THE PATCH FRAME AND ITS RELATIONS WITH SEPARATION IN POINT-FREE TOPOLOGY

ABSTRACT.

1. INTRODUCTION

Aquí va la introducción.

2. PRELIMINARIES

3. HAUSDORFF PROPERTIES IMPLIES PATCH TRIVIALITY

Gadgets:
A the base frame
Its point space $S = \mathrm{pt}(A)$.
NA is the assembly of nucleus of A.
The compact saturated sets of S,

$$\mathcal{Q}(S).$$

The preframe of open filters of A,

$$A^\wedge.$$

The preframe of open filters of $\Omega(S)$.

$$\Omega(S)^\wedge.$$

The set of compact quotients

$$KA = \{j \in NA \mid A_j \text{ is compact}\}.$$

First we recall that every open filter $F \in A^\wedge$ has three faces, that is, determines (and its determine) by :

- The compact saturated $Q \in \mathcal{Q}S$.
- $\nabla \in \Omega(S)^\wedge$.
- The compact quotient $A \to A_F$.
- The fitted nucleus v_F.

Hoffman-Mislove can be rephrase:

There is a bijection between compact quotients of A and compact saturated sets of S

Definition 3.1. A frame has KC if every compact quotient of A is a closed one. In other words every compact sublocale is close.

Denote by $\mathcal{H}rm$ the subcategory of Frm of Hausdorff frames in the sense of Johnstone and Shu.

Lemma 3.2. *Let $A \in \mathrm{Frm}$ and $j, k \in NA$ be. We consider $F \in A_j^\wedge$ y $g = j_* k j^*$ where $F = \nabla(k)$. Then $\hat{F} = \nabla(g) \in A^\wedge$.*

Proof. $\hat{F}$ is a filter, because g is a nucleus. Let us consider $X \subseteq A$ such that $\bigvee X \in \hat{F}$. We must prove that $X \cap \hat{F} \neq \emptyset$.

If $\bigvee X \in \hat{F}$, then $g(\bigvee X) = (j_* k j^*)(\bigvee X) = 1$. Thus

$$j^*(\bigvee X) \leq j(\bigvee \{j(x) \mid x \in X\}) = j(\bigvee j[X]) = \bigvee_j X$$

and

$$1 = (j_* k)(j^*(\bigvee X)) \leq (j_* k)(\bigvee_j X).$$

$\{j(x) \mid x \in X\}$ is a directed set because X is directed. Then $j(j[X]) \subseteq A_j$, $\bigvee_j X \in F$ and $F \in A_j^\wedge$, so that, exists $x \in X$ such that $x = j(x) \in F$. Therefore $k(j(x)) = 1$ and $(j_* k j^*)(x) = 1$, so that, $x \in \nabla(g) = \hat{F}$. Thus $X \cap \hat{F} \neq \emptyset$. □

Lemma 3.3. *For $F \in A_j^\wedge$ and $\hat{F} \in A^\wedge$ as above, then $F \subseteq \hat{F}$.*

Proof. Let $x \in F$ be, then $j(x) = x$ and $k(x) = 1$. Thus

$$g(x) = (j_* k j^*)(x) = (j_* k)(j(x)) = j_*(k(x)) = j_*(1) = 1.$$

Therefore $x \in \hat{F}$. □

Lemma 3.4. *Let $A \in \mathrm{Frm}$ and $j \in NA$ be. If A is tidy then A_j is tidy.*

The block structure on a frame is an important problem and its related with some separation properties of frames.

Proposition 3.5. *For $F \in A^\wedge$ and $Q \in \mathcal{Q}S$, if $j \in [v_Q, w_Q]$, then $U_* j U^* \in [v_F, w_F]$, where U^* is the morfism spatial reflection U_* is the right adjoint.*

Proof. Since N is a functor, we have

$$\begin{tikzcd} A \arrow[d, "U"'] \arrow[r, mapsto, "N(\_)"] & NA \arrow[d, "N(U)"] \\ \mathcal{O}S & N\mathcal{O}S \end{tikzcd}$$

and $N(U)_*$ is the right adjoint of $N(U)^\wedge$. Note the following:

(1) $N(U)(j) \leq k \Leftrightarrow j \leq N(U)_* k$.
(2) If $k \in N\mathcal{O}S$ then $N(U)(j) \leq k \Leftrightarrow Uj \leq kU$.
(3) $N(U)_* k = U_* k U^*$ and $UN(U)_* k = k(U)$.

