The patch frame
and some separation axioms in Frm

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A little example

Let $S = \mathbb{R}$ be with the topologies

$$\mathcal{O}_l S = \{(-\infty, a)\}, \quad \mathcal{O}_m S = \{(a, b)\}, \quad \mathcal{O}_n S = \{[a, b)\},$$

where $a, b \in S$. Then

$$\mathcal{O}_l S \hookrightarrow \mathcal{O}_m S \hookrightarrow \mathcal{O}_n S$$

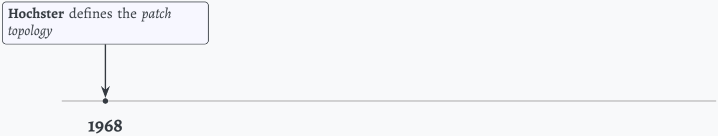
We can see that

$$\mathcal{O}_l^p S = \mathcal{O}_m S \simeq P\mathcal{O}_l S \quad \text{y} \quad \mathcal{O}_l^f S = \mathcal{O}_n S \simeq N\mathcal{O}_l S,$$

that is,

$$\mathcal{O}_l S = A \to PA \hookrightarrow NA$$

The patch construction



The patch construction

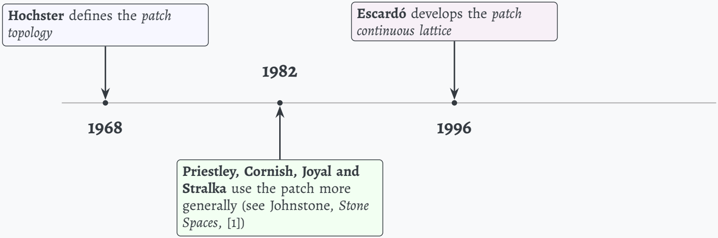
Hochster defines the *patch topology*

1982

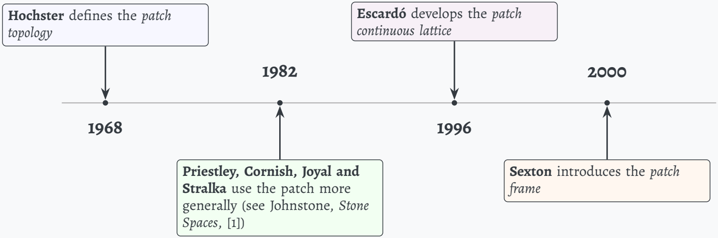
1968

Priestley, Cornish, Joyal and Stralka use the patch more generally (see Johnstone, *Stone Spaces*, [1])

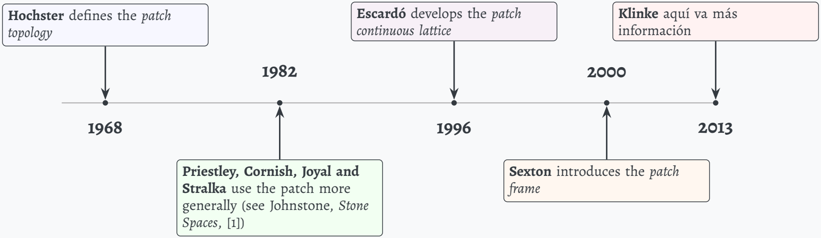
The patch construction



The patch construction



The patch construction



Frame theory

$$\mathsf{Frm} = \begin{cases} \mathsf{Obj}: & (A, \leqslant, \wedge, \bigvee, 1, 0) \\ \text{Arrows:} & f : A \to B \end{cases}$$

For $S \in \mathsf{Top}$,

$$(\mathcal{O}S, \subseteq, \cap, \bigcup, S, \emptyset) \in \mathsf{Frm}$$

Furthermore,

$$\begin{tikzcd} \mathsf{Top} \arrow[rr, bend left, "\mathcal{O}(\_)"] & & \mathsf{Frm} \arrow[ll, bend left, "\mathsf{pt}(\_)"] \end{tikzcd}$$

is an adjunction.

Packed spaces

${}^{p}S = (S, \mathcal{O}^{p}S)$, where $\mathcal{O}^{p}S$ is gerated by

$$\text{pbase} = \{U \cap Q' \mid U \in \mathcal{O}S, Q \in \mathcal{Q}S\}$$

Definition

$S \in \mathsf{Top}$ is **packed** if every compact (saturated) set is closed

$$S \text{ is packed} \iff {}^{p}S = S$$

$$T_2 \Rightarrow \text{packed} \Rightarrow T_1$$

Patch trivial

By *Hoffman-Mislove theorem*[1]

$$\mathrm{Pbase} = \{u_a \wedge v_F \mid a \in A, F \in A^{\wedge}\}$$

Definition

1. The **patch frame** of $A \in \mathsf{Frm}\ (PA)$, is the frame generated by Pbase
2. A is **patch trivial** if $A \simeq PA$.

