

THE PATCH FRAME AND ITS RELATIONS WITH SEPARATION IN POINT-FREE TOPOLOGY

ABSTRACT.

1. INTRODUCTION

Aquí va la introducción.

2. PRELIMINARIES

3. HAUSDORFF PROPERTIES IMPLIES PATCH TRIVIALITY

Gadgets:

A the base frame

Its point space $S = \text{pt}(A)$.

NA is the assembly of nucleus of A .

The compact saturated sets of S ,

$$Q(S).$$

The preframe of open filters of A ,

$$A^\wedge.$$

The preframe of open filters of $\Omega(S)$.

$$\Omega(S)^\wedge.$$

The set of compact quotients

$$KA = \{j \in NA \mid A_j \text{ is compact}\}.$$

First we recall that every open filter $F \in A^\wedge$ has three faces, that is, determines (and its determine) by :

- The compact saturated $Q \in QS$.
- $\nabla \in \Omega(S)^\wedge$.
- The compact quotient $A \rightarrow A_F$.
- The fitted nucleus v_F .

Hoffman-Mislove can be rephrase:

There is a bijection between compact quotients of A and compact saturated sets of S

4. COMPACT QUOTIENTS

Definition 4.1. A frame has KC if every compact quotient of A is a closed one. In other words every compact sublocale is closed.

Denote by \mathcal{Hrm} the subcategory of Frm of Hausdorff frames, that is, $A \in \mathcal{Hrm}$ if and only if:

If $f^*: A \rightarrow B$ is a frame morphism and $F \subseteq A$, $G \subseteq B$ filters in A , B , respectively, we can produce new filters as follows

$$(1) \quad b \in f^*F \Leftrightarrow f_*(b) \in F \quad \text{and} \quad a \in f_*G \Leftrightarrow f^*(a) \in G$$

where $a \in A$, $b \in B$ and f_* is the right adjoint of f^* . Here $f^*F \subseteq B$ and $f_*G \subseteq A$ are filters on B and A , respectively.

Proposition 4.2. For $f = f^*: A \rightarrow B$ a frame morphism and $G \in B^\wedge$, then $f_*G \in A^\wedge$.

Proof. By (1), f_*G is a filter on A . We need f_*G to satisfy the open filter condition. Let $X \subseteq A$ be such that $\bigvee X \in f_*G$, with X directed. Then

$$Y = \{f(x) \mid x \in X\}$$

is directed and $f(\bigvee X) = \bigvee f[X] = \bigvee Y \in G$. Since G is a open filter, exists $y = f(x) \in Y$ such that $y \in G$. Thus $x \in f_*G$, so that, $f_*G \in A^\wedge$. \square

In [Sex03], the autor says that $A \in \mathbf{Frm}$ is *tidy* if for all $F \in A^\wedge$

$$x \in F \Rightarrow u_d(x) = d \vee x = 1$$

where $d = d(\alpha) = f^\alpha(0)$, $f = \bigvee \{v_y \mid y \in F\}$, $v_y \in NA$ and $0 = 0_A$ (the reason for the last two clarifications will be understood later a que te refieres).

We want translate this same notion, but for A_j when $j \in NA$, so that, for all $F \in A_j^\wedge$, if $x \in F$ then $d \vee x = 1$, with d similar to before, because for this case we have that $v_y \in NA_j$ and $0_{A_j} = j(0)$.

In [Sim04, Lemma 8.9 and Corollary 8.10] the author shows, that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f^\infty} & A \\ U_A \downarrow & & \downarrow U_A \\ \mathcal{OS} & \xrightarrow{F^\infty} & \mathcal{OS} \end{array}$$

commutes laxly, that is, $U_A \circ f^\infty \leq F^\infty \circ U_A$. In this diagram U_A is the spatial reflection morphism, f^∞ and F^∞ represent the associated nuclei to the filters $F \in A^\wedge$ and $\nabla \in \mathcal{OS}^\wedge$. Also f^∞ and F^∞ are idempotent closededs associated to the prenuclei f and F respectively.

