

# THE PATCH FRAME AND ITS RELATIONS WITH SEPARATION IN POINT-FREE TOPOLOGY

ABSTRACT.

## 1. INTRODUCTION

Aquí va la introducción.

## 2. PRELIMINARIES

### 3. HAUSDORFF PROPERTIES IMPLIES PATCH TRIVIALITY

Gadgets:

$A$  the base frame

Its point space  $S = \text{pt}(A)$ .

$NA$  is the assembly of nucleus of  $A$ .

The compact saturated sets of  $S$ ,

$$Q(S).$$

The preframe of open filters of  $A$ ,

$$A^\wedge.$$

The preframe of open filters of  $\Omega(S)$ .

$$\Omega(S)^\wedge.$$

The set of compact quotients

$$KA = \{j \in NA \mid A_j \text{ is compact}\}.$$

First we recall that every open filter  $F \in A^\wedge$  has three faces, that is, determines (and its determine) by :

- The compact saturated  $Q \in QS$ .
- $\nabla \in \Omega(S)^\wedge$ .
- The compact quotient  $A \rightarrow A_F$ .
- The fitted nucleus  $v_F$ .

Hoffman-Mislove can be rephrase:

There is a bijection between compact quotients of  $A$  and compact saturated sets of  $S$

## 4. COMPACT QUOTIENTS

**Definition 4.1.** A frame has KC if every compact quotient of  $A$  is a closed one. In other words every compact sublocale is closed.

Denote by  $\mathcal{Hrm}$  the subcategory of  $Frm$  of Hausdorff frames, that is,  $A \in \mathcal{Hrm}$  if and only if:

**Definition 4.2.** A frame is *efficient* if every fitted kq nucleus is closed equivalently the frame is tidy.

If  $f^*: A \rightarrow B$  is a frame morphism and  $F \subseteq A$ ,  $G \subseteq B$  filters in  $A$ ,  $B$ , respectively, we can produce new filters as follows

$$(1) \quad b \in f^*F \Leftrightarrow f_*(b) \in F \quad \text{and} \quad a \in f_*G \Leftrightarrow f^*(a) \in G$$

where  $a \in A$ ,  $b \in B$  and  $f_*$  is the right adjoint of  $f^*$ . Here  $f^*F \subseteq B$  and  $f_*G \subseteq A$  are filters on  $B$  and  $A$ , respectively.

**Proposition 4.3.** For  $f = f^*: A \rightarrow B$  a frame morphism and  $G \in B^\wedge$ , then  $f_*G \in A^\wedge$ .

*Proof.* By (1),  $f_*G$  is a filter on  $A$ . We need  $f_*G$  to satisfy the open filter condition. Let  $X \subseteq A$  be such that  $\bigvee X \in f_*G$ , with  $X$  directed. Then

$$Y = \{f(x) \mid x \in X\}$$

is directed and  $f(\bigvee X) = \bigvee f[X] = \bigvee Y \in G$ . Since  $G$  is a open filter, exists  $y = f(x) \in Y$  such that  $y \in G$ . Thus  $x \in f_*G$ , so that,  $f_*G \in A^\wedge$ .  $\square$

In [Sex03], the autor says that  $A \in \mathbf{Frm}$  is *tidy* if for all  $F \in A^\wedge$

$$x \in F \Rightarrow u_d(x) = d \vee x = 1$$

where  $d = d(\alpha) = f^\alpha(0)$ ,  $f = \bigvee \{v_y \mid y \in F\}$ ,  $v_y \in NA$  and  $0 = 0_A$  (the reason for the last two clarifications will be understood later a que te refieres).

We want translate this same notion, but for  $A_j$  when  $j \in NA$ , so that, for all  $F \in A_j^\wedge$ , if  $x \in F$  then  $d \vee x = 1$ , with  $d$  similar to before, because for this case we have that  $v_y \in NA_j$  and  $0_{A_j} = j(0)$ .

In [Sim04, Lemma 8.9 and Corollary 8.10] the author shows, that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f^\infty} & A \\ U_A \downarrow & & \downarrow U_A \\ \mathcal{OS} & \xrightarrow{F^\infty} & \mathcal{OS} \end{array}$$

commutes laxly, that is,  $U_A \circ f^\infty \leq F^\infty \circ U_A$ . In this diagram  $U_A$  is the spatial reflection morphism,  $f^\infty$  and  $F^\infty$  represent the associated nuclei to the filters  $F \in A^\wedge$  and  $\nabla \in \mathcal{OS}^\wedge$ . Also  $f^\infty$  and  $F^\infty$  are idempotent closed associated to

the prenuclei  $f$  and  $F$  respectively.

