THE PATCH FRAME AND ITS RELATIONS WITH SEPARATION IN POINT-FREE TOPOLOGY

ABSTRACT.

1. INTRODUCTION

Aquí va la introducción.

2. PRELIMINARIES

3. HAUSDORFF PROPERTIES IMPLIES PATCH TRIVIALITY

Gadgets:
A the base frame
Its point space $S = \mathrm{pt}(A)$.
NA is the assembly of nucleus of A.
The compact saturated sets of S,

$$\mathcal{Q}(S).$$

The preframe of open filters of A,

$$A^{\wedge}.$$

The preframe of open filters of $\Omega(S)$.

$$\Omega(S)^{\wedge}.$$

The set of compact quotients

$$KA = \{j \in NA \mid A_j \text{ is compact}\}.$$

First we recall that every open filter $F \in A^{\wedge}$ has three faces, that is, determines (and its determine) by :

- The compact saturated $Q \in \mathcal{Q}S$.
- $\nabla \in \Omega(S)^{\wedge}$.
- The compact quotient $A \to A_F$.
- The fitted nucleus v_F.

Hoffman-Mislove can be rephrase:

There is a bijection between compact quotients of A and compact saturated sets of S

Definition 3.1. A frame has KC if every compact quotient of A is a closed one. In other words every compact sublocale is close.

Denote by $\mathcal{H}rm$ the subcategory of Frm of Hausdorff frames in the sense of Johnstone and Shu.

The block structure on a frame is an important problem and its related with some separation properties of frames.

Proposition 3.2. *For every $A \in \mathcal{H}rm$ the interval corresponding to the block determined by a open filter $F \in A^\wedge$ is trivial, that is,*

$$[v_F, w_F] = \{*\}$$

Proof. We know that for all $F \in A^\wedge$ the following holds: $v_F \leq w_F$. As a contradition, suppose that exists $F \in A^\wedge$ such that $w_F \nleq v_F$, that is, exists $a \in A$ such that $w_F(a) \nleq v_F(a)$.

Note that $w_F(a) \neq 1$, otherwise

$$1 = w_F(a) = \bigwedge\{p \in M \mid a \leq p\} \leq p$$

and this is a contradition because $p \neq 1$.

Then $1 \neq w_F(a) \nleq v_F(a)$ and for the property (**H**), exists $u \in A$ such that

$$u \nleq w_F(a) \quad \text{y} \quad \neg u \nleq v_F(a) \tag{1}$$

Note that $0 \leq a$, then $w_F(0) \leq w_F(a)$ and $v_F(0) \leq v_F(a)$- Thus, for 1 we have that

$$i)\, u \nleq w_F(0) \quad \text{y} \quad ii)\, \neg u \nleq v_F(0). \tag{2}$$

For 2-(i) is true that $u \nleq \bigwedge M$, in particular, $u \nleq p$ for all $p \in M$. Therefore, $\neg u \leq p$ and $\neg u \leq w_F(0)$. If 2-(ii) is true, then $u \notin F$, in otherwise

$$u \in F \Rightarrow v_u \leq f \Rightarrow v_u(0) = \neg u \leq f(0)$$

and this is a contradition. Thus, for the Birkhoff's separation lemma, exists a completely prime filter G such that $u \notin G \supseteq F$. We take

$$q = \bigvee\{y \in A \mid y \notin G\}$$

the point corresponding to G. Thus, $u \notin G$, $u \leq q$. If $q \notin F$, then exists $m \in M$ such that $q \leq m$. Sinse q is maximum, $q = m$ or $m = 1$, but $m \neq 1$ ($1 \in F$ and $M = A \setminus F$), then $m = q \in M$. Hence $u \nleq q$ and this is a contradition. Therefore $v_F = w_F$. ∎

A consequence of the Proposition 3.2 is that $v_F = w_F$, so that, $A_{v_F} = A_{w_F}$ and $A_{w_F} \simeq \mathcal{O}M \simeq \mathcal{O}Q$. Thus, for all $j \in KA$ we have that $j = v_F$. Then in the Huasdorff case

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_F \arrow[d, "g"] \\
\mathcal{O}S \arrow[r] & \mathcal{O}S_\nabla
\end{tikzcd}$$

where g is an isomorphism and $A_F \simeq \mathcal{O}Q \simeq \mathcal{O}S_{\nabla}$.

On the other hand, $U_* u_{Q'} U^* = v_F$ if and only if $u_{Q'} U^* = U^* v_F$, for the adjuntion properties and U^* the spatial reflection morphism. Therefore

$$\begin{tikzcd} A \arrow[r, "v_F"] \arrow[d, shift left, "U"] & A \arrow[d, shift right, "U"'] \\ \mathcal{O}S \arrow[r, "v_{\nabla}"'] \arrow[u, shift left, "U_*"] & \mathcal{O}S \arrow[u, shift right, "U_*"'] \end{tikzcd}$$

so that, if $A \in \mathcal{H}rm$ then patch trivial implies KC.

The above is the proof of the following theorem.

Theorem 3.3. *C.Hausdorff If $A \in \mathcal{H}rm$. then every compact quotient is isomrphic to a closed quotient of the topology of a Hausdorff space.*

Proposition 3.4. *Every Hausdorff frame A (in the sense of Johnstone and Shou) is tidy, that is, A is patch trivial.*

Proof. □

REFERENCES