THE PATCH FRAME AND ITS RELATIONS WITH SEPARATION IN POINT-FREE TOPOLOGY

ABSTRACT.

1. INTRODUCTION

Aquí va la introducción.

2. PRELIMINARIES

3. HAUSDORFF PROPERTIES IMPLIES PATCH TRIVIALITY

Gadgets:
A the base frame
Its point space $S = \mathrm{pt}(A)$.
NA is the assembly of nucleus of A.
The compact saturated sets of S,

$$\mathcal{Q}(S).$$

The preframe of open filters of A,

$$A^\wedge.$$

The preframe of open filters of $\Omega(S)$.

$$\Omega(S)^\wedge.$$

The set of compact quotients

$$KA = \{j \in NA \mid A_j \text{ is compact}\}.$$

First we recall that every open filter $F \in A^\wedge$ has three faces, that is, determines (and its determine) by :

- The compact saturated $Q \in \mathcal{Q}S$.
- $\nabla \in \Omega(S)^\wedge$.
- The compact quotient $A \to A_F$.
- The fitted nucleus v_F.

Hoffman-Mislove can be rephrase:

There is a bijection between compact quotients of A and compact saturated sets of S

4. COMPACT QUOTIENTS

Definition 4.1. A frame has KC if every compact quotient of A is a closed one. In other words every compact sublocale is closed.

Denote by $\mathcal{H}$ the subcategory of Frm of Hausdorff frames, that is, $A \in \mathcal{H}$ if and only if:

Definition 4.2. A frame is *efficient* if every fitted kq nucleus is closed equivalently the frame is tidy.

If $f^*\colon A \to B$ is a frame morphism and $F \subseteq A$, $G \subseteq B$ filters in A, B, respectively, we can produce new filters as follows

$$b \in f^*F \Leftrightarrow f_*(b) \in F \quad \text{and} \quad a \in f_*G \Leftrightarrow f^*(a) \in G \tag{1}$$

where $a \in A, b \in B$ and f_* is the right adjoint of f^*. Here $f^*F \subseteq B$ and $f_*G \subseteq A$ are filters on B and A, respectively.

Proposition 4.3. *For $f = f^*\colon A \to B$ a frame morphism and $G \in B^\wedge$, then $f_*G \in A^\wedge$.*

Proof. By (1), f_*G is a filter on A. We need f_*G to satisfy the open filter condition. Let $X \subseteq A$ be such that $\bigvee X \in f_*G$, with X directed. Then

$$Y = \{f(x) \mid x \in X\}$$

is directed and $f(\bigvee X) = \bigvee f[X] = \bigvee Y \in G$. Since G is a open filter, exists $y = f(x) \in Y$ such that $y \in G$. Thus $x \in f_*G$, so that, $f_*G \in A^\wedge$. □

In [Sex03], the autor says that $A \in \mathbf{Frm}$ is *tidy* if for all $F \in A^\wedge$

$$x \in F \Rightarrow u_d(x) = d \vee x = 1$$

where $d = d(\alpha) = f^\alpha(0)$, $f = \dot{\bigvee}\{v_y \mid y \in F\}$, $v_y \in NA$ and $0 = 0_A$ (the reason for the last two clarifications will be understood later a que te refieres).

We want translate this same notion, but for A_j when $j \in NA$, so that, for all $F \in A_j^\wedge$, if $x \in F$ then $d \vee x = 1$, with d similar to before, because for this case we have that $v_y \in NA_j$ and $0_{A_j} = j(0)$.

In [Sim04, Lemma 8.9 and Corollary 8.10] the author shows, that the diagram

$$\begin{tikzcd} A \arrow[r, "f^\infty"] \arrow[d, "U_A"'] & A \arrow[d, "U_A"] \\ \mathcal{O}S \arrow[r, "F^\infty"'] & \mathcal{O}S \end{tikzcd}$$

commutes laxly, that is,

$$U_A \circ f^\infty \leq F^\infty \circ U_A.$$

In this diagram U_A is the spatial reflection morphism, f^∞ and F^∞ represent the associated nuclei to the filters $F \in A^\wedge$ and $\nabla \in \mathcal{O}S^\wedge$. Also f^∞ and F^∞ are idempotent closeds associated to the prenuclei f and F respectively.

We prove something more general here, since we consider the diagram

$$\begin{tikzcd} A \arrow[r, "\hat{f}^\infty"] \arrow[d, "j"'] & A \arrow[d, "j"] \\ A_j \arrow[r, "f^\infty"'] & A_j \end{tikzcd}$$

where $\hat{f}^\infty$ is the nuclei associated to the filter $j_* F \in A^\wedge$ and $j \in NA$.

Lemma 4.4. *For j, f and $\hat{f}$ as above, it holds that $j \circ \hat{f} \leq f \circ j$.*

Proof. By (1) is true that

$$\hat{f} = \dot{\bigvee}\{v_y \mid y \in j_* F\} \quad \text{and} \quad f = \dot{\bigvee}\{v_{j(y)} \mid j(y) \in F\}.$$

then, for $a \in A$ it is hold

$$v_y(a) = (y \succ a) \leq \hat{f}(a) \leq j(\hat{f}(a)).$$

Also, for all $a, y \in A$, $(y \succ a) \wedge y = y \wedge a$ and

$$\begin{aligned} j((y \succ a) \wedge y) \leq j(a) &\Leftrightarrow j(y \succ a) \wedge j(y) \leq j(a) \\ &\Leftrightarrow j(y \succ a) \leq (j(y) \succ j(a)). \end{aligned}$$

Thus

$$v_y(a) \leq j(\hat{f}(a)) \leq (j(y) \succ j(a)) = v_{j(y)}(j(a)) \leq f(j(a)).$$

Therefore $j \circ \hat{f} \leq f \circ j$. □

Now, we prove the above, but for all α-ordinals.

