tau_{k:n}(\mathbf{x}\vee\mathbf{y})} > \phi_{\tau_{k:n}(\mathbf{x})\vee\tau_{k:n}(\mathbf{y})}$$

holds iff

$$\begin{aligned}\phi_{\tau_{k:n}(\mathbf{x}\vee\mathbf{y})} &= 1\\ \phi_{\tau_{k:n}(\mathbf{x})\vee\tau_{k:n}(\mathbf{y})} &= 0,\end{aligned}$$

or equivalently,

$$\begin{aligned}\phi_{\tau_{k:n}(\mathbf{x}\vee\mathbf{y})} &= 1\\ \phi_{\tau_{k:n}(\mathbf{x})} &= 0\\ \phi_{\tau_{k:n}(\mathbf{y})} &= 0\end{aligned}$$

or equivalently, the following system of inequalities is satisfied.

$$\begin{aligned}\vee(x_1,y_1) + \vee(x_2,y_2) + \cdots + \vee(x_n,y_n) &\geq k\\ x_1 + x_2 + \cdots + x_n &\leq k-1\\ y_1 + y_2 + \cdots + y_n &\leq k-1,\end{aligned}$$

where $x_i, y_i \in \{0,1\}$ for all $i = 1,2,\ldots,n$. One can easily verify that the above system has a solution except for $k = 1$. Thus,

$$P\left[\tau_{1:n}(\mathbf{X} \vee \mathbf{Y}) > \tau_{1:n}(\mathbf{X}) \vee \tau_{1:n}(\mathbf{Y})\right] = 0, \tag{8}$$

and for $k = 2,\ldots,n$,

$$P\left[\tau_{k:n}(\mathbf{X} \vee \mathbf{Y}) > \tau_{k:n}(\mathbf{X}) \vee \tau_{k:n}(\mathbf{Y})\right] > 0. \tag{9}$$

Therefore, on using (7), (8), and (9), we have

$$\begin{aligned}P&\left[\tau_{k:n}(\mathbf{X} \vee \mathbf{Y}) > \tau_{k:n}(\mathbf{X}) \vee \tau_{k:n}(\mathbf{Y})\right]\\ &\geq P\left[\tau_{k:n}(\mathbf{X}) \vee \tau_{k:n}(\mathbf{Y}) > \tau_{k:n}(\mathbf{X} \vee \mathbf{Y})\right],\end{aligned}$$

where the equality holds for $k = 1$. Hence, the result follows. □

In Example 1 of Boland and El-Neweihi [1], it is mentioned that in the case of nonmatching spares, the above result may not hold for hazard rate order. This result motivates us to investigate whether the result holds under some conditions. In the following theorem, we show that, for a 2-out-of-2 system, this result holds under certain conditions.

Theorem 7: Let X_1, and X_2 be the lifetimes of the components; and let Y_1, and Y_2 be the lifetimes of the respective spares. Further, let X_i and Y_i be statistically independent, $i = 1, 2$. Suppose that one of the following conditions holds.

(a) $X_1 \leq_{hr} Y_1$ *and* $X_2 \geq_{hr} Y_2$,

(b) $X_1 \geq_{hr} Y_1$ *and* $X_2 \leq_{hr} Y_2$.

Then $\tau_{2:2}(\mathbf{X} \vee \mathbf{Y}) \geq_{hr} \tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})$.

Proof: We prove the result under (*a*). The result under (*b*) follows similarly. We see that

$$\begin{aligned}&\bar{F}_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t)\\ &\quad= [1 - F_{X_1}(t)F_{Y_1}(t)][1 - F_{X_2}(t)F_{Y_2}(t)],\end{aligned}$$

and

$$\begin{aligned}&\bar{F}_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t)\\ &\quad= 1 - [F_{X_1}(t) + F_{X_2}(t) - F_{X_1}(t)F_{X_2}(t)]\\ &\qquad\cdot [F_{Y_1}(t) + F_{Y_2}(t) - F_{Y_1}(t)F_{Y_2}(t)].\end{aligned}$$

Writing $\alpha_2(t) = \bar{F}_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t)/\bar{F}_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t)$, we get the equation at the bottom of the page. Differentiating $\alpha_2(t)$ with respect to t, and performing some algebraic calculations, we get

$$\begin{aligned}\alpha_2'(t) \overset{\text{sign}}{=} &\bar{F}_{X_2}(t)\bar{F}_{Y_2}(t)[1 - F_{X_1}(t)F_{Y_1}(t)]\\ &[r_{Y_2}(t)\{F_{X_1}(t)\bar{F}_{Y_1}(t) - F_{X_2}(t)F_{Y_1}(t)\bar{F}_{X_1}(t)\}\\ &- r_{X_2}(t)\{F_{Y_2}(t)F_{X_1}(t)\bar{F}_{Y_1}(t) - \bar{F}_{X_1}(t)F_{Y_1}(t)\}]\\ &+ \bar{F}_{X_1}(t)\bar{F}_{Y_1}(t)[1 - F_{X_2}(t)F_{Y_2}(t)]\\ &[r_{X_1}(t)\{F_{Y_2}(t)\bar{F}_{X_2}(t) - F_{Y_1}(t)F_{X_2}(t)\bar{F}_{Y_2}(t)\}\\ &- r_{Y_1}(t)\{F_{X_1}(t)F_{Y_2}(t)\bar{F}_{X_2}(t) - F_{X_2}(t)\bar{F}_{Y_2}(t)\}].\end{aligned}$$