We prove something more general here, since we consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\hat{f}^\infty} & A \\ j \downarrow & & \downarrow j \\ A_j & \xrightarrow{f^\infty} & A_j \end{array}$$

where \hat{f}^∞ is the nuclei associated to the filter $j_*F \in A^\wedge$ and $j \in NA$.

Lemma 4.3. *For j , f and \hat{f} as above, it holds that $j \circ \hat{f} \leq f \circ j$.*

Proof. By (1) is true that

$$\hat{f} = \bigvee \{v_y \mid y \in j_*F\} \quad \text{and} \quad f = \bigvee \{v_{j(y)} \mid j(y) \in F\}.$$

then, for $a \in A$ it is hold

$$v_y(a) = (y \succ a) \leq \hat{f}(a) \leq j(\hat{f}(a)).$$

Also, for all $a, y \in A$, $(y \succ a) \wedge y = y \wedge a$ and

$$\begin{aligned} j((y \succ a) \wedge y) \leq j(a) &\Leftrightarrow j(y \succ a) \wedge j(y) \leq j(a) \\ &\Leftrightarrow j(y \succ a) \leq (j(y) \succ j(a)). \end{aligned}$$

Thus

$$v_y(a) \leq j(\hat{f}(a)) \leq (j(y) \succ j(a)) = v_{j(y)}(j(a)) \leq f(j(a)).$$

Therefore $j \circ \hat{f} \leq f \circ j$. □

Now, we prove the above, but for all α -ordinals.

Corollary 4.4. *For j , f and \hat{f} as before, it is hold that $j \circ \hat{f}^\alpha \leq f^\alpha \circ j$*

Proof. For an ordinal α we will check that $j \circ \hat{f}^\alpha \leq f^\alpha \circ j$. We will do it by transfinite induction.

If $\alpha = 0$, it is trivial.

For the induction step, we assume that for α it holds. Then

$$j \circ \hat{f}^{\alpha+1} = j \circ \hat{f} \circ \hat{f}^\alpha \leq f \circ j \circ \hat{f}^\alpha \leq f \circ f^\alpha \circ j = f^{\alpha+1} \circ j,$$

where the first inequality is Lemma 4.3 and the second is true by the induction hypothesis.

If λ is a limit ordinal, then

$$\hat{f}^\lambda = \bigvee \{\hat{f}^\alpha \mid \alpha < \lambda\}, \quad f^\lambda = \bigvee \{f^\alpha \mid \alpha < \lambda\}$$

and

$$j \circ \hat{f}^\lambda = j \circ \bigvee_{\alpha < \lambda} \hat{f}^\alpha \leq \bigvee_{\alpha < \lambda} j \circ \hat{f}^\alpha.$$

Thus, by the induction hypothesis, we have that

$$j \circ \hat{f}^\alpha \leq f^\alpha \circ j \Rightarrow \bigvee_{\alpha < \lambda} j \circ \hat{f}^\alpha \leq \bigvee_{\alpha < \lambda} f^\alpha \circ j.$$

Therefore $j \circ \hat{f}^\lambda \leq f^\lambda \circ j$. \square

By the Corollary 4.4, we have that $j \circ \hat{f}^\infty \leq f^\infty \circ j$ is true. Furthermore, by H-M Theorem (preliminaries con la idea de la prueba nueva), $f^\infty = v_F$ and $\hat{f}^\infty = v_{j_*F}$. With this in mind, we have the following diagram

$$\begin{array}{ccc} A & \xrightleftharpoons[(v_{j_*F})_*]{(v_{j_*F})^*} & A_{j_*F} \\ j \downarrow & \searrow H & \\ A_j & \xrightleftharpoons[(v_F)^*]{(v_F)_*} & A_F \end{array}$$

ES este diagrama hay que poner punteada la flecha que eiria en los cocientes Here, A_F and A_{j_*F} are the compact quotients produced by v_F and v_{j_*F} , respectively. The morfism $H: A \rightarrow A_F$ is defined by $H = v_F \circ j$. Furthermore, $(v_F)_*$ and $(v_{j_*F})_*$ are inclusions.

