

# THE PATCH FRAME AND ITS RELATIONS WITH SEPARATION IN POINT-FREE TOPOLOGY

ABSTRACT.

## 1. INTRODUCTION

Aquí va la introducción.

## 2. PRELIMINARIES

### 3. HAUSDORFF PROPERTIES IMPLIES PATCH TRIVIALITY

Gadgets:

$A$  the base frame

Its point space  $S = \text{pt}(A)$ .

$NA$  is the assembly of nucleus of  $A$ .

The compact saturated sets of  $S$ ,

$$Q(S).$$

The preframe of open filters of  $A$ ,

$$A^\wedge.$$

The preframe of open filters of  $\Omega(S)$ .

$$\Omega(S)^\wedge.$$

The set of compact quotients

$$KA = \{j \in NA \mid A_j \text{ is compact}\}.$$

First we recall that every open filter  $F \in A^\wedge$  has three faces, that is, determines (and its determine) by :

- The compact saturated  $Q \in QS$ .
- $\nabla \in \Omega(S)^\wedge$ .
- The compact quotient  $A \rightarrow A_F$ .
- The fitted nucleus  $v_F$ .

Hoffman-Mislove can be rephrase:

There is a bijection between compact quotients of  $A$  and compact saturated sets of  $S$

## 4. COMPACT QUOTIENTS

**Definition 4.1.** A frame has KC if every compact quotient of  $A$  is a closed one. In other words every compact sublocale is closed.

Denote by  $\mathcal{Hrm}$  the subcategory of  $\mathcal{Frm}$  of Hausdorff frames in the sense of Johnstone and Shu.

If  $f^*: A \rightarrow B$  is a frame morphism and  $F \subseteq A$ ,  $G \subseteq B$  filters in  $A$ ,  $B$ , respectively, we can produce new filters as follows

$$(1) \quad b \in f^*F \Leftrightarrow f_*(b) \in F \quad \text{and} \quad a \in f_*G \Leftrightarrow f^*(a) \in G$$

where  $a \in A$ ,  $b \in B$  and  $f_*$  is the right adjoint of  $f^*$ . Here  $f^*F \subseteq B$  and  $f_*G \subseteq A$  are filters on  $B$  and  $A$ , respectively.

**Proposition 4.2.** For  $f = f^*: A \rightarrow B$  a frame morphism and  $G \in B^\wedge$ , then  $f_*G \in A^\wedge$ .

*Proof.* By (1),  $f_*G$  is a filter on  $A$ . We need  $f_*G$  to satisfy the open filter condition. Let  $X \subseteq A$  be such that  $\bigvee X \in f_*G$ , with  $X$  directed. Then

$$Y = \{f(x) \mid x \in X\}$$

is directed and  $f(\bigvee X) = \bigvee f[X] = \bigvee Y \in G$ . Since  $G$  is a open filter, exists  $y = f(x) \in Y$  such that  $y \in G$ . Thus  $x \in f_*G$ , so that,  $f_*G \in A^\wedge$ .  $\square$

In [4], the autor says that  $A \in \mathbf{Frm}$  is *tidy* if for all  $F \in A^\wedge$

$$x \in F \Rightarrow u_d(x) = d \vee x = 1$$

where  $d = d(\alpha) = f^\alpha(0)$ ,  $f = \bigvee \{v_y \mid y \in F\}$ ,  $v_y \in NA$  and  $0 = 0_A$  (the reason for the last two clarifications will be understood later).

We want translate this same notion, but for  $A_j$  when  $j \in NA$ , so that, for all  $F \in A_j^\wedge$ , if  $x \in F$  then  $d \vee x = 1$ , with  $d$  similar to before, because for this case we have that  $v_y \in NA_j$  and  $0_{A_j} = j(0)$ .