We prove something more general here, since we consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\hat{f}^\infty} & A \\ j \downarrow & & \downarrow j \\ A_j & \xrightarrow{f^\infty} & A_j \end{array}$$

where  $\hat{f}^\infty$  is the nuclei associated to the filter  $j_*F \in A^\wedge$  and  $j \in NA$ .

**Lemma 4.4.** *For  $j$ ,  $f$  and  $\hat{f}$  as above, it holds that  $j \circ \hat{f} \leq f \circ j$ .*

*Proof.* By (1) is true that

$$\hat{f} = \bigvee \{v_y \mid y \in j_*F\} \quad \text{and} \quad f = \bigvee \{v_{j(y)} \mid j(y) \in F\}.$$

then, for  $a \in A$  it is hold

$$v_y(a) = (y \succ a) \leq \hat{f}(a) \leq j(\hat{f}(a)).$$

Also, for all  $a, y \in A$ ,  $(y \succ a) \wedge y = y \wedge a$  and

$$\begin{aligned} j((y \succ a) \wedge y) \leq j(a) &\Leftrightarrow j(y \succ a) \wedge j(y) \leq j(a) \\ &\Leftrightarrow j(y \succ a) \leq (j(y) \succ j(a)). \end{aligned}$$

Thus

$$v_y(a) \leq j(\hat{f}(a)) \leq (j(y) \succ j(a)) = v_{j(y)}(j(a)) \leq f(j(a)).$$

Therefore  $j \circ \hat{f} \leq f \circ j$ . □

Now, we prove the above, but for all  $\alpha$ -ordinals.

**Corollary 4.5.** *For  $j$ ,  $f$  and  $\hat{f}$  as before, it is hold that  $j \circ \hat{f}^\alpha \leq f^\alpha \circ j$*

*Proof.* For an ordinal  $\alpha$  we will check that  $j \circ \hat{f}^\alpha \leq f^\alpha \circ j$ . We will do it by transfinite induction.

If  $\alpha = 0$ , it is trivial.

For the induction step, we assume that for  $\alpha$  it holds. Then

$$j \circ \hat{f}^{\alpha+1} = j \circ \hat{f} \circ \hat{f}^\alpha \leq f \circ j \circ \hat{f}^\alpha \leq f \circ f^\alpha \circ j = f^{\alpha+1} \circ j,$$

where the first inequality is Lemma 4.4 and the second is true by the induction hypothesis.

If  $\lambda$  is a limit ordinal, then

$$\hat{f}^\lambda = \bigvee \{\hat{f}^\alpha \mid \alpha < \lambda\}, \quad f^\lambda = \bigvee \{f^\alpha \mid \alpha < \lambda\}$$

and

$$j \circ \hat{f}^\lambda = j \circ \bigvee_{\alpha < \lambda} \hat{f}^\alpha \leq \bigvee_{\alpha < \lambda} j \circ \hat{f}^\alpha.$$

Thus, by the induction hypothesis, we have that

$$j \circ \hat{f}^\alpha \leq f^\alpha \circ j \Rightarrow \bigvee_{\alpha < \lambda} j \circ \hat{f}^\alpha \leq \bigvee_{\alpha < \lambda} f^\alpha \circ j.$$

Therefore  $j \circ \hat{f}^\lambda \leq f^\lambda \circ j$ .  $\square$

By the Corollary 4.5, we have that  $j \circ \hat{f}^\infty \leq f^\infty \circ j$  is true. Furthermore, by H-M Theorem (preliminaries con la idea de la prueba nueva),  $f^\infty = v_F$  and  $\hat{f}^\infty = v_{j_*F}$ . With this in mind, we have the following diagram

$$\begin{array}{ccc} A & \xrightleftharpoons[(v_{j_*F})_*]{(v_{j_*F})^*} & A_{j_*F} \\ \downarrow j & \searrow H & \\ A_j & \xrightleftharpoons[(v_F)^*]{(v_F)_*} & A_F \end{array}$$

ES este diagrama hay que poner punteada la flecha que eiria en los cocientes Here,  $A_F$  and  $A_{j_*F}$  are the compact quotients produced by  $v_F$  and  $v_{j_*F}$ , respectively. The morfism  $H: A \rightarrow A_F$  is defined by  $H = v_F \circ j$ . Furthermore,  $(v_F)_*$  and  $(v_{j_*F})_*$  are inclusions.