To prove $\alpha_2'(t) \geq 0$, it suffices to show that

$$\begin{aligned}&r_{Y_2}(t)\{F_{X_1}(t)\bar{F}_{Y_1}(t) - F_{X_2}(t)F_{Y_1}(t)\bar{F}_{X_1}(t)\}\\ &\quad - r_{X_2}(t)\{F_{Y_2}(t)F_{X_1}(t)\bar{F}_{Y_1}(t) - \bar{F}_{X_1}(t)F_{Y_1}(t)\} \geq 0,\end{aligned} \tag{10}$$

and

$$\begin{aligned}&r_{X_1}(t)\{F_{Y_2}(t)\bar{F}_{X_2}(t) - F_{Y_1}(t)F_{X_2}(t)\bar{F}_{Y_2}(t)\}\\ &\quad - r_{Y_1}(t)\{F_{X_1}(t)F_{Y_2}(t)\bar{F}_{X_2}(t) - F_{X_2}(t)\bar{F}_{Y_2}(t)\} \geq 0.\end{aligned} \tag{11}$$

Note that

$$\begin{aligned}&F_{X_1}(t)\bar{F}_{Y_1}(t) - F_{X_2}(t)F_{Y_1}(t)\bar{F}_{X_1}(t)\\ &\quad\geq F_{Y_2}(t)F_{X_1}(t)\bar{F}_{Y_1}(t) - \bar{F}_{X_1}(t)F_{Y_1}(t),\end{aligned} \tag{12}$$

and from condition (*a*), we have

$$r_{Y_2}(t) \geq r_{X_2}(t), \tag{13}$$

and

$$F_{X_1}(t)\bar{F}_{Y_1}(t) - F_{X_2}(t)F_{Y_1}(t)\bar{F}_{X_1}(t) \geq 0. \tag{14}$$

$$\alpha_2(t) = \frac{[1 - F_{X_1}(t)F_{Y_1}(t)][1 - F_{X_2}(t)F_{Y_2}(t)]}{1 - [F_{X_1}(t) + F_{X_2}(t) - F_{X_1}(t)F_{X_2}(t)][F_{Y_1}(t) + F_{Y_2}(t) - F_{Y_1}(t)F_{Y_2}(t)]}.$$

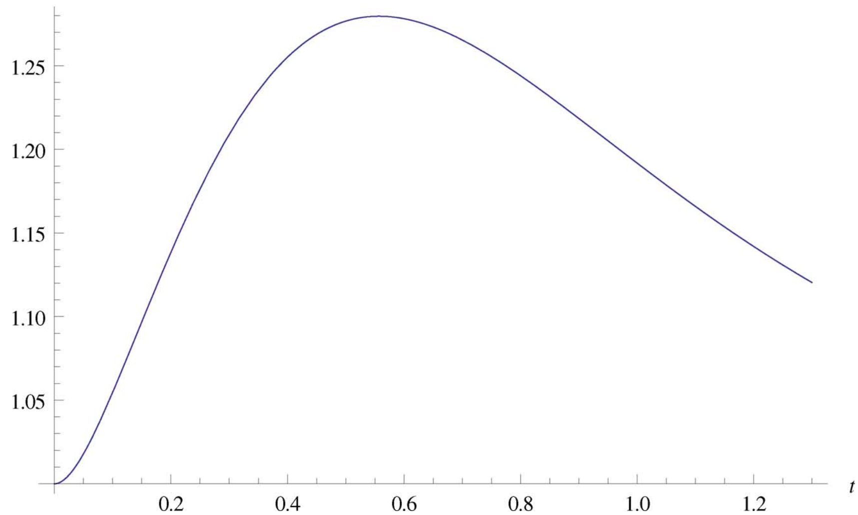

Fig. 6. Plot of $k_6(t)$ against $t \in [0, 1.30]$ (Counterexample 6).

