THE PATCH FRAME AND ITS RELATIONS WITH SEPARATION IN POINT-FREE TOPOLOGY

ABSTRACT.

1. INTRODUCTION

Aquí va la introducción.

2. PRELIMINARIES

3. HAUSDORFF PROPERTIES IMPLIES PATCH TRIVIALITY

Gadgets:
A the base frame
Its point space $S = \mathrm{pt}(A)$.
NA is the assembly of nucleus of A.
The compact saturated sets of S,

$$\mathcal{Q}(S).$$

The preframe of open filters of A,

$$A^\wedge.$$

The preframe of open filters of $\Omega(S)$.

$$\Omega(S)^\wedge.$$

The set of compact quotients

$$KA = \{j \in NA \mid A_j \text{ is compact}\}.$$

First we recall that every open filter $F \in A^\wedge$ has three faces, that is, determines (and its determine) by :

- The compact saturated $Q \in \mathcal{Q}S$.
- $\nabla \in \Omega(S)^\wedge$.
- The compact quotient $A \to A_F$.
- The fitted nucleus v_F.

Hoffman-Mislove can be rephrase:

There is a bijection between compact quotients of A and compact saturated sets of S

4. COMPACT QUOTIENTS

Definition 4.1. A frame has KC if every compact quotient of A is a closed one. In other words every compact sublocale is closed.

Denote by $\mathcal{H}rm$ the subcategory of Frm of Hausdorff frames in the sense of Johnstone and Shu.

If $f^*\colon A \to B$ is a frame morphism and $F \subseteq A$, $G \subseteq B$ filters in A, B, respectively, we can produce new filters as follows

$$b \in f^*F \Leftrightarrow f_*(b) \in F \quad \text{and} \quad a \in f_*G \Leftrightarrow f^*(a) \in G \tag{1}$$

where $a \in A, b \in B$ and f_* is the right adjoint of f^*. Here $f^*F \subseteq B$ and $f_*G \subseteq A$ are filters on B and A, respectively.

Proposition 4.2. *For $f = f^*\colon A \to B$ a frame morphism and $G \in B^\wedge$, then $f_*G \in A^\wedge$.*

Proof. By (1), f_*G is a filter on A. We need f_*G to satisfy the open filter condition. Let $X \subseteq A$ be such that $\bigvee X \in f_*G$, with X directed. Then

$$Y = \{f(x) \mid x \in X\}$$

is directed and $f(\bigvee X) = \bigvee f[X] = \bigvee Y \in G$. Since G is a open filter, exists $y = f(x) \in Y$ such that $y \in G$. Thus $x \in f_*G$, so that, $f_*G \in A^\wedge$. $\square$

In [4], the autor says that $A \in \mathbf{Frm}$ is *tidy* if for all $F \in A^\wedge$

$$x \in F \Rightarrow u_d(x) = d \vee x = 1$$

where $d = d(\alpha) = f^\alpha(0)$, $f = \dot{\bigvee}\{v_y \mid y \in F\}$, $v_y \in NA$ and $0 = 0_A$ (the reason for the last two clarifications will be understood later).

We want translate this same notion, but for A_j when $j \in NA$, so that, for all $F \in A_j^\wedge$, if $x \in F$ then $d \vee x = 1$, with d similar to before, because for this case we have that $v_y \in NA_j$ and $0_{A_j} = j(0)$.

Simmons proves in [6] (Lamma 8.9 and Corollary 8.10), that the diagram

$$\begin{tikzcd} A \arrow[r, "f^\infty"] \arrow[d, "U_A"'] & A \arrow[d, "U_A"] \\ \mathcal{O}S \arrow[r, "F^\infty"'] & \mathcal{O}S \end{tikzcd}$$

commutes laxly, so that, $U_A \circ f^\infty \leq F^\infty \circ U_A$. In this diagram U_A is the spatial reflection morphism, f^∞ and F^∞ represent the associated nuclei asociados to the filters $F \in A^\wedge$ and $\nabla \in \mathcal{O}S^\wedge$. Also f^∞ and F^∞ are idempotent closeds associated to the prenucleis f and F respectively.