Let $h: A_{j_*F} \rightarrow A_j$ be such that, for $x \in A_{j_*F}$, $h(x) = H(x)$. Therefore, if $h = H|_{A_{j_*F}}$, then the above diagram commutes.

We need that h to be a frame morphism. First, by the difinition of h , this is \wedge -morphism. It remains to be seen that h is \vee -morphism.

The joins in A_{j_*F} and A_F are calculated differently. Thus, let $\hat{\vee}$ be join in A_{j_*F} and let $\tilde{\vee}$ be join in A_F . Therefore

$$\hat{\vee} = v_{j_*F} \circ \vee \quad \text{and} \quad \tilde{\vee} = v_F \circ \vee,$$

that is, for $X \subseteq A$, $Y \subseteq A_j$,

$$\hat{\vee} X = v_{j_*F}(\vee X) \quad \text{and} \quad \tilde{\vee} Y = v_F(\vee Y).$$

Since H is a frame morphism, then $H \circ \vee = \tilde{\vee} \circ H$. Let us get something similar to h .

Lemma 4.5. $h \circ \hat{\vee} = \tilde{\vee} \circ h$.

Proof. It is enough to check the comparison $h \circ \hat{\vee} \leq \tilde{\vee} \circ h$. Thus

$$h \circ \hat{\vee} = H \circ v_{j_*F} \circ \vee = v_F \circ j \circ v_{j_*F} \circ \vee \leq v_F \circ v_F \circ j \circ \vee$$

where the inequality is the Corollary 4.4. Furthermore, $v_F \circ v_F = v_F$, then

$$h \circ \hat{\vee} \leq v_F \circ j \circ \vee = H \circ \vee = \tilde{\vee} \circ H = \tilde{\vee} \circ h.$$

Therefore $h \circ \hat{V} = \tilde{V} \circ h$. \square

With this we prove the following.

Proposition 4.6. *The diagram*

$$\begin{array}{ccc} A & \xrightarrow{v_{j_*F}} & A_{j_*F} \\ j \downarrow & & \downarrow h \\ A_j & \xrightarrow{v_F} & A_F \end{array}$$

is commutative.

HAY QUE PONER LA PRUEBA With the above diagram, we could analyze some compact quotients, for example, closed compact quotients.

Definition 4.7. Let A be a frame and $F \in A^\wedge$. The compact quotient A_F is closed if $A_F = A_{u_d}$ for some $d \in A$.

Proposition 4.8. *If A is a tidy frame, then A_j is tidy.*

Proof. It is easy to prove that $F \subseteq j_*F$. Since A is tidy and $F \in A^\wedge$, it is true that

$$x \in F \Rightarrow \hat{d} \vee x = 1,$$

where $\hat{d} = d(\alpha) = f^\alpha(0)$.

If $\hat{d} \leq d$, then $d \vee x = 1$, for $d = d(\alpha) = f^\alpha(j(0))$.

Thus, for Corollary 4.4

$$\hat{d} = \hat{d}(\alpha) \leq j(\hat{d}(\alpha)) = j(\hat{f}^\alpha(0)) \leq f^\alpha(j(0)) = d(\alpha) = d.$$

Therefore if $x \in F$, then $d \vee x = 1$ and A_j is tidy. \square

Proposition 4.9. *If A has KC, then A_j has KC for every $j \in N(A)$.*

Proof. We consider $k \in NA_j$ such that $(A_j)_k$ is compact. Since any open filter is admissible, we have $\nabla(k) \in A_j^\wedge$ and by Proposition 4.2 $j_*\nabla(k) \in A^\wedge$.

Let $l = j_* \circ k \circ j^* \in NA$ be, then A_l is a compact quotient of A and exists $a \in A$ such that $l = u_a$. Thus, we have

$$\begin{array}{ccccccc} & & & l & & & \\ & & & \curvearrowright & & & \\ A & \xrightarrow{j^*} & A_j & \xrightarrow{k} & (A_j)_k & \xrightarrow{j_*} & A_j \subseteq A \end{array}$$

and $a \vee x = k(j(x))$. Therefore, if $x = a$, $k(j(x)) = a$.

