The patch frame
and some separation axioms in Frm

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A little example

Let $S = \mathbb{R}$ be with the topologies

$$\mathcal{O}_l S = \{(-\infty, a)\}, \quad \mathcal{O}_m S = \{(a, b)\}, \quad \mathcal{O}_n S = \{[a, b)\},$$

where $a, b \in S$. Then

$$\mathcal{O}_l S \hookrightarrow \mathcal{O}_m S \hookrightarrow \mathcal{O}_n S$$

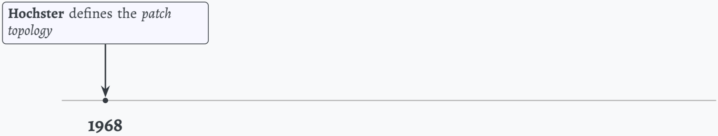
We can see that

$$\mathcal{O}_l^p S = \mathcal{O}_m S \simeq P\mathcal{O}_l S \quad \text{y} \quad \mathcal{O}_l^f S = \mathcal{O}_n S \simeq N\mathcal{O}_l S,$$

that is,

$$\mathcal{O}_l S = A \to PA \hookrightarrow NA$$

The patch construction



The patch construction

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Hochster defines the *patch topology*

1982

Priestley, Cornish, Joyal and Stralka use the patch more generally (see Johnstone, *Stone Spaces*, [1])

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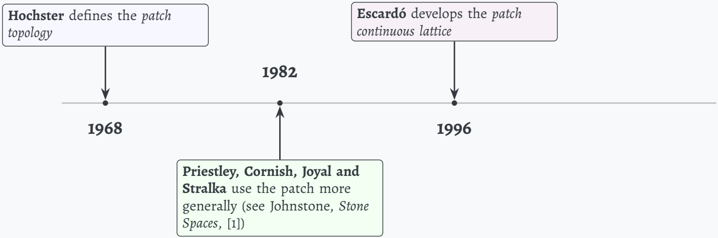


The patch construction

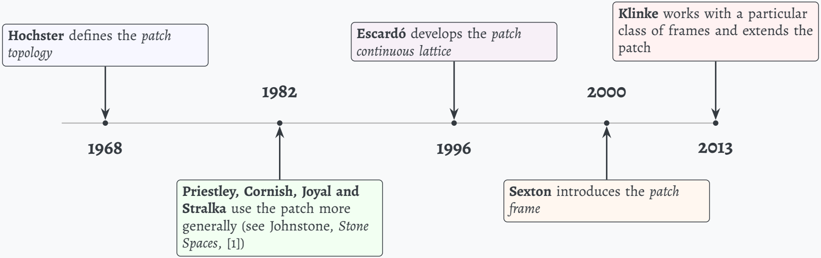
1968 — **Hochster** defines the *patch topology*

1982 — **Priestley, Cornish, Joyal and Stralka** use the patch more generally (see Johnstone, *Stone Spaces*, [1])

1996 — **Escardó** develops the *patch continuous lattice*

2000 — **Sexton** introduces the *patch frame*

The patch construction



Frame theory

$$\mathsf{Frm} = \begin{cases} \mathsf{Obj}: & (A, \leqslant, \wedge, \bigvee, 1, 0) \\ \text{Arrows:} & f : A \to B \end{cases}$$

For $S \in \mathsf{Top}$,

$$(\mathcal{O}S, \subseteq, \cap, \bigcup, S, \emptyset) \in \mathsf{Frm}$$

Furthermore,

$$\begin{tikzcd} \mathsf{Top} \arrow[rr, bend left, "\mathcal{O}(\_)"] & & \mathsf{Frm} \arrow[ll, bend left, "\mathsf{pt}(\_)"] \end{tikzcd}$$

is an adjunction.