[1]**Thm:** There is a bijective correspondence between $F \in A^{\wedge}$ and $Q \in \mathcal{Q}S$

∘ $F \in A^{\wedge}$ if F is a filter in A and $\forall X \subseteq A$, with X directed, if $\bigvee X \in F$, then $a \in F$ for some $a \in X$.

∘ For $a \in A$, $u_a(x) = a \vee x$ and $v_a(x) = (a \succ x)$ are nuclei in A.

∘ $v_F = f^{\infty}$, where $f = \dot{\bigvee}\{v_a \mid a \in F\}$

Tidy frames

Definition [[8], Def. 8.2.1]

Let $A \in \mathsf{Frm}$, $F \in A^{\wedge}$ and $\alpha \in \mathsf{Ord}$ be. We say that:

1. F is α-tidy if for $x \in F$, $d \vee x = 1$, where
$$d = d(\alpha) = f^{\alpha}(0).$$
2. A is α-tidy if every $F \in A^{\wedge}$ is α-tidy.
3. A is **tidy** if it is α-tidy for some $\alpha \in \mathsf{Ord}$.

Proposition [[8], Lemma 8.2.2]

$$A \text{ is tidy} \quad \Longleftrightarrow \quad A \text{ is patch trivial.}$$

Objectives

1. Understand tidy frames in more detail.
2. To explore the relationship with some separation axioms in Frm.
3. Provide tools to study the tidy frames.
4. Give examples.

Separation axioms in Frm

$$\begin{tikzcd}
 & (\mathbf{reg}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
(\mathbf{fit}) \arrow[d, Rightarrow] \arrow[dr, Rightarrow] \arrow[r, dotted, no head] & (\mathbf{sH}) \arrow[d, Rightarrow] \arrow[dr, Rightarrow] \arrow[r, dotted, no head] \arrow[dl, dotted, no head] & (\mathbf{H}) + (\mathbf{sfit}) \arrow[d, Rightarrow] \\
(\mathbf{sfit}) \arrow[r, dotted, no head] & T_1 & (\mathbf{H}) \arrow[l, Rightarrow]
\end{tikzcd}$$

∘$\forall\, a \not\leqslant b \in A$, then

∘**(reg):** $\exists\, x, y \in A$ such that $a \vee x = 1, y \not\leqslant b$ and $x \wedge y = 0$.

∘**(H):** $\exists\, c \in A$ such that $c \not\leqslant a$ and $\neg c \leqslant b$.

∘**(fit):** $\exists\, x, y \in A$ such that $x \vee a = 1, y \not\leqslant b$ and $x \wedge y \leqslant b$.

∘**(sfit):** $\exists\, c \in A$ such that $c \vee a = 1 \neq c \vee b$.

∘**(sH)** and T_1 are notion some different. All this can be found in [5].

Properties of the tidy frames

This is a summary of the properties that Sexton includes in [8]

- In the spatial case ($A = \mathcal{O}S$),

$$\begin{aligned} \mathcal{O}S \text{ is 0-tidy} &\iff S = \emptyset \\ \mathcal{O}S \text{ is 1-tidy} &\iff S \text{ is } T_2 \\ \mathcal{O}S \text{ is tidy} &\iff S \text{ is packed and stacked.} \end{aligned}$$

- For $A \in \mathrm{Frm}$ arbitrary

$$\begin{aligned} A \text{ is } (\mathbf{reg}) &\Rightarrow A \text{ is tidy} \\ A \text{ is } (\mathbf{fit}) &\Rightarrow A \text{ is tidy} \\ A \text{ is tidy} &\Rightarrow A \text{ is } T_1 \end{aligned}$$

Some results

If $(f : A \to B) \in \mathsf{Frm}$, $G \in A^\wedge$ and $F \in B^\wedge$, then

$$b \in f[G] \iff f_*(b) \in G \quad \text{and} \quad a \in f_*[F] \iff f(a) \in F.$$

Also, if $F \in B^\wedge$, then $f_*(F) \in A^\wedge$.