In 3), if $j = k$, $N(U)_*(j) = U_* j U^*$ and $UN(U)_* j = jU$. For $x \in F$

$$x \in A \xrightarrow{U^*} \mathcal{O}S \xrightarrow{j} \mathcal{O}S \xrightarrow{U_*} A$$

and $U_*(j(U(x)) = \bigwedge(S \setminus j(U(x)))$. Note that $U_*(j(U^*(x))) \subseteq \mathrm{pt}\, A$. Thus

$$\begin{aligned} x \in F \Leftrightarrow Q \subseteq U(x) \Leftrightarrow U(x) \in \nabla(j) = \nabla(Q) \Leftrightarrow S \setminus j(U(x)) = \emptyset \\ \Leftrightarrow \bigwedge(S \setminus j(U(x))) = 1 = (U_* j U^*)(x) \\ \Leftrightarrow x \in \nabla(U_* j U^*) \end{aligned}$$

Therefor $F = \nabla(U_* j U^*)$. □

In this way we have a function

$$\mho\colon [V_Q, W_Q] \to [V_F, W_F]$$

Proposition 3.6. *For every $A \in \mathcal{H}rm$ the interval corresponding to the block determined by a open filter $F \in A^\wedge$ is trivial, that is,*

$$[v_F, w_F] = \{*\}$$

Proof. We know that for all $F \in A^\wedge$ the following holds: $v_F \leq w_F$. As a contradition, suppose that exists $F \in A^\wedge$ such that $w_F \nleq v_F$, that is, exists $a \in A$ such that $w_F(a) \nleq v_F(a)$.

Note that $w_F(a) \neq 1$, otherwise

$$1 = w_F(a) = \bigwedge\{p \in M \mid a \leq p\} \leq p$$

and this is a contradition because $p \neq 1$.

Then $1 \neq w_F(a) \nleq v_F(a)$ and for the property (**H**), exists $u \in A$ such that

$$u \nleq w_F(a) \quad \text{y} \quad \neg u \nleq v_F(a) \tag{1}$$

Note that $0 \leq a$, then $w_F(0) \leq w_F(a)$ and $v_F(0) \leq v_F(a)$- Thus, for 1 we have that

$$i)\, u \nleq w_F(0) \quad \text{y} \quad ii)\, \neg u \nleq v_F(0). \tag{2}$$

For 2-(i) is true that $u \nleq \bigwedge M$, in particular, $u \nleq p$ for all $p \in M$. Therefore, $\neg u \leq p$ and $\neg u \leq w_F(0)$. If 2-(ii) is true, then $u \notin F$, in otherwise

$$u \in F \Rightarrow v_u \leq f \Rightarrow v_u(0) = \neg u \leq f(0)$$

and this is a contradition. Thus, for the Birkhoff's separation lemma, exists a completely prime filter G such that $u \notin G \supseteq F$. We take

$$q = \bigvee\{y \in A \mid y \notin G\}$$

the point corresponding to G. Thus, $u \notin G$, $u \leq q$. If $q \notin F$, then exists $m \in M$ such that $q \leq m$. Sinse q is maximum, $q = m$ or $m = 1$, but $m \neq 1$ ($1 \in F$ and $M = A \setminus F$), then $m = q \in M$. Hence $u \nleq q$ and this is a contradition. Therefore $v_F = w_F$. □

A consequence of the Proposition 3.6 is that $v_F = w_F$, so that, $A_{v_F} = A_{w_F}$ and $A_{w_F} \simeq \mathcal{O}M \simeq \mathcal{O}Q$. Thus, for all $j \in KA$ we have that $j = v_F$. Then in the Huasdorff case

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_F \arrow[d, "g"] \\ \mathcal{O}S \arrow[r] & \mathcal{O}S_{\nabla} \end{tikzcd}$$

where g is an isomorphism and $A_F \simeq \mathcal{O}Q \simeq \mathcal{O}S_{\nabla}$.

On the other hand, $U_* u_{Q'} U^* = v_F$ if and only if $u_{Q'} U^* = U^* v_F$, for the adjuntion properties and U^* the spatial reflection morphism. Therefore

$$\begin{tikzcd} A \arrow[r, "v_F"] \arrow[d, shift left, "U"] & A \arrow[d, shift right, "U"'] \\ \mathcal{O}S \arrow[r, "v_{\nabla}"'] \arrow[u, shift left, "U_*"] & \mathcal{O}S \arrow[u, shift right, "U_*"'] \end{tikzcd}$$

so that, if $A \in \mathcal{H}rm$ then patch trivial implies KC.

The above is the proof of the following theorem.

Theorem 3.7. *If $A \in \mathcal{H}rm$. then every compact quotient is isomrphic to a closed quotient of the topology of a Hausdorff space.*

Proposition 3.8. *Every Hausdorff frame A (in the sense of Johnstone and Shou) is tidy, that is, A is patch trivial.*

Proof. □

REFERENCES