THE PATCH FRAME AND ITS RELATIONS WITH SEPARATION IN POINT-FREE TOPOLOGY

ABSTRACT.

1. INTRODUCTION

Aquí va la introducción.

2. PRELIMINARIES

3. HAUSDORFF PROPERTIES IMPLIES PATCH TRIVIALITY

Gadgets:
A the base frame
Its point space $S = \mathrm{pt}(A)$.
NA is the assembly of nucleus of A.
The compact saturated sets of S,

$$\mathcal{Q}(S).$$

The preframe of open filters of A,

$$A^\wedge.$$

The preframe of open filters of $\Omega(S)$.

$$\Omega(S)^\wedge.$$

The set of compact quotients

$$KA = \{j \in NA \mid A_j \text{ is compact}\}.$$

First we recall that every open filter $F \in A^\wedge$ has three faces, that is, determines (and its determine) by :

- The compact saturated $Q \in \mathcal{Q}S$.
- $\nabla \in \Omega(S)^\wedge$.
- The compact quotient $A \to A_F$.
- The fitted nucleus v_F.

Hoffman-Mislove can be rephrase:

There is a bijection between compact quotients of A and compact saturated sets of S

4. COMPACT QUOTIENTS

Definition 4.1. A frame has KC if every compact quotient of A is a closed one. In other words every compact sublocale is closed.

Denote by $\mathcal{H}rm$ the subcategory of Frm of Hausdorff frames in the sense of Johnstone and Shu.

If $j^*\colon A \to B$ is a frame morphism and $F \subseteq A$, $G \subseteq B$ filters in A, B, respectively, we can produce new filters as follows

$$b \in j^*F \Leftrightarrow j_*(b) \in F \quad \text{and} \quad a \in j_*G \Leftrightarrow j^*(a) \in G \tag{1}$$

where $a \in A, b \in B$ and j_* is the right adjoint of j^*. Here $j^*F \subseteq B$ and $j_*G \subseteq A$ are filters on B and A, respectively.

Proposition 4.2. *For $j = j^*\colon A \to B$ a frame morphism and $G \in B^\wedge$, then $j_*G \in A^\wedge$.*

Proof. By (1), j_*G is a filter on A. We need j_*G to satisfy the open filter condition. Let $X \subseteq A$ be such that $\bigvee X \in j_*G$, with X directed. Then

$$Y = \{j(x) \mid x \in X\}$$

is directed and $j(\bigvee X) = \bigvee j[X] = \bigvee Y \in G$. Since G is a open filter, exists $y = j(x) \in Y$ such that $y \in G$. Thus $x \in j_*G$, so that, $j_*G \in A^\wedge$. □

In [4], the autor says that $A \in \mathbf{Frm}$ is *tidy* if for all $F \in A^\wedge$

$$x \in F \Rightarrow u_d(x) = d \vee x = 1$$

where $d = d(\alpha) = f^\alpha(0)$, $f = \dot{\bigvee}\{v_y \mid y \in F\}$, $v_y \in NA$ and $0 = 0_A$ (the reason for the last two clarifications will be understood later).

We want translate this same notion, but for A_j when $j \in NA$, so that, for all $F \in A_j^\wedge$, if $x \in F$ then $d \vee x = 1$, with d similar to before, because for this case we have that $v_y \in NA_j$ and $0_{A_j} = j(0)$.

Let is note that we need open filters F in A and in A_j, prenuclei f in A and A_j and elements d in A and in A_j. To make writing easier, we will denote by $\hat{F}$, $\hat{f}$ and $\hat{d}$, to the open filter, prenuclei and element associated with the prenucleus, respectively, in the frame A. In the frame A_j we use the usual notation, so that, F, f and d for the open filter, prenuclei and element associated to the prenucleus.

Simmons proves in [6] (Lamma 8.9 and Corollary 8.10), that the diagram

$$\begin{tikzcd} A \arrow[r, "f^\infty"] \arrow[d, "U_A"'] & A \arrow[d, "U_A"] \\ \mathcal{O}S \arrow[r, "F^\infty"'] & \mathcal{O}S \end{tikzcd}$$

commutes laxly, so that, $U_A \circ f^\infty \leq F^\infty \circ U_A$.

In this diagram U_A is the spatial reflection morphism, f^∞ and F^∞ represent the associated nuclei asociados to the filters $F \in A^\wedge$ and $\nabla \in \mathcal{O}S^\wedge$.