We need that $k = u_b$ for some $b \in A_j$. For $x \in A_j$ and $b = j(a)$

$$\begin{aligned} u_b(x) &= b \vee x = b \vee j(x) = j(j(a) \vee j(x)) \\ &= j(k(j(a)) \vee x) \\ &= j(u_a(x)) \\ &= j(k(x)) \\ &= k(x). \end{aligned}$$

Therefore $u_b = k$. □

Proposition 4.10. *If A is a KC frame, the A is a T_1 frame.*

Proof. Let $p \in \text{pt } A$ and $a \in A$ be such that $p \leq a \leq 1$. We consider

$$w_p(x) = \begin{cases} 1 & \text{si } x \not\leq p \\ p & \text{si } x \leq p \end{cases}$$

for $x \in A$. $P = \nabla(w_p) = \{x \in A \mid x \not\leq p\}$ is a filter completely prime (in particular, $P \in A^\wedge$). Since A is KC , then A_{w_p} is a closed compact quotient. Thus $u_p = w_p$, futhermore

$$u_p(a) = a \quad \text{and} \quad w_p(a) = 1.$$

that is, $a = 1$. Therefore p is maximal. □

5. ADMISSIBILITY INTERVALS

The block structure on a frame is an important problem and its related with some separation properties of frames.

Proposition 5.1. *For $F \in A^\wedge$ and $Q \in \mathcal{QS}$, if $j \in [v_Q, w_Q]$, then $U_* j U^* \in [v_F, w_F]$, where U^* is the morfism spatial reflection U_* is the right adjoint.*

Proof. Since N is a functor, we have

$$\begin{array}{ccc} A & & NA \\ \downarrow U & \xrightarrow{N(-)} & \downarrow N(U) \\ \mathcal{OS} & & N\mathcal{OS} \end{array}$$

and $N(U)_*$ is the right adjoint of $N(U)^\wedge$. Note the following:

- (1) $N(U)(j) \leq k \Leftrightarrow j \leq N(U)_* k$.
- (2) If $k \in N\mathcal{OS}$ then $N(U)(j) \leq k \Leftrightarrow Uj \leq kU$.
- (3) $N(U)_* k = U_* k U^*$ and $UN(U)_* k = k(U)$.

In 3), if $j = k$, $N(U)_*(j) = U_* j U^*$ and $UN(U)_* j = jU$. For $x \in F$

$$x \in A \xrightarrow{U^*} \mathcal{OS} \xrightarrow{j} \mathcal{OS} \xrightarrow{U_*} A$$

and $U_*(j(U(x))) = \bigwedge (S \setminus j(U(x)))$. Note that $U_*(j(U^*(x))) \subseteq \text{pt } A$. Thus

$$\begin{aligned}
x \in F &\Leftrightarrow Q \subseteq U(x) \Leftrightarrow U(x) \in \nabla(j) = \nabla(Q) \Leftrightarrow S \setminus j(U(x)) = \emptyset \\
&\Leftrightarrow \bigwedge (S \setminus j(U(x))) = 1 = (U_* j U^*)(x) \\
&\Leftrightarrow x \in \nabla(U_* j U^*)
\end{aligned}$$

Therefor $F = \nabla(U_* j U^*)$. □

In this way we have a function

$$\mathcal{U}: [V_Q, W_Q] \rightarrow [V_F, W_F]$$

Theorem 5.2. *Let $A \in \mathcal{Hrm}$ then for every $F \in A^\wedge$ with corresponding \mathcal{Q} compact saturated we have*

$$\mathcal{O}\mathcal{Q} \cong \uparrow \mathcal{Q}'$$

, that is, the frame of opens of the point space of A_F is isomorphic to a compact closed quotient of a Hausdorff space.

Proof. □

EJEMPLOS DE marcos pt que no sean KC

HAY que COMENTAR LAS COSAS QUE ESTAN MAL comentar me refiero a ponerlas entre

Trivially KC implies patch trivial (or equivalently tidy) we want some converse of this fact.

Following articulo de igor.,

Definition 5.3. A frame A has *fitted points* (p-fit for short) if for every point $p \in \text{pt}(A)$ the nucleus

$$w_p \text{ is fitted}$$

that is, to said for every point p the nucleus w_p is alone in its block.