Therefore, on using (12), (13), and (14), we get

$$r_{Y_2}(t)\{F_{X_1}(t)\bar{F}_{Y_1}(t) - F_{X_2}(t)F_{Y_1}(t)\bar{F}_{X_1}(t)\} \\ -r_{X_2}(t)\{F_{Y_2}(t)F_{X_1}(t)\bar{F}_{Y_1}(t) - \bar{F}_{X_1}(t)F_{Y_1}(t)\} \geq 0,$$

which proves (10). Further, one can verify that

$$F_{Y_2}(t)\bar{F}_{X_2}(t) - F_{Y_1}(t)F_{X_2}(t)\bar{F}_{Y_2}(t) \\ \geq F_{X_1}(t)F_{Y_2}(t)\bar{F}_{X_2}(t) - F_{X_2}(t)\bar{F}_{Y_2}(t). \quad (15)$$

Because $X_1 \leq_{hr} Y_1$, and $X_2 \geq_{hr} Y_2$, we have

$$r_{X_1}(t) \geq r_{Y_1}(t), \quad (16)$$

and

$$\bar{F}_{X_2}(t)F_{Y_2}(t) - \bar{F}_{Y_2}(t)F_{X_2}(t)F_{Y_1}(t) \geq 0. \quad (17)$$

Therefore, on using (15), (16), and (17), we get

$$r_{X_1}(t)\{F_{Y_2}(t)\bar{F}_{X_2}(t) - F_{Y_1}(t)F_{X_2}(t)\bar{F}_{Y_2}(t)\} \\ -r_{Y_1}(t)\{F_{X_1}(t)F_{Y_2}(t)\bar{F}_{X_2}(t) - F_{X_2}(t)\bar{F}_{Y_2}(t)\} \geq 0,$$

which proves (11). So, $\alpha'_2(t) \geq 0$ for all $t > 0$. Thus, $\tau_{2:2}(\mathbf{X} \vee \mathbf{Y}) \geq_{hr} \tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})$. □

As a consequence of Theorem 7, we have the following corollary which is given in Boland and El-Neweihi [1].

Corollary 1: If $X_1 =_{st} Y_1$, and $X_2 =_{st} Y_2$, then $\tau_{2:2}(\mathbf{X} \vee \mathbf{Y}) \geq_{hr} \tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})$. □

The following example illustrates the result given in Theorem 7.

Example 1: Let X_1, X_2, Y_1, and Y_2 be statistically independent random variables having distributions respectively $GP(k_1, \sigma_1, \sigma_1/k_1)$, $GP(k_2, \sigma_2, \sigma_2/k_2)$, $GP(k_3, \sigma_3, \sigma_3/k_3)$, and $GP(k_4, \sigma_4, \sigma_4/k_4)$, where $k_1 \leq k_3$, and $k_2 \geq k_4$. Note that $X_1 \leq_{hr} Y_1$, and $X_2 \geq_{hr} Y_2$. Thus, from Theorem 7, it follows that $\tau_{2:2}(\mathbf{X} \vee \mathbf{Y}) \geq_{hr} \tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})$. □

The following counterexample shows that conditions $X_1 \leq_{hr} Y_1$ and $X_2 \geq_{hr} Y_2$ given in Theorem 7 cannot be dropped.

Counterexample 6: Let X_1, X_2, Y_1, and Y_2 be statistically independent random variables with hazard rates $2, 1, 4, 3$, respectively. It is clear that $X_1 \nleq_{hr} Y_1$, and $X_2 \geq_{hr} Y_2$. Note that

$$\bar{F}_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t) = e^{-3t} + 2e^{-5t} - e^{-6t} - e^{-8t} - e^{-9t} + e^{-10t},$$
and
$$\bar{F}_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})} = e^{-3t} + e^{-7t} - e^{-10t}.$$

Writing $k_6(t) = \bar{F}_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t)/\bar{F}_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t)$, we get

$$k_6(t) = \frac{e^{-3t} + 2e^{-5t} - e^{-6t} - e^{-8t} - e^{-9t} + e^{-10t}}{e^{-3t} + e^{-7t} - e^{-10t}},$$

which is not monotone as Fig. 6 shows. Thus, $\tau_{2:2}(\mathbf{X} \vee \mathbf{Y}) \ngeq_{hr} \tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})$. □

In the case of 2-component non-matching spares of iid components and iid spares, Gupta and Nanda [3] proved that component redundancy is more reliable than system redundancy in reversed hazard rate order. Then a natural question arises, which is whether this result holds for not necessarily iid components and spares. Below, we cite a counterexample which answers this question in negative.

Counterexample 7: Let X_1, X_2, Y_1, and Y_2 be statistically independent random variables with survival functions given by $\bar{F}_{X_1}(t) = e^{-0.01t^{0.04}}$, $t > 0$; $\bar{F}_{X_2}(t) = e^{-0.02t^{0.7}}$, $t > 0$; $\bar{F}_{Y_1}(t) = e^{-0.6t^{0.02}}$, $t > 0$ and $\bar{F}_{Y_2}(t) = e^{-0.7t^{0.5}}$, $t > 0$, respectively. Note that

$$F_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t) = \left(1 - e^{-0.01t^{0.04}}\right)\left(1 - e^{-0.6t^{0.02}}\right) \\ + \left(1 - e^{-0.02t^{0.7}}\right)\left(1 - e^{-0.7t^{0.5}}\right) \\ - \left(1 - e^{-0.01t^{0.04}}\right)\left(1 - e^{-0.6t^{0.02}}\right) \\ \cdot \left(1 - e^{-0.02t^{0.7}}\right)\left(1 - e^{-0.7t^{0.5}}\right),$$

and

$$F_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t) \\ = \left(1 - e^{-(0.01t^{0.04}+0.02t^{0.7})}\right)\left(1 - e^{-(0.6t^{0.02}+0.7t^{0.5})}\right).$$