Simmons proves in [6] (Lamma 8.9 and Corollary 8.10), that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f^\infty} & A \\ U_A \downarrow & & \downarrow U_A \\ \mathcal{O}S & \xrightarrow{F^\infty} & \mathcal{O}S \end{array}$$

commutes laxly, so that,  $U_A \circ f^\infty \leq F^\infty \circ U_A$ . In this diagram  $U_A$  is the spatial reflection morphism,  $f^\infty$  and  $F^\infty$  represent the associated nuclei asociados to the filters  $F \in A^\wedge$  and  $\nabla \in \mathcal{O}S^\wedge$ . Also  $f^\infty$  and  $F^\infty$  are idempotent closedos associated to the prenucleis  $f$  and  $F$  respectively.

We prove something more general here, since we consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\hat{f}^\infty} & A \\ j \downarrow & & \downarrow j \\ A_j & \xrightarrow{f^\infty} & A_j \end{array}$$

where  $\hat{f}^\infty$  is the nuclei associated to the filters  $j_*F \in A^\wedge$  and  $j \in NA$ .

**Lemma 4.3.** *For  $j$ ,  $f$  and  $\hat{f}$  as before, it holds that  $j \circ \hat{f} \leq f \circ j$ .*

*Proof.* By (1) is true that

$$\hat{f} = \bigvee \{v_y \mid y \in j_*F\} \quad \text{and} \quad f = \bigvee \{v_{j(y)} \mid j(y) \in F\}.$$

then, for  $a \in A$  it is hold

$$v_y(a) = (y \succ a) \leq \hat{f}(a) \leq j(\hat{f}(a)).$$

Also, for all  $a, y \in A$ ,  $(y \succ a) \wedge y = y \wedge a$  and

$$\begin{aligned} j((y \succ a) \wedge y) \leq j(a) &\Leftrightarrow j(y \succ a) \wedge j(y) \leq j(a) \\ &\Leftrightarrow j(y \succ a) \leq (j(y) \succ j(a)). \end{aligned}$$

Thus

$$v_y(a) \leq j(\hat{f}(a)) \leq (j(y) \succ j(a)) = v_{j(y)}(j(a)) \leq f(j(a)).$$

Therefore  $j \circ \hat{f} \leq f \circ j$ . □

Now, we prove the above, but for all  $\alpha$ -ordinals.

**Corollary 4.4.** *For  $j$ ,  $f$  and  $\hat{f}$  as before, it is hold that  $j \circ \hat{f}^\alpha \leq f^\alpha \circ j$*

*Proof.* For an ordinal  $\alpha$  we will check that  $j \circ \hat{f}^\alpha \leq f^\alpha \circ j$ . We will do it by transfinite induction.

If  $\alpha = 0$ , it is trivial.

For the induction step, we assume that for  $\alpha$  it holds. Then

$$j \circ \hat{f}^{\alpha+1} = j \circ \hat{f} \circ \hat{f}^\alpha \leq f \circ j \circ \hat{f}^\alpha \leq f \circ f^\alpha \circ j = f^{\alpha+1} \circ j,$$

where the first inequality is Lemma 4.3 and the second is true by the induction hypothesis.

If  $\lambda$  is a limit ordinal, then

$$\hat{f}^\lambda = \bigvee \{\hat{f}^\alpha \mid \alpha < \lambda\}, \quad f^\lambda = \bigvee \{f^\alpha \mid \alpha < \lambda\}$$

and

$$j \circ \hat{f}^\lambda = j \circ \bigvee_{\alpha < \lambda} \hat{f}^\alpha \leq \bigvee_{\alpha < \lambda} j \circ \hat{f}^\alpha.$$

Thus, by the induction hypothesis, we have that

$$j \circ \hat{f}^\alpha \leq f^\alpha \circ j \Rightarrow \bigvee_{\alpha < \lambda} j \circ \hat{f}^\alpha \leq \bigvee_{\alpha < \lambda} f^\alpha \circ j.$$