In general for each $p \in \text{pt}(A)$, the nucleus w_p is the largest memeber of his block, that is,

$$[v_{\mathcal{P}}, w_p]$$

the corresponding block, here $\mathcal{P} = \{x \in A \mid x \not\leq p\}$ in this case we know how to calculate

$$v_{\mathcal{P}}.$$

using the prenucleus $f_{\mathcal{P}}$ we know that

$$v_{\mathcal{P}} = f_{\mathcal{P}}^\infty = (\bigvee \{v_x \mid x \in \mathcal{P}\})^\infty$$

moreover:

$$f_{\mathcal{P}}(x) = \begin{cases} 1 & \text{si } x \not\leq p \\ \leq p & \text{si } x \leq p \end{cases}$$

for $x \in A$.

and in fact $w_p = u_p \vee v_p = f_p \circ u_p$. If w_p is fitted, that is,

$$w_p = v_p$$

then one need to have $u_p \leq v_p$ then

$$p \leq v_p(0)$$

by the equation of f_p we have

$$0 \leq \dots \leq f_p^\alpha(0) \leq \dots \leq$$

Proposition 5.4. *Let A be a frame for each $p \in \text{pt}(A)$ the following are equivalent:*

- (i) w_p is fitted.
- (ii) w_p is alone in its block.
- (iii) $u_p \leq v_p$.
- (iv) $u_p \leq f_p$.
- (v) $f_p \circ u_p = v_p$.
- (vi) *aquí debe de ir una formula de primer de orden.*

Proposition 5.5. *In a p-fit frame for each $p \in \text{pt}(A)$ the nucleus w_p is a maximal element in pA .*

Proof. First we dealing with the basics v_F for $F \in A^\wedge$ of the patch frame, given any w_p suppose that $w_p \leq v_F$ then by (propiedades generales de los w) $v_F = w_b$ where $b = v_F(0)$ thus

$$w_p \leq w_b \Leftrightarrow w_p(b) = b$$

since w_p is two valuated we have $b = 1$ or $b = p$ if the first case occur then we are done, for the case $b = p$ we have $v_f(p) = p$ that is, to say, $p \notin F$, then by the Birkhoff's separation lemma we can find a completely prime filter D such that

$$F \subseteq D \not\ni p$$

let q the corresponding point associated to D , then $p \leq q$ since A is p-fit $v_D = w_q$ and thus $w_p \leq w_q$ wich is equivalent to $w_p(q) = q$ again since we are dealing with points one necesary has $p = q$.

Now consider the basic any closed u_c such that, $w_p \leq u_c$ then $w_p(c) = 1$ and thus $1 = c$.

Therefore in basics of the patch the nuclei w_p are maximal, now consider any $k \in pA$ then $k = \bigvee \mathcal{C}$ for some $\mathcal{C} \subseteq p\text{base}$ then

$$w_p \leq u_a \wedge v_F \leq k$$

for some open filter F and some $a \in A$, thus by the above reasoning we have $w_p = u_a \wedge v_F \leq k$ □

Definition 5.6. A frame A is *tame* if does not have wild points.

Proposition 5.7. *In a tame p-fit frame the patch frame pA is T_1 .*

Since every hausdorff frame is tame and p-fit we have:

Corollary 5.8. *If $A \in \mathcal{Hrm}$ then, the patch frame pA is T_1 .*

Definition 5.9. Let A be a frame a nucleus k on A it said to be kq if A_j is a compact frame.

Denote by

$$\mathfrak{K}A = \{j \in NA \mid j \text{ is } kq\}.$$

Definition 5.10. A frame A is *compact closed Hausdorff* (KCH for short) if every compact quotient of A is closed and Hausdorff.

Denote by $\mathfrak{Q}A = \{kq \text{ fitted nuclei}\} = \{v_F \mid F \in A^\wedge\}$

denote by $\mathfrak{C}A = \{a \in A \mid u_a \in \mathfrak{K}A\}$

[Esc01] [Esc06]

[SS06]

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