Proposition

For f^∞ and f_j^∞ the nuclei associated to F and j_*F, respectively, we have

$$j \circ f_j^\infty \leqslant f^\infty \circ j$$

Proof

By induction transfinite. $\square$

More properties of the tidy frames

Proposition

If $A \in \mathsf{Frm}$ is tidy and $j \in NA$, then A_j is tidy.

Proof

- We take $x \in F \in A_j^{\wedge}$ and $F \subseteq j_*[F] \in A^{\wedge}$.
- For f^{∞} and f_j^{∞} as before, we have

$$d = d(\alpha) \geqslant d_j(\alpha) = d_j$$

- Since A is tidy, then $d_j \vee x = 1$, for all $x \in j_*[F]$. In particular, for all $x \in F$.
- Therefore, $d \vee x = 1$.

$\square$

Corollary

If A is $(\mathbf{sH})$, A is tidy.

Proof

In $(\mathbf{sH})$ all compact quotient is closed. $\square$

$$
\begin{tikzcd}
 & (\mathbf{H}) \arrow[dr, Rightarrow, dotted] & \\
(\mathbf{reg}) \arrow[ur, Rightarrow] \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & (\mathbf{sH}) \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{Tidy} \\
 & (\mathbf{fit}) \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

Compact quotients

$$\text{Tidy} \iff \text{P. trivial} \iff u_d = v_F$$

Theorem

Let $A \in \mathsf{Frm}$ and $j \in NA$. Then

$$A_j \text{ is compact} \quad \iff \quad \nabla(j) \in A^{\wedge}.$$

Then

$$A_{u_d} \text{ is a closed quotient} \quad \text{and} \quad A_{v_F} \text{ is a compact quotient.}$$

If A is tidy, we have a compact closed quotient.

°$\nabla(j) = \{a \in A \mid j(a) = 1\}$ is a filter in A *(Admissibility filter)*.
°With $\nabla(j)$ we can define a "$\sim$" in NA: $j \sim k \iff \nabla(j) = \nabla(k)$.

KC frames

In [13], Wilansky defines a space S to be **KC** if every compact set is closed.

Definition

$A \in \mathsf{Frm}$ is a **KC frame** if every compact quotient is closed.

$$\mathbf{KC} \Rightarrow \text{Tidy}$$

Proposition

If $A \in \mathsf{Frm}$ es **KC** and $j \in NA$, then A_j is **KC**.

Proof

- Consider $k \in NA_j$ such that $\nabla(k) \in A_j^{\wedge}$.
- If $\nabla(k) \in A_j^{\wedge} \Rightarrow j_*[\nabla(k)] \in A^{\wedge}$.
- We take $l = j_* \circ k \circ j \in NA$ and $\nabla(l) \in A^{\wedge} \Rightarrow l = u_a$ for some $a \in A$.
- Furthermore $a = k(j(a))$.
- For $x, b \in A_j$ with $b = j(a)$ we have $u_b(x) = k(x)$.

$\square$

°**Proposition:** For $j \in NA$ and $k \in NA_j$. If $\nabla(k) \in A_j^{\wedge}$, $\nabla(j_*kj) \in A^{\wedge}$.

We can build the diagram (see [12])

$$\begin{tikzcd}
A \arrow[r, "v_F"] \arrow[dd, "U_A"'] & A_F \arrow[dd, "g"] \arrow[dr, "\simeq"] & \\
& & \mathcal{O}Q \\
\mathcal{O}S \arrow[r, "v_\nabla"'] & \mathcal{O}S_\nabla \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

What happens if A has property ($\mathbf{H}$)?

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[1] $g = (u_A)_* \circ (v_\nabla)_{|A_F}$.

Theorem

Let A be a frame with property $(\mathbf{H})$ then for every $F \in A^{\wedge}$ with corresponding $Q \in \mathcal{Q}S$ compact we have

$$\mathcal{O}Q \simeq \uparrow Q',$$

that is, the frame of opens of the point space of A_F is isomorphic to a compact closed quotient of a Hausdorff space.

Some examples

- With the cofinite topology we look that $PA = NA$.
- With the cocountable topology we look that $\mathrm{pt}\, NA \subseteq \mathrm{pt}\, PA$.
- With a subregular topology on the real we have a 1-tidy frame that is not regular.
- With the maximal compact topology we have a 2-tidy frame that is not 1-tidy.
- With the boss topology on a tree we look that exist α-tidy frames.

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