Packed spaces

${}^{p}S = (S, \mathcal{O}^{p}S)$, where $\mathcal{O}^{p}S$ is gerated by

$$\text{pbase} = \{U \cap Q' \mid U \in \mathcal{O}S, Q \in \mathcal{Q}S\}$$

Definition

$S \in \mathsf{Top}$ is **packed** if every compact (saturated) subset is closed

$$S \text{ is packed} \quad \iff \quad {}^{p}S = S$$

$$T_2 \Rightarrow \text{packed} \Rightarrow T_1$$

Patch trivial

By *Hoffman-Mislove theorem*[1]

$$\mathrm{Pbase} = \{u_a \wedge v_F \mid a \in A, F \in A^\wedge\}$$

Definition

1. The **patch frame** of $A \in \mathsf{Frm}\,(PA)$, is the frame generated by Pbase
2. A is **patch trivial** if $A \simeq PA$.

[1]**Thm:** There is a bijective correspondence between $F \in A^\wedge$ and $Q \in \mathcal{Q}S$

$\circ F \in A^\wedge$ if F is a filter in A and $\forall\, X \subseteq A$, with X directed, if $\bigvee X \in F$, then $a \in F$ for some $a \in X$.

$\circ$For $a \in A$, $u_a(x) = a \vee x$ and $v_a(x) = (a \succ x)$ are nuclei in A.

$\circ v_F = f^\infty$, where $f = \dot{\bigvee}\{v_a \mid a \in F\}$

Tidy frames

Definition [[6], Def. 8.2.1]

Let $A \in \mathsf{Frm}$, $F \in A^{\wedge}$ and $\alpha \in \mathsf{Ord}$ be. We say that:

1. F is α-tidy if for $x \in F$, $d \vee x = 1$, where
$$d = d(\alpha) = f^{\alpha}(0).$$
2. A is α-tidy if every $F \in A^{\wedge}$ is α-tidy.
3. A is **tidy** if it is α-tidy for some $\alpha \in \mathsf{Ord}$.

Proposition [[6], Lemma 8.2.2]

$$A \text{ is tidy} \iff A \text{ is patch trivial.}$$

Objectives

1. Understand tidy frames in more detail.
2. To explore the relationship with some separation axioms in Frm.
3. Provide tools to study the tidy frames.
4. Give examples.

Separation axioms in Frm

$$
\begin{tikzcd}
 & (\mathbf{reg}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
(\mathbf{fit}) \arrow[r, dash, dotted] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & (\mathbf{sH}) \arrow[r, dash, dotted] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] \arrow[dl, dash, dotted] & (\mathbf{H}) + (\mathbf{sfit}) \arrow[d, Rightarrow] \\
(\mathbf{sfit}) \arrow[r, dash, dotted] & T_1 & (\mathbf{H}) \arrow[l, Rightarrow]
\end{tikzcd}
$$

°$\forall\, a \nleq b \in A$, then

°**(reg):** $\exists\, x, y \in A$ such that $a \vee x = 1, y \nleq b$ and $x \wedge y = 0$.

°**(H):** $\exists\, c \in A$ such that $c \nleq a$ and $\neg c \leqslant b$.

°**(fit):** $\exists\, x, y \in A$ such that $x \vee a = 1, y \nleq b$ and $x \wedge y \leqslant b$.

°**(sfit):** $\exists\, c \in A$ such that $c \vee a = 1 \neq c \vee b$.

°**(sH)** and T_1 are notion some different. All this can be found in [3].

Properties of the tidy frames

This is a summary of the properties that Sexton includes in [6]

- In the spatial case ($A = \mathcal{O}S$),

$$\begin{aligned} \mathcal{O}S \text{ is 0-tidy} &\iff S = \emptyset \\ \mathcal{O}S \text{ is 1-tidy} &\iff S \text{ is } T_2 \\ \mathcal{O}S \text{ is tidy} &\iff S \text{ is packed and stacked.} \end{aligned}$$

- For $A \in \mathrm{Frm}$ arbitrary

$$\begin{aligned} A \text{ is } (\mathbf{reg}) &\Rightarrow A \text{ is tidy} \\ A \text{ is } (\mathbf{fit}) &\Rightarrow A \text{ is tidy} \\ A \text{ is tidy} &\Rightarrow A \text{ is } T_1 \end{aligned}$$

Some results

If $(f : A \to B) \in \mathsf{Frm}$, $G \in A^{\wedge}$ and $F \in B^{\wedge}$, then

$$b \in f[G] \iff f_*(b) \in G \quad \text{and} \quad a \in f_*[F] \iff f(a) \in F.$$

Also, if $F \in B^{\wedge}$, then $f_*(F) \in A^{\wedge}$.