Let  $h: A_{j_*F} \rightarrow A_j$  be such that, for  $x \in A_{j_*F}$ ,  $h(x) = H(x)$ . Therefore, if  $h = H|_{A_{j_*F}}$ , then the above diagram commutes.

We need that  $h$  to be a frame morphism. First, by the difinition of  $h$ , this is  $\wedge$ -morphism. It remains to be seen that  $h$  is  $\bigvee$ -morphism.

The joins in  $A_{j_*F}$  and  $A_F$  are calculated differently. Thus, let  $\hat{\bigvee}$  be join in  $A_{j_*F}$  and let  $\tilde{\bigvee}$  be join in  $A_F$ . Therefore

$$\hat{\bigvee} = v_{j_*F} \circ \bigvee \quad \text{and} \quad \tilde{\bigvee} = v_F \circ \bigvee,$$

that is, for  $X \subseteq A$ ,  $Y \subseteq A_j$ ,

$$\hat{\bigvee} X = v_{j_*F}(\bigvee X) \quad \text{and} \quad \tilde{\bigvee} Y = v_F(\bigvee Y).$$

Since  $H$  is a frame morphism, then  $H \circ \bigvee = \tilde{\bigvee} \circ H$ . Let us get something similar to  $h$ .

**Lemma 4.6.**  $h \circ \hat{\bigvee} = \tilde{\bigvee} \circ h$ .

*Proof.* It is enough to check the comparison  $h \circ \hat{\bigvee} \leq \tilde{\bigvee} \circ h$ . Thus

$$h \circ \hat{\bigvee} = H \circ v_{j_*F} \circ \bigvee = v_F \circ j \circ v_{j_*F} \circ \bigvee \leq v_F \circ v_F \circ j \circ \bigvee$$

where the inequality is the Corollary 4.5. Furthermore,  $v_F \circ v_F = v_F$ , then

$$h \circ \hat{\bigvee} \leq v_F \circ j \circ \bigvee = H \circ \bigvee = \tilde{\bigvee} \circ H = \tilde{\bigvee} \circ h.$$

Therefore  $h \circ \hat{\bigvee} = \tilde{\bigvee} \circ h$ . □

With this we prove the following.

**Proposition 4.7.** *The diagram*

$$\begin{array}{ccc} A & \xrightarrow{v_{j_* F}} & A_{j_* F} \\ j \downarrow & & \downarrow h \\ A_j & \xrightarrow{v_F} & A_F \end{array}$$

is commutative.

HAY QUE PONER LA PRUEBA With the above diagram, we could analyze some compact quotients, for example, closed compact quotients.

**Definition 4.8.** Let  $A$  be a frame and  $F \in A^\wedge$ . The compact quotient  $A_F$  is closed if  $A_F = A_{u_d}$  for some  $d \in A$ .

**Proposition 4.9.** *If  $A$  is a tidy frame, then  $A_j$  is tidy.*

*Proof.* It is easy to prove that  $F \subseteq j_* F$ . Since  $A$  is tidy and  $F \in A^\wedge$ , it is true that

$$x \in F \Rightarrow \hat{d} \vee x = 1,$$

where  $\hat{d} = d(\alpha) = f^\alpha(0)$ .

If  $\hat{d} \leq d$ , then  $d \vee x = 1$ , for  $d = d(\alpha) = f^\alpha(j(0))$ .

Thus, for Corollary 4.5

$$\hat{d} = \hat{d}(\alpha) \leq j(\hat{d}(\alpha)) = j(\hat{f}^\alpha(0)) \leq f^\alpha(j(0)) = d(\alpha) = d.$$

Therefore if  $x \in F$ , then  $d \vee x = 1$  and  $A_j$  is tidy. □

**Proposition 4.10.** *If  $A$  has KC, then  $A_j$  has KC for every  $j \in N(A)$ .*

*Proof.* We consider  $k \in NA_j$  such that  $(A_j)_k$  is compact. Since any open filter is admissible, we have  $\nabla(k) \in A_j^\wedge$  and by Proposition 4.3  $j_* \nabla(k) \in A^\wedge$ .

Let  $l = j_* \circ k \circ j^* \in NA$  be, then  $A_l$  is a compact quotient of  $A$  and exists  $a \in A$  such that  $l = u_a$ . Thus, we have

$$\begin{array}{ccccccc} & & & l & & & \\ & \nearrow & & \searrow & & & \\ A & \xrightarrow{j^*} & A_j & \xrightarrow{k} & (A_j)_k & \xrightarrow{j_*} & A_j \subseteq A \end{array}$$

and  $a \vee x = k(j(x))$ . Therefore, if  $x = a$ ,  $k(j(x)) = a$ .