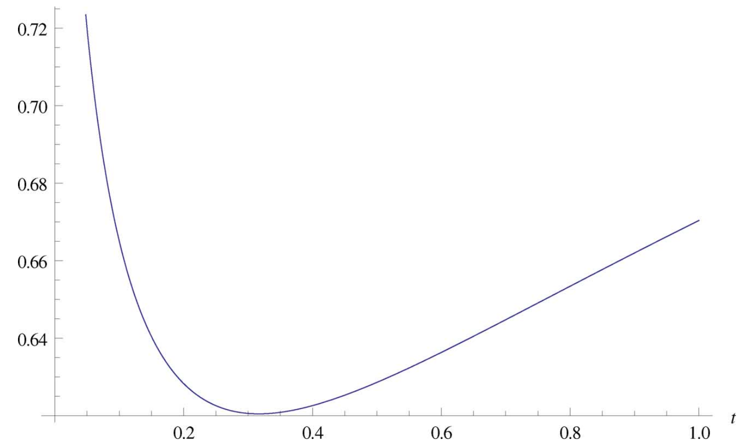

Fig. 7. Plot of $k_7(t)$ against $t \in [0,1]$ (Counterexample 7).

Writing $k_7(t) = F_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t)/F_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t)$, *we get*

$$\begin{aligned}k_7(t) = &\Big[\big(1-e^{-0.01t^{0.04}}\big)\big(1-e^{-0.6t^{0.02}}\big)\\&+\big(1-e^{-0.02t^{0.7}}\big)\big(1-e^{-0.7t^{0.5}}\big)\\&-\big(1-e^{-0.01t^{0.04}}\big)\big(1-e^{-0.6t^{0.02}}\big)\\&\cdot\big(1-e^{-0.02t^{0.7}}\big)\big(1-e^{-0.7t^{0.5}}\big)\Big]\\&\Big[\big(1-e^{-(0.01t^{0.04}+0.02t^{0.7})}\big)\big(1-e^{-(0.6t^{0.02}+0.7t^{0.5})}\big)\Big]^{-1}.\end{aligned}$$

From Fig. 7, it is seen that $k_7(t)$ *is not monotone in* $t > 0$. *Hence,* $\tau_{2:2}(\mathbf{X}\vee\mathbf{Y}) \ngeq_{rhr} \tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})$. □

Counterexample 7 motivates us to find some sufficient conditions under which component redundancy offers better reliability than system redundancy in reversed hazard rate order. In the following theorem, we prove a result for 2-component nonmatching spares of not necessarily iid components and spares.

Theorem 8: Let X_1, *and* X_2 *be the lifetimes of the components; and let* Y_1, *and* Y_2 *be the lifetimes of the respective spares. Further, let* X_i *and* Y_i *be statistically independent,* $i = 1, 2$. *Suppose that one of the following conditions holds.*

(a) $X_1 \geq_{rhr} X_2$ *and* $Y_1 \leq_{rhr} Y_2$.

(b) $X_1 \leq_{rhr} X_2$ *and* $Y_1 \geq_{rhr} Y_2$.

Then $\tau_{2:2}(\mathbf{X}\vee\mathbf{Y}) \geq_{rhr} \tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})$.

Proof: We prove the result under (*a*), and it follows similarly under (*b*). We see that

$$\begin{aligned}F_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t) = 1 - &[\bar F_{X_1}(t)+\bar F_{Y_1}(t)-\bar F_{X_1}(t)\bar F_{Y_1}(t)]\\&\cdot[\bar F_{X_2}(t)+\bar F_{Y_2}(t)-\bar F_{X_2}(t)\bar F_{Y_2}(t)],\end{aligned}$$

and

$$F_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t) = [1-\bar F_{X_1}(t)\bar F_{X_2}(t)][1-\bar F_{Y_1}(t)\bar F_{Y_2}(t)].$$

Writing $\alpha_3(t) = F_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t)/F_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t)$, we get the equation at the bottom of the page. Differentiating the above expression with respect to t, an extensive algebra shows that

$$\begin{aligned}\alpha_3'(t) \overset{\text{sign}}{=} &[1-\bar F_{Y_1}(t)\bar F_{Y_2}(t)]F_{X_1}(t)F_{X_2}(t)\\&[\tilde r_{X_1}(t)\{\bar F_{Y_1}(t)\bar F_{X_2}(t)F_{Y_2}(t)-\bar F_{Y_2}(t)F_{Y_1}(t)\}\\&+\tilde r_{X_2}(t)\{\bar F_{X_1}(t)\bar F_{Y_2}(t)F_{Y_1}(t)-\bar F_{Y_1}(t)F_{Y_2}(t)\}]\\&+[1-\bar F_{X_1}(t)\bar F_{X_2}(t)]F_{Y_1}(t)F_{Y_2}(t)\\&[\tilde r_{Y_1}(t)\{\bar F_{X_1}(t)\bar F_{Y_2}(t)F_{X_2}(t)-\bar F_{X_2}(t)F_{X_1}(t)\}\\&+\tilde r_{Y_2}(t)\{\bar F_{Y_1}(t)\bar F_{X_2}(t)F_{X_1}(t)-\bar F_{X_1}(t)F_{X_2}(t)\}].\end{aligned}$$