We prove something more general here, since we consider the diagram

$$\begin{tikzcd} A \arrow[r, "\hat{f}^{\infty}"] \arrow[d, "j"'] & A \arrow[d, "j"] \\ A_j \arrow[r, "f^{\infty}"'] & A_j \end{tikzcd}$$

where $\hat{f}^{\infty}$ is the nuclei associated to the filters $j_*F \in A^{\wedge}$ and $j \in NA$.

Lemma 4.3. *For j, f and $\hat{f}$ as before, it holds that $j \circ \hat{f} \leq f \circ j$.*

Proof. By (1) is true that

$$\hat{f} = \dot{\bigvee}\{v_y \mid y \in j_*F\} \quad \text{and} \quad f = \dot{\bigvee}\{v_{j(y)} \mid j(y) \in F\}.$$

then, for $a \in A$ it is hold

$$v_y(a) = (y \succ a) \leq \hat{f}(a) \leq j(\hat{f}(a)).$$

Also, for all $a, y \in A$, $(y \succ a) \wedge y = y \wedge a$ and

$$\begin{aligned} j((y \succ a) \wedge y) \leq j(a) &\Leftrightarrow j(y \succ a) \wedge j(y) \leq j(a) \\ &\Leftrightarrow j(y \succ a) \leq (j(y) \succ j(a)). \end{aligned}$$

Thus

$$v_y(a) \leq j(\hat{f}(a)) \leq (j(y) \succ j(a)) = v_{j(y)}(j(a)) \leq f(j(a)).$$

Therefore $j \circ \hat{f} \leq f \circ j$. □

Now, we prove the above, but for all α-ordinals.

Corollary 4.4. *For j, f and $\hat{f}$ as before, it is hold that $j \circ \hat{f}^{\alpha} \leq f^{\alpha} \circ j$*

Proof. For an ordinal α we will check that $j \circ \hat{f}^{\alpha} \leq f^{\alpha} \circ j$. We will do it by transfinite induction.

If $\alpha = 0$, it is trivial.

For the induction step, we assume that for α it holds. Then

$$j \circ \hat{f}^{\alpha+1} = j \circ \hat{f} \circ \hat{f}^{\alpha} \leq f \circ j \circ \hat{f}^{\alpha} \leq f \circ f^{\alpha} \circ j = f^{\alpha+1} \circ j,$$

where the first inequality is Lemma 4.3 and the second is true by the induction hypothesis.

If λ is a limit ordinal, then

$$\hat{f}^{\lambda} = \bigvee\{\hat{f}^{\alpha} \mid \alpha < \lambda\}, \quad f^{\lambda} = \bigvee\{f^{\alpha} \mid \alpha < \lambda\}$$

and

$$j \circ \hat{f}^{\lambda} = j \circ \bigvee_{\alpha<\lambda} \hat{f}^{\alpha} \leq \bigvee_{\alpha<\lambda} j \circ \hat{f}^{\alpha}.$$

Thus, by the induction hypothesis, we have that

$$j \circ \hat{f}^{\alpha} \leq f^{\alpha} \circ j \Rightarrow \bigvee_{\alpha<\lambda} j \circ \hat{f}^{\alpha} \leq \bigvee_{\alpha<\lambda} f^{\alpha} \circ j.$$

Therefore $j \circ \hat{f}^{\lambda} \leq f^{\lambda} \circ j$. $\square$

By the Corollary 4.4, we have that $j \circ \hat{f}^{\infty} \leq f^{\infty} \circ j$ is true. Futhermore, by H-M Theorem, $f^{\infty} = v_F$ and $\hat{f}^{\infty} = v_{j_*F}$. With this in mind, we have the following diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "(v_{j_*F})^*"] \arrow[d, "j"'] \arrow[dr, "H"] & A_{j_*F} \arrow[l, shift left, "(v_{j_*F})_*"] \\
A_j \arrow[r, shift right, "(v_F)^*"'] & A_F \arrow[l, shift right, "(v_F)_*"']
\end{tikzcd}$$

Here, A_F and A_{j_*F} are the compact quotients produced by v_F and v_{j+F}, respectively. The morfism $H\colon A \to A_F$ is defined by $H = v_F \circ j$. Futhermore, $(v_F)_*$ and $(v_{j_*F})_*$ are inclusions.

