SDS LU of a penta-diagonal matrix

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November 3, 2021

1 Describe LU Factorization

Original matrix:

$$\begin{pmatrix} 2 & 0 & -1 & & & \\ 0 & 2 & 0 & -1 & & \\ -1 & 0 & 2 & 0 & -1 & \\ & -1 & 0 & 2 & 0 & -1 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

To do the A = LU Factorization process, the L will be the lower triangular matrix with a leading diagonal of ones. To find U, which is the upper triangular matrix, we use row operations on the original matrix.

What R3 of U will be after adding $\frac{1}{2}$R1 + R3:

$$\begin{pmatrix} 2 & 0 & -1 & & & \\ 0 & 2 & 0 & -1 & & \\ 0 & 0 & 3/2 & 0 & -1 & \\ & -1 & 0 & 2 & 0 & -1 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

What R4 of U will be after adding $\frac{1}{2}$R2 + R4:

$$\begin{pmatrix} 2 & 0 & -1 & & & \\ 0 & 2 & 0 & -1 & & \\ 0 & 0 & 3/2 & 0 & -1 & \\ & 0 & 0 & 3/2 & 0 & -1 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Now that we have U, we can find L using the opposite of the operations that were applied. Rows 1 and 2 will remain unchanged from what the initial matrix was since no row operations were applied on them. Row's 3 and 4 will have a -1/2 replace the -1 from the original matrix.

L=

$$\begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ -1/2 & 0 & 1 & & & \\ & -1/2 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Therefore the result of A = LU is:

$$\begin{pmatrix} 2 & 0 & -1 & & & & & \\ & 2 & 0 & -1 & & & & \\ & & 3/2 & 0 & -1 & & & \\ & & & 3/2 & 0 & -1 & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & \ddots & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ -1/2 & 0 & 1 & & & & \\ & -1/2 & 0 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & 0 & 1 & \end{pmatrix}$$

2 No need for fill-in

There is no need for fill-in since when Gaussian Elimination is performed on the original matrix, there is only one column that needs to be changed to 0 for every column. For example, row three only needs column one to be changed to zero, and row four only needs column two to be changed to zero. Therefore, no fill-in is needed because no indexes that are originally zero will be updated to non-zero.

Inductive Proof: From the matrix we were given in the beginning of the problem, in R3, we need the first value to be zero, and we can use a row operation of $\frac{1}{2}$R1 + R3 to make it zero. This would result in R3 being:

$$\begin{pmatrix} 2 & 0 & -1 & & & & \\ 0 & 2 & 0 & -1 & & & \\ 0 & 0 & 3/2 & 0 & -1 & & \\ & -1 & 0 & 2 & 0 & -1 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

This shows that there was no need for fill-in in this line since you only needed one row operation to make the column zero. There was no need to make the index that is originally zero to be non-zero.

Another example would be R4, which also just needs one row operation that does not need to change an index to be non-zero.

$$\begin{pmatrix} 2 & 0 & -1 & & & \\ 0 & 2 & 0 & -1 & & \\ 0 & 0 & 3/2 & 0 & -1 & \\ & 0 & 0 & 3/2 & 0 & -1 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Describe the LU factorizations of matrix

1. Convince yourself there will be no fill in. Give inductive proof

Inductive Proof: Fill in - entries changing from inital zero to nonzero value

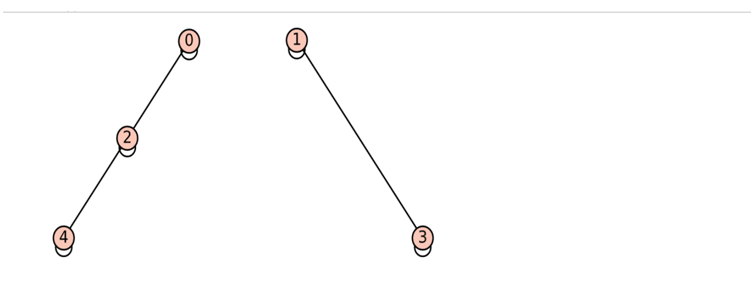
$$\underbrace{\begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}}_{\text{Step 1}} \xRightarrow{\frac{1}{2}R_1+R_3} \underbrace{\begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 2+\frac{1}{2} & 0 & -1 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}}_{\text{Step 2}} \xRightarrow{\frac{1}{2}R_2+R_4} \underbrace{\begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 2-\frac{1}{2} & 0 & -1 \\ 0 & 0 & 0 & 2-\frac{1}{2} & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}}_{\text{Step 3}}$$

In our first step of Gaussian elimination, no inital zeroes change to non zero values in step 1 to Step 2. Therefore, the $\boxed{\text{condition holds}}$

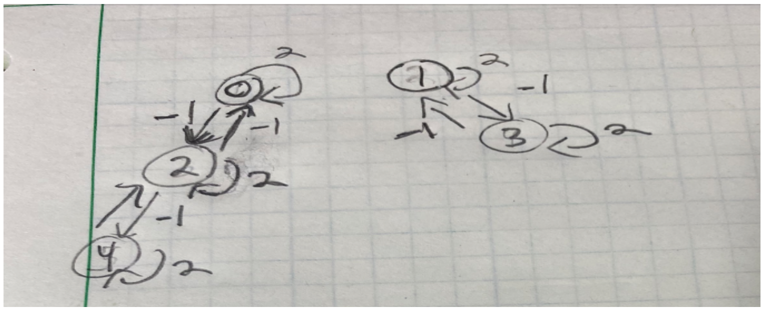
From Step 2 to Step 3 the $\boxed{\text{Condition holds}}$ after performing Gaussian elimination so therefore there will be no further fill-in!

3 Graph of Matrix

We used an open source tool to generate the visual representation of the matrix.



This is a graph we sketched ourselves showing the directions of the connected nodes and the weightings.



We believe the graph is a directed bipartite graph that is disconnected. The zero entries in the penta-diagonal adjacency matrix produce a graph whose adjacent nodes are formed from node numbers that do not lie in succession over all natural numbers (i.e. 0 connects to 2, 2 connects to 4, and 1 connects to 3). The node pairs (0,2), (2,4), and (1,3) constitute adjacent nodes in the graph. This follows from the adjacency matrix where the entries in A(1,3), A(2,4), A(2,4), A(3,1), A(3,5), A(4,2), A(5,3) are all non-zero meaning that the nodes associated with those entries are connected. The slightly off-center circles depicted below the shaded orange nodes represent cyclical paths that connect the nodes to themselves. This follows from the adjacency matrix as each element along the main diagonal is non-zero, meaning that the nodes in the ith row and ith column are connected (for i = 0:4). Therefore, the adjacency matrix produces a graph with two branches, the first of which connects nodes 0,2, and 4, and the second, connects nodes 1 and 3 as shown. Lastly, the reason why performing Gaussian Elimination did not result in fill-in (conversion of zero entries to non-zero entries) can be characterized by no successive numbered nodes being connected in the graph, namely, the adjacent nodes are not numbered in succession.