We need that  $k = u_b$  for some  $b \in A_j$ . For  $x \in A_j$  and  $b = j(a)$

$$\begin{aligned} u_b(x) &= b \vee x = b \vee j(x) = j(j(a) \vee j(x)) \\ &= j(k(j(a)) \vee x) \\ &= j(u_a(x)) \\ &= j(k(x)) \\ &= k(x). \end{aligned}$$

Therefore  $u_b = k$ . □

**Proposition 4.11.** *If  $A$  is a  $KC$  frame, the  $A$  is a  $T_1$  frame.*

*Proof.* Let  $p \in \text{pt } A$  and  $a \in A$  be such that  $p \leq a \leq 1$ . We consider

$$w_p(x) = \begin{cases} 1 & \text{si } x \not\leq p \\ p & \text{si } x \leq p \end{cases}$$

for  $x \in A$ .  $P = \nabla(w_p) = \{x \in A \mid x \not\leq p\}$  is a filter completely prime (in particular,  $P \in A^\wedge$ ). Since  $A$  is  $KC$ , then  $A_{w_p}$  is a closed compact quotient. Thus  $u_p = w_p$ , futhermore

$$u_p(a) = a \quad \text{and} \quad w_p(a) = 1.$$

that is,  $a = 1$ . Therefore  $p$  is maximal. □

**Proposition 4.12.** *The following holds:*

- (1) *The class of tidy frames is closed under coproducts.*
- (2) *The class of  $KC$  frames is closed under coproducts.*

*Proof.* □

## 5. ADMISSIBILITY INTERVALS

The block structure on a frame is an important problem and its related with some separation properties of frames.

**Proposition 5.1.** *For  $F \in A^\wedge$  and  $Q \in \mathcal{QS}$ , if  $j \in [v_Q, w_Q]$ , then  $U_* j U^* \in [v_F, w_F]$ , where  $U^*$  is the morfism spatial reflection  $U_*$  is the right adjoint.*

*Proof.* Since  $N$  is a functor, we have

$$\begin{array}{ccc} A & & NA \\ \downarrow U & \xrightarrow{N(-)} & \downarrow N(U) \\ \mathcal{OS} & & N\mathcal{OS} \end{array}$$

and  $N(U)_*$  is the right adjoint of  $N(U)^\wedge$ . Note the following:

- (1)  $N(U)(j) \leq k \Leftrightarrow j \leq N(U)_* k$ .
- (2) If  $k \in N\mathcal{OS}$  then  $N(U)(j) \leq k \Leftrightarrow Uj \leq kU$ .
- (3)  $N(U)_* k = U_* k U^*$  and  $U N(U)_* k = k(U)$ .

In 3), if  $j = k$ ,  $N(U)_*(j) = U_*jU^*$  and  $UN(U)_*j = jU$ . For  $x \in F$

$$x \in A \xrightarrow{U^*} \mathcal{O}S \xrightarrow{j} \mathcal{O}S \xrightarrow{U_*} A$$

and  $U_*(j(U(x))) = \bigwedge(S \setminus j(U(x)))$ . Note that  $U_*(j(U^*(x))) \subseteq \text{pt } A$ . Thus

$$\begin{aligned} x \in F &\Leftrightarrow Q \subseteq U(x) \Leftrightarrow U(x) \in \nabla(j) = \nabla(Q) \Leftrightarrow S \setminus j(U(x)) = \emptyset \\ &\Leftrightarrow \bigwedge(S \setminus j(U(x))) = 1 = (U_*jU^*)(x) \\ &\Leftrightarrow x \in \nabla(U_*jU^*) \end{aligned}$$

Therefor  $F = \nabla(U_*jU^*)$ . □

In this way we have a function

$$\mathcal{U}: [V_Q, W_Q] \rightarrow [V_F, W_F]$$

**Theorem 5.2.** *Let  $A \in \mathcal{H}rm$  then for every  $F \in A^\wedge$  with corresponding  $\mathcal{Q}$  compact saturated we have*

$$\mathcal{O}\mathcal{Q} \cong \uparrow \mathcal{Q}'$$

, that is, the frame of opens of the point space of  $A_F$  is isomorphic to a compact closed quotient of a Hausdorff space.

*Proof.* □

EJEMPLOS DE marcos pt que no sean KC

HAY que COMENTAR LAS COSAS QUE ESTAN MAL comentar me refiero a ponerlas entre

Trivially KC implies patch trivial (or equivalently tidy) we want some converse of this fact.