Let $h\colon A_{j_*F} \to A_j$ be such that, for $x \in A_{j_*F}$, $h(x) = H(x)$. Therefore, if $h = H_{|A_{j_*F}}$, then the above diagram commutes.

We need that h to be a frame morphism. First, by the difinition of h, this is $\wedge$-morphism. It remains to be seen that h is $\bigvee$-morphism.

The joins in A_{j_*F} and A_F are calculated differently. Thus, let $\hat{\bigvee}$ be join in A_{j_*F} and let $\tilde{\bigvee}$ be join in A_F. Therefore

$$\hat{\bigvee} = v_{j_*F} \circ \bigvee \quad \text{and} \quad \tilde{\bigvee} = v_F \circ \bigvee,$$

that is, for $X \subseteq A$, $Y \subseteq A_j$,

$$\hat{\bigvee} X = v_{j_*F}(\bigvee X) \quad \text{and} \quad \tilde{\bigvee} Y = v_F(\bigvee Y).$$

Since H is a frame morphism, then $H \circ \bigvee = \tilde{\bigvee} \circ H$. Let us get something similar to h.

Lemma 4.5. $h \circ \hat{\bigvee} = \tilde{\bigvee} \circ h$.

Proof. It is enough to check the comparison $h \circ \hat{\bigvee} \leq \tilde{\bigvee} \circ h$. Thus

$$h \circ \hat{\bigvee} = H \circ v_{j_*F} \circ \bigvee = v_F \circ j \circ v_{j_*F} \circ \bigvee \leq v_F \circ v_F \circ j \circ \bigvee$$

where the inequality is the Corollary 4.4. Futhermore, $v_F \circ v_F = v_F$, Then

$$h \circ \hat{\bigvee} \leq v_F \circ j \circ \bigvee = H \circ \bigvee = \tilde{\bigvee} \circ H = \tilde{\bigvee} \circ h.$$

Therefore $h \circ \hat{\bigvee} = \tilde{\bigvee} \circ h$. $\square$

The following is the commutative diagram

$$\begin{tikzcd} A \arrow[r, "v_{j_*F}"] \arrow[d, "j"'] & A_{j_*F} \arrow[d, "h"] \\ A_j \arrow[r, "v_F"'] & A_F \end{tikzcd}$$

and with this we could analyze some compact quotients, for example, closed compact quotients.

Definition 4.6. Let A be a frame and $F \in A^\wedge$. The compact quotient A_F is closed if $A_F = A_{u_d}$ for some $d \in A$.

Proposition 4.7. *If A is a tidy frame, then A_j is tidy.*

Proof. It is easy to prove that $F \subseteq j_*F$. Since A is tidy and $F \in A^\wedge$, it is true that

$$x \in F \Rightarrow \hat{d} \vee x = 1,$$

where $\hat{d} = d(\alpha) = f^\alpha(0)$.
If $\hat{d} \leq d$, then $d \vee x = 1$, for $d = d(\alpha) = f^\alpha(j(0))$.

Thus, for Corollary 4.4

$$\hat{d} = \hat{d}(\alpha) \leq j(\hat{d}(\alpha)) = j(\hat{f}^\alpha(0)) \leq f^\alpha(j(0)) = d(\alpha) = d.$$

Therefore if $x \in F$, then $d \vee x = 1$ and A_j is tidy. □

Proposition 4.8. *If A has KC, then A_j has KC for every $j \in N(A)$.*

Proof. We consider $k \in NA_j$ such that $(A_j)_k$ is compact. Since any open filter is admissible, we have $\nabla(k) \in A_j^\wedge$ and by Proposition 4.2 $j_*\nabla(K) \in A^\wedge$.