To prove $\alpha_3'(t) \leq 0$, it suffices to show that

$$\begin{aligned}&\tilde r_{X_1}(t)\{\bar F_{Y_1}(t)\bar F_{X_2}(t)F_{Y_2}(t)-\bar F_{Y_2}(t)F_{Y_1}(t)\}\\&+\tilde r_{X_2}(t)\{\bar F_{X_1}(t)\bar F_{Y_2}(t)F_{Y_1}(t)-\bar F_{Y_1}(t)F_{Y_2}(t)\} \leq 0,\end{aligned}\tag{18}$$

and

$$\begin{aligned}&\tilde r_{Y_1}(t)\{\bar F_{X_1}(t)\bar F_{Y_2}(t)F_{X_2}(t)-\bar F_{X_2}(t)F_{X_1}(t)\}\\&+\tilde r_{Y_2}(t)\{\bar F_{Y_1}(t)\bar F_{X_2}(t)F_{X_1}(t)-\bar F_{X_1}(t)F_{X_2}(t)\} \leq 0.\end{aligned}\tag{19}$$

Note that

$$\begin{aligned}&\bar F_{Y_2}(t)F_{Y_1}(t)-\bar F_{Y_1}(t)\bar F_{X_2}(t)F_{Y_2}(t)\\&\geq \bar F_{X_1}(t)\bar F_{Y_2}(t)F_{Y_1}(t)-\bar F_{Y_1}(t)F_{Y_2}(t).\end{aligned}\tag{20}$$

Because $X_1 \geq_{rhr} X_2$, and $Y_1 \leq_{rhr} Y_2$, it gives

$$\tilde r_{X_1}(t) \geq \tilde r_{X_2}(t),\tag{21}$$

and

$$\bar F_{Y_2}(t)F_{Y_1}(t)-\bar F_{Y_1}(t)F_{Y_2}(t)\bar F_{X_2}(t) \geq 0.\tag{22}$$

$$\alpha_3(t) = \frac{[1-\bar F_{X_1}(t)\bar F_{X_2}(t)][1-\bar F_{Y_1}(t)\bar F_{Y_2}(t)]}{1-[\bar F_{X_1}(t)+\bar F_{Y_1}(t)-\bar F_{X_1}(t)\bar F_{Y_1}(t)][\bar F_{X_2}(t)+\bar F_{Y_2}(t)-\bar F_{X_2}(t)\bar F_{Y_2}(t)]}.$$

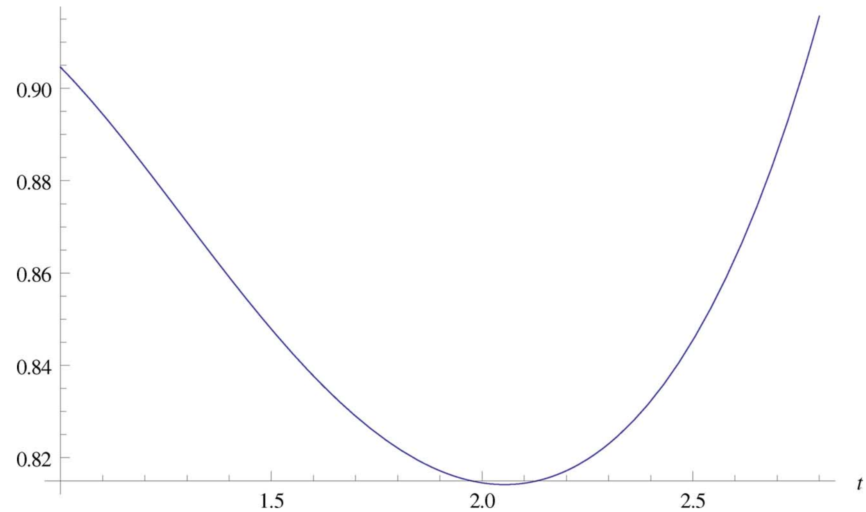

Fig. 8. Plot of $k_8(t)$ against $t \in [1, 2.7]$ (Counterexample 8).