Proposition

For f^{∞} and f_j^{∞} the nuclei associated to F and j_*F, respectively, we have

$$j \circ f_j^{\infty} \leqslant f^{\infty} \circ j$$

Proof

By transfinite induction. $\square$

More properties of the tidy frames

Proposition

If $A \in \mathsf{Frm}$ is tidy and $j \in NA$, then A_j is tidy.

Proof

- We take $x \in F \in A_j^{\wedge}$ and $F \subseteq j_*[F] \in A^{\wedge}$.
- For f^{∞} and f_j^{∞} as before, we have

$$d = d(\alpha) \geqslant d_j(\alpha) = d_j$$

- Since A is tidy, then $d_j \vee x = 1$, for all $x \in j_*[F]$. In particular, for all $x \in F$.
- Therefore, $d \vee x = 1$.

$\square$

Corollary

If A is $(\mathbf{sH})$, A is tidy.

Proof

In $(\mathbf{sH})$ all compact quotient is closed. $\square$

$$\begin{tikzcd}
 & (\mathbf{H}) \arrow[dr, Rightarrow, dotted] & \\
(\mathbf{reg}) \arrow[ur, Rightarrow] \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & (\mathbf{sH}) \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{Tidy} \\
 & (\mathbf{fit}) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Compact quotients

$$\text{Tidy} \iff \text{P. trivial} \iff u_d = v_F$$

Theorem

Let $A \in \mathrm{Frm}$ and $j \in NA$. Then

$$A_j \text{ is compact} \quad \iff \quad \nabla(j) \in A^{\wedge}.$$

Then

$$A_{u_d} \text{ is a closed quotient} \quad \text{and} \quad A_{v_F} \text{ is a compact quotient.}$$

If A is tidy, we have a compact closed quotient.

°$\nabla(j) = \{a \in A \mid j(a) = 1\}$ is a filter in A *(Admissibility filter)*.

°With $\nabla(j)$ we can define a "$\sim$" in NA: $j \sim k \iff \nabla(j) = \nabla(k)$.

KC frames

In [10], Wilansky defines a space S to be **KC** if every compact set is closed.

Definition

$A \in \mathsf{Frm}$ is a **KC frame** if every compact quotient is closed.

$$\mathbf{KC} \Rightarrow \text{Tidy}$$

Proposition

If $A \in \mathsf{Frm}$ es **KC** and $j \in NA$, then A_j is **KC**.

Proof

- Consider $k \in NA_j$ such that $\nabla(k) \in A_j^{\wedge}$.
- If $\nabla(k) \in A_j^{\wedge} \Rightarrow j_*[\nabla(k)] \in A^{\wedge}$.
- We take $l = j_* \circ k \circ j \in NA$ and $\nabla(l) \in A^{\wedge} \Rightarrow l = u_a$ for some $a \in A$.
- Furthermore $a = k(j(a))$.
- For $x, b \in A_j$ with $b = j(a)$ we have $u_b(x) = k(x)$.

$\square$

°**Proposition:** For $j \in NA$ and $k \in NA_j$. If $\nabla(k) \in A_j^{\wedge}$, $\nabla(j_* kj) \in A^{\wedge}$.

We can build the diagram (see [9])

$$\begin{tikzcd}
A \arrow[r, "v_F"] \arrow[d, "U_A"'] & A_F \arrow[d, "g"] \arrow[dr, "\simeq"] & \\
 & & \mathcal{O}Q \\
\mathcal{O}S \arrow[r, "v_\nabla"'] & \mathcal{O}S_\nabla \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

What happens if A has property (**H**)?

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[1] $g = (u_A)_* \circ (v_\nabla)_{|A_F}$.