Following articulo de igor.,

**Definition 5.3.** A frame  $A$  has *fitted points* (p-fit for short) if for every point  $p \in \text{pt}(A)$  the nucleus

$$w_p \text{ is fitted}$$

that is, to said for every point  $p$  the nucleus  $w_p$  is alone in its block.

In general for each  $p \in \text{pt}(A)$ , the nucleus  $w_p$  is the largest memeber of his block, that is,

$$[v_{\mathcal{P}}, w_p]$$

the corresponding block, here  $\mathcal{P} = \{x \in A \mid x \not\leq p\}$  in this case we know how to calculate

$$v_{\mathcal{P}}.$$

using the prenucleus  $f_{\mathcal{P}}$  we know that

$$v_{\mathcal{P}} = f_{\mathcal{P}}^\infty = (\bigvee \{v_x \mid x \in \mathcal{P}\})^\infty$$

moreover:

$$f_{\mathcal{P}}(x) = \begin{cases} 1 & \text{si } x \not\leq p \\ \leq p & \text{si } x \leq p \end{cases}$$

for  $x \in A$ .

and in fact  $w_p = u_p \vee v_p = f_{\mathcal{P}} \circ u_p$ . If  $w_p$  is fitted, that is,

$$w_p = v_p$$

then one need to have  $u_p \leq v_p$  then

$$p \leq v_p(0)$$

by the equation of  $f_{\mathcal{P}}$  we have

$$0 \leq \dots \leq f_{\mathcal{P}}^\alpha(0) \leq \dots \leq$$

**Proposition 5.4.** *Let  $A$  be a frame for each  $p \in \text{pt}(A)$  the following are equivalent:*

- (i)  $w_p$  is fitted.
- (ii)  $w_p$  is alone in its block.
- (iii)  $u_p \leq v_p$ .
- (iv)  $u_p \leq f_{\mathcal{P}}$ .
- (v)  $f_{\mathcal{P}} \circ u_p = v_p$ .
- (vi) *aqui debe de ir una formula de primer de orden.*

**Proposition 5.5.** *In a p-fit frame for each  $p \in \text{pt}(A)$  the nucleus  $w_p$  is a maximal element in  $pA$ .*

*Proof.* First we dealing with the basics  $v_F$  for  $F \in A^\wedge$  of the patch frame, given any  $w_p$  suppose that  $w_p \leq v_F$  then by (propiedades generales de los w)  $v_F = w_b$  where  $b = v_F(0)$  thus

$$w_p \leq w_b \Leftrightarrow w_p(b) = b$$

since  $w_p$  is two valuated we have  $b = 1$  or  $b = p$  if the first case occur then we are done, for the case  $b = p$  we have  $v_f(p) = p$  that is, to say,  $p \notin F$ , then by the Birkhoff's separation lemma we can find a completely prime filter  $D$  such that

$$F \subseteq G \not\ni p$$

let  $q$  the corresponding point associated to  $G$ , then  $p \leq q$  since  $A$  is p-fit  $v_G = w_q$  and thus  $w_p \leq w_q$  wich is equivalent to  $w_p(q) = q$  again since we are dealing with points one neccesary has  $p = q$ .

Now consider any closed  $u_c$  such that,  $w_p \leq u_c$  then  $w_p(c) = 1$  and thus  $1 = c$ .

Therefore in basics of the patch the nuclei  $w_p$  are maximal, now consider any  $k \in pA$  such that  $k \in \mathfrak{K}A$  □

**Proposition 5.6.** *Let  $A$  be a frame then if*

$$v_F \neq v_G$$

**Definition 5.7.** A frame  $A$  is *tame* if does not have wild points.

**Proposition 5.8.** *In a tame p-fit frame the patch frame  $pA$  is  $T_1$ .*



Since every hausdorff frame is tame and p-fit we have:

**Corollary 5.9.** *If  $A \in \mathcal{Hrm}$  then, the patch frame  $pA$  is  $T_1$ .*

**Definition 5.10.** Let  $A$  be a frame a nucleus  $k$  on  $A$  it said to be  $kq$  if  $A_j$  is a compact frame.

Denote by

$$\mathfrak{K}A = \{j \in NA \mid j \text{ is } kq\}.$$

**Definition 5.11.** A frame  $A$  is *compact closed Hausdorff* (KCH for short) if every compact quotient of  $A$  is closed and Hausdorff.

Denote by  $\mathfrak{Q}A = \{kq \text{ fitted nuclei}\} = \{v_F \mid F \in A^\wedge\}$

denote by  $\mathfrak{C}A = \{a \in A \mid u_a \in \mathfrak{K}A\}$

[Esc01] [Esc06]

[SS06]

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