We prove something more general here, since we consider the diagram

$$\begin{tikzcd} A \arrow[r, "\hat{f}^\infty"] \arrow[d, "j"'] & A \arrow[d, "j"] \\ A_j \arrow[r, "f^\infty"'] & A_j \end{tikzcd}$$

Lemma 4.3. *For j, f and $\hat{f}$ as before, it holds that $j \circ \hat{f} \leq f \circ j$.*

Proof. By (1) is true that

$$\hat{f} = \dot{\bigvee}\{v_y \mid y \in j_* F\} \quad \text{and} \quad f = \dot{\bigvee}\{v_{j(y)} \mid j(y) \in F\}.$$

then, for $a \in A$ it is hold

$$v_y(a) = (y \succ a) \leq \hat{f}(a) \leq j(\hat{f}(a)).$$

Also, for all $a, y \in A$, $(y \succ a) \wedge y = y \wedge a$ and

$$\begin{aligned} j((y \succ a) \wedge y) \leq j(a) &\Leftrightarrow j(y \succ a) \wedge j(y) \leq j(a) \\ &\Leftrightarrow j(y \succ a) \leq (j(y) \succ j(a)). \end{aligned}$$

Thus

$$v_y(a) \leq j(\hat{f}(a)) \leq (j(y) \succ j(a)) = v_{j(y)}(j(a)) \leq f(j(a)).$$

Therefore $j \circ \hat{f} \leq f \circ j$. □

For $\hat{f}$ and f be nuclei, we need their idempotent closeds.

Corollary 4.4. *For j, f and $\hat{f}$ as before, it is hold that $j \circ \hat{f}^\infty \leq f^\infty \circ j$*

Proof. For an ordinal α we will check that $j \circ \hat{f}^\alpha \leq f^\alpha \circ j$. We will do it by transfinite induction.

If $\alpha = 0$, it is trivial.

For the induction step, we assume that for α it holds. Then

$$j \circ \hat{f}^{\alpha+1} = j \circ \hat{f} \circ \hat{f}^\alpha \leq f \circ j \circ \hat{f}^\alpha \leq f \circ f^\alpha \circ j = f^{\alpha+1} \circ j,$$

where the first inequality is Lemma 4 and the second is true by the induction hypothesis.

Ifλ is a limit ordinal, then

$$\hat{f}^\lambda = \bigvee\{\hat{f}^\alpha \mid \alpha < \lambda\}, \quad f^\lambda = \bigvee\{f^\alpha \mid \alpha < \lambda\}$$

and

$$j \circ \hat{f}^{\lambda} = j \circ \bigvee_{\alpha<\lambda} \hat{f}^{\alpha} \leq \bigvee_{\alpha<\lambda} j \circ \hat{f}^{\alpha}.$$

Thus, by the induction hypothesis, we have that

$$j \circ \hat{f}^{\alpha} \leq f^{\alpha} \circ j \Rightarrow \bigvee_{\alpha<\lambda} j \circ \hat{f}^{\alpha} \leq \bigvee_{\alpha<\lambda} f^{\alpha} \circ j.$$

Therefore $j \circ \hat{f}^{\lambda} \leq f^{\lambda} \circ j$. □

We now have the tools to prove the following:

Proposition 4.5. *If A is a tidy frame, then A_j is tidy.*

Proof. It is easy to prove that $F \subseteq j_* F$. Since A is tidy and $F \in A^{\wedge}$, it is true that

$$x \in F \Rightarrow \hat{d} \vee x = 1,$$

where $\hat{d} = d(\alpha) = f^{\alpha}(0)$.
If $\hat{d} \leq d$, then $d \vee x = 1$, for $d = d(\alpha) = f^{\alpha}(j(0))$.

Thus, for Corollary 4.4

$$\hat{d} = \hat{d}(\alpha) \leq j(\hat{d}(\alpha)) = j(\hat{f}^{\alpha}(0)) \leq f^{\alpha}(j(0)) = d(\alpha) = d.$$

Therefore if $x \in F$, then $d \vee x = 1$ and A_j is tidy. □

Proposition 4.6. *If A has KC, then A_j has KC for every $j \in N(A)$.*

5. Admissibility intervals

The block structure on a frame is an important problem and its related with some separation properties of frames.

Proposition 5.1. *For $F \in A^{\wedge}$ and $Q \in \mathcal{Q}S$, if $j \in [v_Q, w_Q]$, then $U_* j U^* \in [v_F, w_F]$, where U^* is the morfism spatial reflection U_* is the right adjoint.*

Proof. Since N is a functor, we have

$$\begin{array}{ccc} A & & NA \\ U \downarrow & \xmapsto{N(\_)} & \downarrow N(U) \\ \mathcal{O}S & & N\mathcal{O}S \end{array}$$

and $N(U)_*$ is the right adjoint of $N(U)^{\wedge}$. Note the following:

1. $N(U)(j) \leq k \Leftrightarrow j \leq N(U)_* k$.
2. If $k \in N\mathcal{O}S$ then $N(U)(j) \leq k \Leftrightarrow Uj \leq kU$.
3. $N(U)_* k = U_* k U^*$ and $UN(U)_* k = k(U)$.