Let $l = j_* \circ k \circ j^* \in NA$ be, then A_l is a compact quotient of A and exists $a \in A$ such that $l = u_a$.

We need that $k = u_b$ for some $b \in A_j$. For $x \in A_j$ and $b = j(a)$

$$\begin{aligned} u_b(x) = b \vee x = b \vee j(x) &= j(j(a) \vee j(x)) \\ &= j(k(j(a)) \vee x) \\ &= j(u_a(x)) \\ &= j(k(x)) \\ &= k(x). \end{aligned}$$

Therefore $u_b = k$. □

Proposition 4.9. *If A is a KC frame, the A is a T_1 frame.*

Proof. A frame is T_1 if and only if for all $p \in \operatorname{pt} A$, p is maximum (de donde leiste que asi se esvruibe maximo en ingles?, se dice maximal ve como lo escribe rossy harold o yo). Let $p \in \mathrm{pt}\, A$ and $a \in A$ be such that $p \leq a \leq 1$. We consider

$$w_p(x) = \begin{cases} 1 & \text{si} \quad x \nleq p \\ \\ p & \text{si} \quad x \leq p \end{cases}$$

for $x \in A$. $P = \nabla(w_p) = \{x \in A \mid x \nleq p\}$ is a filter completely prime (in particular, $P \in A^\wedge$). Since A is KC, then A_{w_p} is a closed compact quotient. Thus $u_p = w_p$, futhermore

$$u_p(a) = a \quad \text{and} \quad w_p(a) = 1.$$

that is, $a = 1$. Therefore p is maximum. □

5. Admissibility intervals

The block structure on a frame is an important problem and its related with some separation properties of frames.

Proposition 5.1. *For $F \in A^\wedge$ and $Q \in \mathcal{Q}S$, if $j \in [v_Q, w_Q]$, then $U_* j U^* \in [v_F, w_F]$, where U^* is the morfism spatial reflection U_* is the right adjoint.*

Proof. Since N is a functor, we have

$$\begin{array}{ccc} A & & NA \\ {\scriptstyle U}\downarrow & \xrightarrow{N(\_)} & \downarrow{\scriptstyle N(U)} \\ \mathcal{O}S & & N\mathcal{O}S \end{array}$$

and $N(U)_*$ is the right adjoint of $N(U)^\wedge$. Note the following:

(1) $N(U)(j) \leq k \Leftrightarrow j \leq N(U)_* k$.
(2) If $k \in N\mathcal{O}S$ then $N(U)(j) \leq k \Leftrightarrow Uj \leq kU$.
(3) $N(U)_* k = U_* k U^*$ and $UN(U)_* k = k(U)$.

In 3), if $j = k$, $N(U)_*(j) = U_* j U^*$ and $UN(U)_* j = jU$. For $x \in F$

$$x \in A \xrightarrow{U^*} \mathcal{O}S \xrightarrow{j} \mathcal{O}S \xrightarrow{U_*} A$$

and $U_*(j(U(x)) = \bigwedge(S \setminus j(U(x)))$. Note that $U_*(j(U^*(x))) \subseteq \mathrm{pt}\, A$. Thus

$$\begin{aligned} x \in F \Leftrightarrow Q \subseteq U(x) \Leftrightarrow U(x) \in \nabla(j) = \nabla(Q) \Leftrightarrow S \setminus j(U(x)) = \emptyset \\ \Leftrightarrow \bigwedge(S \setminus j(U(x))) = 1 = (U_* j U^*)(x) \\ \Leftrightarrow x \in \nabla(U_* j U^*) \end{aligned}$$

Therefor $F = \nabla(U_* j U^*)$. □

In this way we have a function

$$\mho \colon [V_Q, W_Q] \to [V_F, W_F]$$

Proposition 5.2. *For every $A \in \mathcal{H}rm$ the interval corresponding to the block determined by a open filter $F \in A^{\wedge}$ is trivial, that is,*

$$[v_F, w_F] = \{*\}$$

Proof. We know that for all $F \in A^{\wedge}$ the following holds: $v_F \leq w_F$. As a contradition, suppose that exists $F \in A^{\wedge}$ such that $w_F \nleq v_F$, that is, exists $a \in A$ such that $w_F(a) \nleq v_F(a)$.