Therefore  $j \circ \hat{f}^\lambda \leq f^\lambda \circ j$ .  $\square$

By the Corollary 4.4, we have that  $j \circ \hat{f}^\infty \leq f^\infty \circ j$  is true. Furthermore, by H-M Theorem,  $f^\infty = v_F$  and  $\hat{f}^\infty = v_{j_*F}$ . With this in mind, we have the following diagram

$$\begin{array}{ccc} A & \xrightleftharpoons[(v_{j_*F})_*]{(v_{j_*F})^*} & A_{j_*F} \\ \downarrow j & \searrow H & \\ A_j & \xrightleftharpoons[(v_F)^*]{(v_F)_*} & A_F \end{array}$$

Here,  $A_F$  and  $A_{j_*F}$  are the compact quotients produced by  $v_F$  and  $v_{j_*F}$ , respectively. The morfism  $H: A \rightarrow A_F$  is defined by  $H = v_F \circ j$ . Furthermore,  $(v_F)_*$  and  $(v_{j_*F})_*$  are inclusions.

Let  $h: A_{j_*F} \rightarrow A_j$  be such that, for  $x \in A_{j_*F}$ ,  $h(x) = H(x)$ . Therefore, if  $h = H|_{A_{j_*F}}$ , then the above diagram commutes.

We need that  $h$  to be a frame morphism. First, by the difinition of  $h$ , this is  $\wedge$ -morphism. It remains to be seen that  $h$  is  $\bigvee$ -morphism.

The joins in  $A_{j_*F}$  and  $A_F$  are calculated differently. Thus, let  $\hat{\bigvee}$  be join in  $A_{j_*F}$  and let  $\tilde{\bigvee}$  be join in  $A_F$ . Therefore

$$\hat{\bigvee} = v_{j_*F} \circ \bigvee \quad \text{and} \quad \tilde{\bigvee} = v_F \circ \bigvee,$$

that is, for  $X \subseteq A$ ,  $Y \subseteq A_j$ ,

$$\hat{\bigvee} X = v_{j_*F}(\bigvee X) \quad \text{and} \quad \tilde{\bigvee} Y = v_F(\bigvee Y).$$

Since  $H$  is a frame morphism, then  $H \circ \bigvee = \tilde{\bigvee} \circ H$ . Let us get something similar to  $h$ .

**Lemma 4.5.**  $h \circ \hat{\bigvee} = \tilde{\bigvee} \circ h$ .

*Proof.* It is enough to check the comparison  $h \circ \hat{\bigvee} \leq \tilde{\bigvee} \circ h$ . Thus

$$h \circ \hat{\bigvee} = H \circ v_{j_*F} \circ \bigvee = v_F \circ j \circ v_{j_*F} \circ \bigvee \leq v_F \circ v_F \circ j \circ \bigvee$$

where the inequality is the Corollary 4.4. Furthermore,  $v_F \circ v_F = v_F$ , Then

$$h \circ \hat{\bigvee} \leq v_F \circ j \circ \bigvee = H \circ \bigvee = \tilde{\bigvee} \circ H = \tilde{\bigvee} \circ h.$$

Therefore  $h \circ \hat{\bigvee} = \tilde{\bigvee} \circ h$ .  $\square$

The following is the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{v_{j*}F} & A_{j*}F \\ j \downarrow & & \downarrow h \\ A_j & \xrightarrow{v_F} & A_F \end{array}$$

and with this we could analyze some compact quotients, for example, closed compact quotients.

**Definition 4.6.** Let  $A$  be a frame and  $F \in A^\wedge$ . The compact quotient  $A_F$  is closed if  $A_F = A_{u_d}$  for some  $d \in A$ .

**Proposition 4.7.** If  $A$  is a tidy frame, then  $A_j$  is tidy.

*Proof.* It is easy to prove that  $F \subseteq j_*F$ . Since  $A$  is tidy and  $F \in A^\wedge$ , it is true that

$$x \in F \Rightarrow \hat{d} \vee x = 1,$$

where  $\hat{d} = d(\alpha) = f^\alpha(0)$ .