Therefore, on using (20), (21), and (22), we have

$$\begin{aligned}&\tilde{r}_{X_1}(t)\{\bar{F}_{Y_1}(t)\bar{F}_{X_2}(t)F_{Y_2}(t) - \bar{F}_{Y_2}(t)F_{Y_1}(t)\}\\&\quad+\tilde{r}_{X_2}(t)\{\bar{F}_{X_1}(t)\bar{F}_{Y_2}(t)F_{Y_1}(t) - \bar{F}_{Y_1}(t)F_{Y_2}(t)\} \le 0,\end{aligned}$$

which proves (18). Further,

$$\begin{aligned}&\bar{F}_{X_1}(t)F_{X_2}(t) - \bar{F}_{Y_1}(t)\bar{F}_{X_2}(t)F_{X_1}(t)\\&\quad\ge \bar{F}_{X_1}(t)\bar{F}_{Y_2}(t)F_{X_2}(t) - \bar{F}_{X_2}(t)F_{X_1}(t). \qquad (23)\end{aligned}$$

Because $Y_1 \le_{rhr} Y_2$, we get

$$\tilde{r}_{Y_2}(t) \ge \tilde{r}_{Y_1}(t). \qquad (24)$$

Further, $X_1 \ge_{rhr} X_2$ implies

$$\bar{F}_{X_1}(t)F_{X_2}(t) - \bar{F}_{Y_1}(t)\bar{F}_{X_2}(t)F_{X_1}(t) \ge 0. \qquad (25)$$

Therefore, on using (23), (24), and (25), we get

$$\begin{aligned}&\tilde{r}_{Y_1}(t)\{\bar{F}_{X_1}(t)\bar{F}_{Y_2}(t)F_{X_2}(t) - \bar{F}_{X_2}(t)F_{X_1}(t)\}\\&\quad+\tilde{r}_{Y_2}(t)\{\bar{F}_{Y_1}(t)\bar{F}_{X_2}(t)F_{X_1}(t) - \bar{F}_{X_1}(t)F_{X_2}(t)\} \le 0,\end{aligned}$$

which proves (19). Hence, $\alpha'_3(t) \le 0$ for all $t > 0$. Thus, $\tau_{2:2}(\mathbf{X} \vee \mathbf{Y}) \ge_{rhr} \tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})$. □

The following corollary proved by Gupta and Nanda [3] directly follows from Theorem 8.

Corollary 2: If $X_1 =_{st} X_2$, *and* $Y_1 =_{st} Y_2$, *then* $\tau_{2:2}(\mathbf{X} \vee \mathbf{Y}) \ge_{rhr} \tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})$. □

The illustration of Theorem 8 is revealed through the following example.

Example 2: Let X_1, X_2, Y_1, *and* Y_2 *be statistically independent random variables having distributions respectively* $GP(k_1, \sigma_1, \sigma_1/k_1)$, $GP(k_2, \sigma_2, \sigma_2/k_2)$, $GP(k_3, \sigma_3, \sigma_3/k_3)$, *and* $GP(k_4, \sigma_4, \sigma_4/k_4)$, *where* $k_1 \ge k_2$, *and* $k_3 \le k_4$. *Note that* $X_1 \ge_{rhr} X_2$, *and* $Y_1 \le_{rhr} Y_2$. *Thus,* $\tau_{2:2}(\mathbf{X} \vee \mathbf{Y}) \ge_{rhr} \tau_{2:2}(\mathbf{X}) \vee \tau_{2:2}(\mathbf{Y})$ *follows from Theorem 8.* □

The following counterexample shows that the conditions $X_1 \ge_{rhr} X_2$ and $Y_1 \le_{rhr} Y_2$ given in Theorem 8 cannot be relaxed.

Counterexample 8: Let X_1, X_2, Y_1, *and* Y_2 *be statistically independent random variables with cumulative distribution functions given by*

$$F_{X_1}(t) = \begin{cases}\frac{t^2}{25} & \text{for } 0 \le t \le 1\\ \frac{t^3-t+1}{25} & \text{for } 1 \le t \le 3\\ 1 & \text{for } t \ge 3,\end{cases}$$

$$F_{X_2}(t) = \begin{cases}\frac{3t^2}{11} & \text{for } 0 \le t \le 1\\ \frac{t^2+2}{11} & \text{for } 1 \le t \le 3\\ 1 & \text{for } t \ge 3,\end{cases}$$

$$F_{Y_1}(t) = 1 - e^{-0.01t},\ t > 0,$$

and

$$F_{Y_2}(t) = 1 - e^{-100t},\ t > 0.$$

It is easy to verify that $X_1 \ge_{rhr} X_2$, *and* $Y_1 \ge_{rhr} Y_2$. *Note that, for* $1 \le t \le 3$, *we have, by writing* $k_8(t) = F_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t)/F_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t)$,

$$\begin{aligned}k_8(t) = &\left[11\left(t^3-t+1\right)\left(1-e^{-0.01t}\right)+25\left(t^2+2\right)\left(1-e^{-100t}\right)\right.\\&\left.-\left(t^3-t+1\right)\left(t^2+2\right)\left(1-e^{-0.01t}\right)\left(1-e^{-100t}\right)\right]\\&\left[\left\{275-\left(24-t^3+t\right)\left(9-t^2\right)\right\}\left(1-e^{-100.01t}\right)\right]^{-1}.\end{aligned}$$

Fig. 8 shows that $k_8(t)$ *is not monotone. Thus,* $F_{\tau_{2:2}(\mathbf{X}\vee\mathbf{Y})}(t) \not\ge_{rhr} F_{\tau_{2:2}(\mathbf{X})\vee\tau_{2:2}(\mathbf{Y})}(t)$. □

III. Standby Redundancy

In the case of standby redundancy, Boland and El-Neweihi [1] showed that component redundancy is more reliable than system redundancy in the usual stochastic order for series systems, whereas the reverse principle holds for parallel systems. Below in Theorem 9 we prove that component redundancy offers better reliability than system redundancy in stochastic precedence order, for k-out-of-n systems.