Note that $w_F(a) \neq 1$, otherwise

$$1 = w_F(a) = \bigwedge\{p \in M \mid a \leq p\} \leq p$$

and this is a contradition because $p \neq 1$.

Then $1 \neq w_F(a) \nleq v_F(a)$ and for the property (**H**), exists $u \in A$ such that

$$u \nleq w_F(a) \quad \text{and} \quad \neg u \nleq v_F(a) \tag{2}$$

Due to monotony, $w_F(0) \leq w_F(a)$ and $v_F(0) \leq v_F(a)$- Thus, for 2 we have that

$$i)\, u \nleq w_F(0) \quad \text{and} \quad ii)\, \neg u \nleq v_F(0). \tag{3}$$

For 3-(i) is true that $u \nleq \bigwedge M$, in particular, $u \nleq p$ for all $p \in M$. Therefore, $\neg u \leq p$ and $\neg u \leq w_F(0)$. If 3-(ii) is true, then $u \notin F$, in otherwise

$$u \in F \Rightarrow v_u \leq f \Rightarrow v_u(0) = \neg u \leq f(0)$$

and this is a contradition. Thus, for the Birkhoff's separation lemma, exists a completely prime filter G such that $u \notin G \supseteq F$. We take

$$q = \bigvee\{y \in A \mid y \notin G\}$$

the point corresponding to G. Thus, $u \notin G$, $u \leq q$. If $q \notin F$, then $q \in M$ and $u \nleq q$. Hence $u \leq q$, $u \nleq q$ and this is a contradition. $\square$

A consequence of the Proposition 5.2 is that $v_F = w_F$, so that, $A_{v_F} = A_{w_F}$ and $A_{w_F} \simeq \mathcal{O}M \simeq \mathcal{O}Q$. Thus, for all $j \in KA$ we have that $j = v_F$. Then in the Huasdorff case

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_F \arrow[d, "g"] \\ \mathcal{O}S \arrow[r] & \mathcal{O}S_{\nabla} \end{tikzcd}$$

where g is an isomorphism and $A_F \simeq \mathcal{O}Q \simeq \mathcal{O}S_{\nabla}$.

On the other hand, $U_* u_{Q'} U^* = v_F$ if and only if $u_{Q'} U^* = U^* v_F$, for the adjuntion properties and U^* the spatial reflection morphism. Therefore

$$\begin{tikzcd} A \arrow[r, "v_F"] \arrow[d, shift left, "U"] & A \arrow[d, shift right, "U"'] \\ \mathcal{O}S \arrow[u, shift left, "U_*"] \arrow[r, "v_\nabla"'] & \mathcal{O}S \arrow[u, shift right, "U_*"'] \end{tikzcd}$$

so that, if $A \in \mathcal{H}rm$ then patch trivial implies KC.

The above is the proof of the following theorem.

Theorem 5.3. *If $A \in \mathcal{H}rm.$ then every compact quotient is isomrphic to a closed quotient of the topology of a Hausdorff space.*

Corollary 5.4. *If $A \in \mathcal{H}rm.$*

$$\mathcal{Q}(S) \cong \mathrm{pt}(V(A))$$

Proposition 5.5. *Every Hausdorff frame A (in the sense of Johnstone and Shou) is tidy, that is, A is patch trivial.*

Proof. □

LA bibliografia hay que ponerla con bibtex, no a mano EJEMPLOS DE marcos ptrivial que no sean KC

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