In 3), if $j = k$, $N(U)_*(j) = U_*jU^*$ and $UN(U)_*j = jU$. For $x \in F$

$$x \in A \xrightarrow{U^*} \mathcal{O}S \xrightarrow{j} \mathcal{O}S \xrightarrow{U_*} A$$

and $U_*(j(U(x)) = \bigwedge(S \setminus j(U(x)))$. Note that $U_*(j(U^*(x))) \subseteq \operatorname{pt} A$. Thus

$$\begin{aligned} x \in F \Leftrightarrow Q \subseteq U(x) \Leftrightarrow U(x) \in \nabla(j) = \nabla(Q) &\Leftrightarrow S \setminus j(U(x)) = \emptyset \\ &\Leftrightarrow \bigwedge(S \setminus j(U(x))) = 1 = (U_*jU^*)(x) \\ &\Leftrightarrow x \in \nabla(U_*jU^*) \end{aligned}$$

Therefor $F = \nabla(U_*jU^*)$. □

In this way we have a function

$$\mho\colon [V_Q, W_Q] \to [V_F, W_F]$$

Proposition 5.2. *For every $A \in \mathcal{H}rm$ the interval corresponding to the block determined by a open filter $F \in A^\wedge$ is trivial, that is,*

$$[v_F, w_F] = \{*\}$$

Proof. We know that for all $F \in A^\wedge$ the following holds: $v_F \leq w_F$. As a contradition, suppose that exists $F \in A^\wedge$ such that $w_F \not\leq v_F$, that is, exists $a \in A$ such that $w_F(a) \not\leq v_F(a)$.

Note that $w_F(a) \neq 1$, otherwise

$$1 = w_F(a) = \bigwedge\{p \in M \mid a \leq p\} \leq p$$

and this is a contradition because $p \neq 1$.

Then $1 \neq w_F(a) \not\leq v_F(a)$ and for the property (**H**), exists $u \in A$ such that

$$u \not\leq w_F(a) \quad \text{and} \quad \neg u \not\leq v_F(a) \tag{2}$$

Due to monotony, $w_F(0) \leq w_F(a)$ and $v_F(0) \leq v_F(a)$- Thus, for 2 we have that

$$i)\, u \not\leq w_F(0) \quad \text{and} \quad ii)\, \neg u \not\leq v_F(0). \tag{3}$$

For 3-(i) is true that $u \not\leq \bigwedge M$, in particular, $u \not\leq p$ for all $p \in M$. Therefore, $\neg u \leq p$ and $\neg u \leq w_F(0)$. If 3-(ii) is true, then $u \notin F$, in otherwise

$$u \in F \Rightarrow v_u \leq f \Rightarrow v_u(0) = \neg u \leq f(0)$$

and this is a contradition. Thus, for the Birkhoff's separation lemma, exists a completely prime filter G such that $u \notin G \supseteq F$. We take

$$q = \bigvee\{y \in A \mid y \notin G\}$$

the point corresponding to G. Thus, $u \notin G$, $u \leq q$. If $q \notin F$, then $q \in M$ and $u \not\leq q$. Hence $u \leq q$, $u \not\leq q$ and this is a contradition. □

A consequence of the Proposition 5.2 is that $v_F = w_F$, so that, $A_{v_F} = A_{w_F}$ and $A_{w_F} \simeq \mathcal{O}M \simeq \mathcal{O}Q$. Thus, for all $j \in KA$ we have that $j = v_F$. Then in the Huasdorff case

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_F \arrow[d, "g"] \\ \mathcal{O}S \arrow[r] & \mathcal{O}S_{\nabla} \end{tikzcd}$$

where g is an isomorphism and $A_F \simeq \mathcal{O}Q \simeq \mathcal{O}S_{\nabla}$.

On the other hand, $U_* u_{Q'} U^* = v_F$ if and only if $u_{Q'} U^* = U^* v_F$, for the adjuntion properties and U^* the spatial reflection morphism. Therefore

$$\begin{tikzcd} A \arrow[r, "v_F"] \arrow[d, shift left, "U"] & A \arrow[d, shift right, "U"'] \\ \mathcal{O}S \arrow[r, "v_{\nabla}"'] \arrow[u, shift left, "U_*"] & \mathcal{O}S \arrow[u, shift right, "U_*"'] \end{tikzcd}$$

so that, if $A \in \mathcal{H}rm$ then patch trivial implies KC.

The above is the proof of the following theorem.

Theorem 5.3. *If $A \in \mathcal{H}rm$. then every compact quotient is isomrphic to a closed quotient of the topology of a Hausdorff space.*

Corollary 5.4. *If $A \in \mathcal{H}rm$.*

$$\mathcal{Q}(S) \cong \mathrm{pt}(V(A))$$

Proposition 5.5. *Every Hausdorff frame A (in the sense of Johnstone and Shou) is tidy, that is, A is patch trivial.*

Proof. □

EJEMPLOS DE marcos ptrivial que no sean KC

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