Theorem 9: Let $X_1, X_2, \dots, X_n$ be component lifetimes, and $Y_1, Y_2, \dots, Y_n$ be those of spares. Further, let X_i and Y_i, $i = 1, 2, \dots, n$, be statistically independent. Then

$$\tau_{k:n}(\mathbf{X}+\mathbf{Y}) \geq_{sp} \tau_{k:n}(\mathbf{X}) + \tau_{k:n}(\mathbf{Y}),\ k = 2, 3, \dots, n,$$

and

$$\tau_{1:n}(\mathbf{X}+\mathbf{Y}) =_{sp} \tau_{1:n}(\mathbf{X}) + \tau_{1:n}(\mathbf{Y}).$$

Proof: Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$, and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ be the state vectors of $\mathbf{X}$, and $\mathbf{Y}$, respectively. We show that

$$\phi_{\tau_{k:n}(\mathbf{x})+\tau_{k:n}(\mathbf{y})} > \phi_{\tau_{k:n}(\mathbf{x}+\mathbf{y})} \tag{26}$$

is never possible. To see this result, note that (26) holds iff

$$\phi_{\tau_{k:n}(\mathbf{x})+\tau_{k:n}(\mathbf{y})} = 1,$$

and

$$\phi_{\tau_{k:n}(\mathbf{x}+\mathbf{y})} = 0.$$

Or equivalently, one of the following two systems of equations holds.

$$\begin{cases} \phi_{\tau_{k:n}(\mathbf{x})} = 1 \\ \phi_{\tau_{k:n}(\mathbf{y})} = 0 \\ \phi_{\tau_{k:n}(\mathbf{x}+\mathbf{y})} = 0 \end{cases}$$
$$\begin{cases} \phi_{\tau_{k:n}(\mathbf{x})} = 0 \\ \phi_{\tau_{k:n}(\mathbf{y})} = 1 \\ \phi_{\tau_{k:n}(\mathbf{x}+\mathbf{y})} = 0. \end{cases}$$

Or equivalently, one of the following two systems of inequalities is satisfied.

$$\begin{cases} x_1 + x_2 + \cdots + x_n \geq k \\ y_1 + y_2 + \cdots + y_n \leq k-1 \\ (x_1+y_1) + (x_2+y_2) + \cdots + (x_n+y_n) \leq k-1 \end{cases}$$
$$\begin{cases} x_1 + x_2 + \cdots + x_n \leq k-1 \\ y_1 + y_2 + \cdots + y_n \geq k \\ (x_1+y_1) + (x_2+y_2) + \cdots + (x_n+y_n) \leq k-1, \end{cases}$$

where $x_i, y_i \in \{0, 1\}$, and $(x_i, y_i) \neq (1, 1)$ for all $i = 1, 2, \dots, n$. It is easy to see that the above two systems do not have any solutions. Thus,

$$P\left[\tau_{k:n}(\mathbf{X}) + \tau_{k:n}(\mathbf{Y}) > \tau_{k:n}(\mathbf{X}+\mathbf{Y})\right] = 0. \tag{27}$$

One may think that $P\left[\tau_{k:n}(\mathbf{X}+\mathbf{Y}) > \tau_{k:n}(\mathbf{X}) + \tau_{k:n}(\mathbf{Y})\right]$ is also zero. Below we show that this is not the case always. Note that the inequality

$$\phi_{\tau_{k:n}(\mathbf{x}+\mathbf{y})} > \phi_{\tau_{k:n}(\mathbf{x})+\tau_{k:n}(\mathbf{y})}$$

holds iff

$$\begin{aligned} \phi_{\tau_{k:n}(\mathbf{x})+\tau_{k:n}(\mathbf{y})} &= 0 \\ &\text{and} \\ \phi_{\tau_{k:n}(\mathbf{x}+\mathbf{y})} &= 1; \end{aligned}$$

or equivalently,

$$\begin{aligned} \phi_{\tau_{k:n}(\mathbf{x})} &= 0 \\ \phi_{\tau_{k:n}(\mathbf{y})} &= 0 \\ \phi_{\tau_{k:n}(\mathbf{x}+\mathbf{y})} &= 1; \end{aligned}$$

or equivalently, the following system holds, for all $x_i, y_i \in \{0, 1\}$, and $(x_i, y_i) \neq (1, 1)$, $i = 1, 2, \dots, n$.