If  $\hat{d} \leq d$ , then  $d \vee x = 1$ , for  $d = d(\alpha) = f^\alpha(j(0))$ .

Thus, for Corollary 4.4

$$\hat{d} = \hat{d}(\alpha) \leq j(\hat{d}(\alpha)) = j(\hat{f}^\alpha(0)) \leq f^\alpha(j(0)) = d(\alpha) = d.$$

Therefore if  $x \in F$ , then  $d \vee x = 1$  and  $A_j$  is tidy.  $\square$

**Proposition 4.8.** If  $A$  has KC, then  $A_j$  has KC for every  $j \in N(A)$ .

## 5. ADMISSIBILITY INTERVALS

The block structure on a frame is an important problem and its related with some separation properties of frames.

**Proposition 5.1.** For  $F \in A^\wedge$  and  $Q \in \mathcal{QS}$ , if  $j \in [v_Q, w_Q]$ , then  $U_*jU^* \in [v_F, w_F]$ , where  $U^*$  is the morfism spatial reflection  $U_*$  is the right adjoint.

*Proof.* Since  $N$  is a functor, we have

$$\begin{array}{ccc} A & & NA \\ U \downarrow & \xrightarrow{N(-)} & \downarrow N(U) \\ \mathcal{OS} & & N\mathcal{OS} \end{array}$$

and  $N(U)_*$  is the right adjoint of  $N(U)^\wedge$ . Note the following:

- (1)  $N(U)(j) \leq k \Leftrightarrow j \leq N(U)_*k$ .
- (2) If  $k \in N\mathcal{OS}$  then  $N(U)(j) \leq k \Leftrightarrow Uj \leq kU$ .
- (3)  $N(U)_*k = U_*kU^*$  and  $UN(U)_*k = k(U)$ .

In 3), if  $j = k$ ,  $N(U)_*(j) = U_*jU^*$  and  $UN(U)_*j = jU$ . For  $x \in F$

$$x \in A \xrightarrow{U^*} \mathcal{O}S \xrightarrow{j} \mathcal{O}S \xrightarrow{U_*} A$$

and  $U_*(j(U(x))) = \bigwedge(S \setminus j(U(x)))$ . Note that  $U_*(j(U^*(x))) \subseteq \text{pt } A$ . Thus

$$\begin{aligned} x \in F &\Leftrightarrow Q \subseteq U(x) \Leftrightarrow U(x) \in \nabla(j) = \nabla(Q) \Leftrightarrow S \setminus j(U(x)) = \emptyset \\ &\Leftrightarrow \bigwedge(S \setminus j(U(x))) = 1 = (U_*jU^*)(x) \\ &\Leftrightarrow x \in \nabla(U_*jU^*) \end{aligned}$$

Therefor  $F = \nabla(U_*jU^*)$ . □

In this way we have a function

$$\mathcal{U}: [V_Q, W_Q] \rightarrow [V_F, W_F]$$

**Proposition 5.2.** *For every  $A \in \mathcal{H}rm$  the interval corresponding to the block determined by a open filter  $F \in A^\wedge$  is trivial, that is,*

$$[v_F, w_F] = \{*\}$$

*Proof.* We know that for all  $F \in A^\wedge$  the following holds:  $v_F \leq w_F$ . As a contradiction, suppose that exists  $F \in A^\wedge$  such that  $w_F \not\leq v_F$ , that is, exists  $a \in A$  such that  $w_F(a) \not\leq v_F(a)$ .

Note that  $w_F(a) \neq 1$ , otherwise

$$1 = w_F(a) = \bigwedge\{p \in M \mid a \leq p\} \leq p$$

and this is a contradiction because  $p \neq 1$ .