Theorem

Let A be a frame with property $(\mathbf{H})$ then for every $F \in A^{\wedge}$ with corresponding $Q \in \mathcal{Q}S$ compact we have

$$\mathcal{O}Q \simeq \uparrow Q',$$

that is, the frame of opens of the point space of A_F is isomorphic to a compact closed quotient of a Hausdorff space.

Some examples

- With the cofinite topology we look that $PA = NA$.
- With the cocountable topology we look that $\mathrm{pt}\, NA \subseteq \mathrm{pt}\, PA$.
- With a subregular topology on the real we have a 1-tidy frame that is not regular.
- With the maximal compact topology we have a 2-tidy frame that is not 1-tidy.
- With the boss topology on a tree we look that exist α-tidy frames.

Bibliografía I

- P. T. Johnstone, *Stone spaces*, Cambridge Studies in Advanced Mathematics, vol. 3, Cambridge University Press, Cambridge, 1982. MR 698074

- J. Picado and A. Pultr, *Frames and locales: Topology without points*, Frontiers in Mathematics, Springer Basel, 2012.

- J. Picado and A. Pultr, *Separation in point-free topology*, Springer, 2021.

- RA Sexton, *A point free and point-sensitive analysis of the patch assembly*, The University of Manchester (United Kingdom), 2003.

Bibliografía II

- RA Sexton, *Frame theoretic assembly as a unifying construct*, The University of Manchester (United Kingdom), 2000.
- RA Sexton and H. Simmons, *Point-sensitive and point-free patch constructions*, Journal of Pure and Applied Algebra **207** (2006), no. 2, 433-468.
- H. Simmons, *An Introduction to Frame Theory*, lecture notes, University of Manchester. Disponible en línea en https://web.archive.org/web/20190714073511/http://staff.cs.manchester.ac.uk/~hsimmons.
- H. Simmons, *Regularity, fitness, and the block structure of frames.* Applied Categorical Structures 14 (2006): 1-34.

Bibliografía III

H. Simmons, *The Vietoris modifications of a frame*. Unpublished manuscript (2004), 79pp., available online at http://www. cs. man. ac. uk/hsimmons.

A. Wilansky, *Between T1 and T2*, MONTHLY (1967): 261-266.

A. Zaldívar, *Introducción a la teoría de marcos* [notas curso], 2025. Universidad de Guadalajara.

Subregular topology on the real

For $S = \mathbb{R}$, let

$$\mathcal{O}S = \{U \cup (\mathbb{Q} \cap V) \mid U, V \in \mathcal{O}_m S\}$$

- S is T_2 (because $\mathcal{O}_m S \subseteq \mathcal{O}S$).
- $\mathcal{O}S$ is not fit (therefore, not regular).

Maximal compact topology

For $S = \mathbb{N}^2 \cup \{x, y\}$ and $R_n = \{(m, n) \mid m \in \mathbb{N}\}$, let

$$\mathcal{O}S = \mathcal{P}\mathbb{N}^2 \cup \mathcal{U}_x \cup \mathcal{V}_y,$$

where

$$\mathcal{U}_x = \{U \subseteq S \mid x \in U \text{ and } U \cap R_n \text{ is cofinite for all } n \in \mathbb{N}\},$$
$$\mathcal{V}_y = \{V \subseteq S \mid y \in V \text{ and } R_n \subseteq V \text{ for all but finitely many } n \in \mathbb{N}\}.$$

- S is packed but not T_2.
- $\mathcal{O}S$ is 2-tidy but not 1-tidy.
- S is compact
- $\mathcal{O}S$ is KC and fit.

Boss topology on a tree

For a tree $\mathbb{T}$, let $S = \mathbb{T} \cup \{*\}$ and $U \in \mathcal{O}S$ if

1. $(\forall x \in \mathbb{T})[x \in U \Rightarrow I(x) \setminus U \text{ is countable}]$,
2. $* \in U \Rightarrow (\forall x \in \mathbb{T})[I(x) \setminus U \text{ is countable}]$

holds, where

$$I(x) = \{y \in \mathbb{T} \mid y \text{ is an immediate successor of } x\}.$$