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &\leq k-1 \\ y_1 + y_2 + \cdots + y_n &\leq k-1 \\ (x_1+y_1) + (x_2+y_2) + \cdots + (x_n+y_n) &\geq k. \end{aligned}$$

One can easily verify that the above system has a solution except for $k = 1$. Thus,

$$P\left[\tau_{1:n}(\mathbf{X}+\mathbf{Y}) > \tau_{1:n}(\mathbf{X}) + \tau_{1:n}(\mathbf{Y})\right] = 0, \tag{28}$$

and for $k = 2, \dots, n$,

$$P\left[\tau_{k:n}(\mathbf{X}+\mathbf{Y}) > \tau_{k:n}(\mathbf{X}) + \tau_{k:n}(\mathbf{Y})\right] > 0. \tag{29}$$

Therefore, on using (27), (28), and (29), we have

$$\begin{aligned} &P\left[\tau_{k:n}(\mathbf{X}+\mathbf{Y}) > \tau_{k:n}(\mathbf{X}) + \tau_{k:n}(\mathbf{Y})\right] \\ &\qquad \geq P\left[\tau_{k:n}(\mathbf{X}) + \tau_{k:n}(\mathbf{Y}) > \tau_{k:n}(\mathbf{X}+\mathbf{Y})\right], \end{aligned}$$

where the equality holds for $k = 1$. Hence, the result follows. □

IV. Data Analysis

The result in Theorem 7 is described through simulation by analyzing a data set as detailed below. Similar kinds of data analysis could be done for the other results as well. We draw samples for X_1, X_2, Y_1, and Y_2, each of size 500,000, respectively from $GP(1/3, 1/6, 1/2)$, $GP(1/2, 1/4, 1/2)$, $GP(1/6, 1/12, 1/2)$, and $GP(1/7, 1/14, 1/2)$. We call the samples x_{1i}, x_{2i}, y_{1i}, and y_{2i}, for $i = 1, 2, \dots, 500{,}000$, respectively. Note that the conditions given in (*a*) of Theorem 7 are satisfied by the concerned random variables. Now, we calculate the lifetimes of the systems where component redundancy, and system redundancy are used; the lives being denoted by c_i, and s_i, respectively. Thus, we have $c_i = \min\{\max\{x_{1i}, y_{1i}\}, \max\{x_{2i}, y_{2i}\}\}$, and $s_i = \max\{\min\{x_{1i}, x_{2i}\}, \min\{y_{1i}, y_{2i}\}\}$, for each i. Then, we sort c_i and s_i together, and corresponding empirical reliability functions $\bar{E}_C(\cdot)$ and $\bar{E}_S(\cdot)$ are calculated. Further, by plotting $\beta(t) = \bar{E}_C(t)/\bar{E}_S(t)$ against t, we get an increasing curve, which is shown in Fig. 9. The small deviation in the monotonicity, as one can see in the depicted figure, occurs only due to sampling fluctuation.

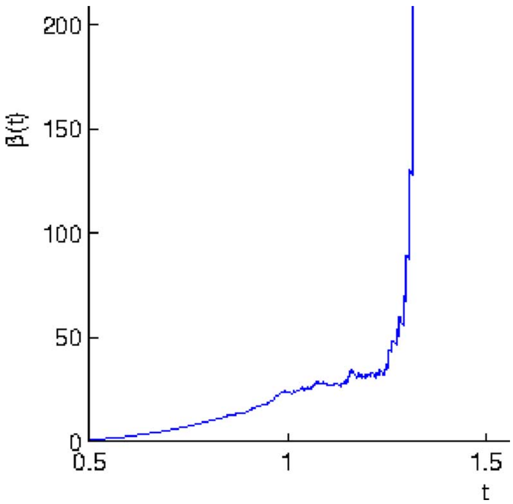

Fig. 9. Plot of $\beta(t)$ against t (Data Analysis).

V. Conclusion

In this paper, we consider the problem of active redundancy as well as standby redundancy allocation at component level versus system level. We mainly concentrate our discussion on active redundancy. We give sufficient conditions under which active redundancy at the component level is superior to that at the system level with respect to the hazard rate, the reversed hazard rate, the up shifted hazard rate, the up shifted reversed hazard rate, and the up shifted likelihood ratio orders, for different types of systems. We also show that this principle holds in stochastic precedence order, for k-out-of-n systems. In addition to this, we show that standby redundancy at the component level is more reliable than that at the system level with respect to the stochastic precedence order, for k-out-of-n systems. We also verify the results through data analysis. Each stochastic order has its individual importance; for example, usual stochastic order compares two reliability functions, whereas hazard rate order compares the failure rates of two systems. The usefulness of redundancy is seen in day-to-day life. The electric inverter is one such example. To see more important use of redundancy let us think of the shadowless lamp used in the operation table where the redundancy is so important that the censoring and switching device is installed to instantaneously activate the redundant component or system used. This case illustrates the usefulness of the redundancy in practical life problems. As a result, our observations may be helpful to different groups of people, say for example to design engineers and reliability analysts to decide on the effective use of redundancy depending on the underlying situation.

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