Then  $1 \neq w_F(a) \not\leq v_F(a)$  and for the property **(H)**, exists  $u \in A$  such that

$$(2) \quad u \not\leq w_F(a) \quad \text{and} \quad \neg u \not\leq v_F(a)$$

Due to monotony,  $w_F(0) \leq w_F(a)$  and  $v_F(0) \leq v_F(a)$ - Thus, for 2 we have that

$$(3) \quad i) u \not\leq w_F(0) \quad \text{and} \quad ii) \neg u \not\leq v_F(0).$$

For 3-(i) is true that  $u \not\leq \bigwedge M$ , in particular,  $u \not\leq p$  for all  $p \in M$ . Therefore,  $\neg u \leq p$  and  $\neg u \leq w_F(0)$ . If 3-(ii) is true, then  $u \notin F$ , in otherwise

$$u \in F \Rightarrow v_u \leq f \Rightarrow v_u(0) = \neg u \leq f(0)$$

and this is a contradiction. Thus, for the Birkhoff's separation lemma, exists a completely prime filter  $G$  such that  $u \notin G \supseteq F$ . We take

$$q = \bigvee\{y \in A \mid y \notin G\}$$

the point corresponding to  $G$ . Thus,  $u \notin G$ ,  $u \leq q$ . If  $q \notin F$ , then  $q \in M$  and  $u \not\leq q$ . Hence  $u \leq q$ ,  $u \not\leq q$  and this is a contradiction. □

A consequence of the Proposition 5.2 is that  $v_F = w_F$ , so that,  $A_{v_F} = A_{w_F}$  and  $A_{w_F} \simeq \mathcal{O}M \simeq \mathcal{O}Q$ . Thus, for all  $j \in KA$  we have that  $j = v_F$ . Then in the Hausdorff case

$$\begin{array}{ccc} A & \longrightarrow & A_F \\ \downarrow & & \downarrow g \\ \mathcal{O}S & \longrightarrow & \mathcal{O}S_{\nabla} \end{array}$$

where  $g$  is an isomorphism and  $A_F \simeq \mathcal{O}Q \simeq \mathcal{O}S_{\nabla}$ .

On the other hand,  $U_*u_QU^* = v_F$  if and only if  $u_QU^* = U^*v_F$ , for the adjunction properties and  $U^*$  the spatial reflection morphism. Therefore

$$\begin{array}{ccc} A & \xrightarrow{v_F} & A \\ U_* \uparrow & \downarrow U & U \downarrow \uparrow U_* \\ \mathcal{O}S & \xrightarrow{v_{\nabla}} & \mathcal{O}S \end{array}$$

so that, if  $A \in \mathcal{H}rm$  then patch trivial implies  $KC$ .

The above is the proof of the following theorem.

**Theorem 5.3.** *If  $A \in \mathcal{H}rm$ . then every compact quotient is isomorphic to a closed quotient of the topology of a Hausdorff space.*

**Corollary 5.4.** *If  $A \in \mathcal{H}rm$ .*

$$\mathcal{Q}(S) \cong \text{pt}(V(A))$$

**Proposition 5.5.** *Every Hausdorff frame  $A$  (in the sense of Johnstone and Shou) is tidy, that is,  $A$  is patch trivial.*

*Proof.*

□

EJEMPLOS DE marcos ptrivial que no sean KC

## REFERENCES

- [1] Johnstone, P. T. (1982). Stone spaces (Vol. 3). Cambridge university press.
- [2] Picado, J., Pultr, A. (2011). Frames and Locales: topology without points. Springer Science and Business Media.
- [3] Picado, J., Pultr, A. (2021). Separation in point-free topology. Switzerland: Birkhäuser.
- [4] Sexton, R. A., Simmons, H. (2006). An ordinal indexed hierarchy of separation properties. In Topology Proc (Vol. 30, No. 2, pp. 585-625).
- [5] Sexton, R. A., Simmons, H. (2006). Point-sensitive and point-free patch constructions. Journal of Pure and Applied Algebra, 207(2), 433-468.
- [6] Simmons, H. (2004). The Vietoris modifications of a frame. Unpublished manuscript, 79pp., available online at <http://www.cs.man.ac.uk/hsimmons>.
- [7] Wilansky, A. (1967). Between T1 and T2, this MONTHLY.