Corollary 4.5. *For j, f and $\hat{f}$ as before, it is hold that $j \circ \hat{f}^\alpha \leq f^\alpha \circ j$*

Proof. For an ordinal α we will check that $j \circ \hat{f}^\alpha \leq f^\alpha \circ j$. We will do it by transfinite induction.

If $\alpha = 0$, it is trivial.

For the induction step, we assume that for α it holds. Then

$$j \circ \hat{f}^{\alpha+1} = j \circ \hat{f} \circ \hat{f}^\alpha \leq f \circ j \circ \hat{f}^\alpha \leq f \circ f^\alpha \circ j = f^{\alpha+1} \circ j,$$

where the first inequality is Lemma 4.4 and the second is true by the induction hypothesis.

If λ is a limit ordinal, then

$$\hat{f}^\lambda = \bigvee\{\hat{f}^\alpha \mid \alpha < \lambda\}, \quad f^\lambda = \bigvee\{f^\alpha \mid \alpha < \lambda\}$$

and

$$j \circ \hat{f}^{\lambda} = j \circ \bigvee_{\alpha<\lambda} \hat{f}^{\alpha} \leq \bigvee_{\alpha<\lambda} j \circ \hat{f}^{\alpha}.$$

Thus, by the induction hypothesis, we have that

$$j \circ \hat{f}^{\alpha} \leq f^{\alpha} \circ j \Rightarrow \bigvee_{\alpha<\lambda} j \circ \hat{f}^{\alpha} \leq \bigvee_{\alpha<\lambda} f^{\alpha} \circ j.$$

Therefore $j \circ \hat{f}^{\lambda} \leq f^{\lambda} \circ j$. ◻

By the Corollary 4.5, we have that $j \circ \hat{f}^{\infty} \leq f^{\infty} \circ j$ is true. Futhermore, by H-M Theorem(preliminares con la idea de la prueba nueva), $f^{\infty} = v_F$ and $\hat{f}^{\infty} = v_{j_*F}$. With this in mind, we have the following diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "(v_{j_*F})^*"] \arrow[d, "j"'] \arrow[dr, "H"] & A_{j_*F} \arrow[l, shift left, "(v_{j_*F})_*"] \\
A_j \arrow[r, shift right, "(v_F)^*"'] & A_F \arrow[l, shift right, "(v_F)_*"']
\end{tikzcd}$$

ES este diagrama hay que poner punteada la flehca qu eiria en los cocientes Here, A_F and A_{j_*F} are the compact quotients produced by v_F and v_{j_*F}, respectively. The morfism $H \colon A \to A_F$ is defined by $H = v_F \circ j$. Futhermore, $(v_F)_*$ and $(v_{j_*F})_*$ are inclusions.

Let $h \colon A_{j_*F} \to A_j$ be such that, for $x \in A_{j_*F}$, $h(x) = H(x)$. Therefore, if $h = H_{|A_{j_*F}}$, then the above diagram commutes.

We need that h to be a frame morphism. First, by the difinition of h, this is $\wedge$-morphism. It remains to be seen that h is $\bigvee$-morphism.

The joins in A_{j_*F} and A_F are calculated differently. Thus, let $\hat{\bigvee}$ be join in A_{j_*F} and let $\tilde{\bigvee}$ be join in A_F. Therefore

$$\hat{\bigvee} = v_{j_*F} \circ \bigvee \quad \text{and} \quad \tilde{\bigvee} = v_F \circ \bigvee,$$

that is, for $X \subseteq A$, $Y \subseteq A_j$,

$$\hat{\bigvee} X = v_{j_*F}(\bigvee X) \quad \text{and} \quad \tilde{\bigvee} Y = v_F(\bigvee Y).$$

Since H is a frame morphism, then $H \circ \bigvee = \tilde{\bigvee} \circ H$. Let us get something similar to h.

Lemma 4.6. $h \circ \hat{\bigvee} = \tilde{\bigvee} \circ h$.

Proof. It is enough to check the comparison $h \circ \hat{\bigvee} \leq \tilde{\bigvee} \circ h$. Thus

$$h \circ \hat{\bigvee} = H \circ v_{j_*F} \circ \bigvee = v_F \circ j \circ v_{j_*F} \circ \bigvee \leq v_F \circ v_F \circ j \circ \bigvee$$

where the inequality is the Corollary 4.5. Futhermore, $v_F \circ v_F = v_F$, then

$$h \circ \hat{\bigvee} \leq v_F \circ j \circ \bigvee = H \circ \bigvee = \tilde{\bigvee} \circ H = \tilde{\bigvee} \circ h.$$

Therefore $h \circ \hat{\bigvee} = \tilde{\bigvee} \circ h$. □

With this we prove the following.

Proposition 4.7. *The diagram*

$$\begin{tikzcd} A \arrow[r, "v_{j_*F}"] \arrow[d, "j"'] & A_{j_*F} \arrow[d, "h"] \\ A_j \arrow[r, "v_F"'] & A_F \end{tikzcd}$$

is commutative.

HAY QUE PONER LA PRUEBA With the above diagram, we could analyze some compact quotients, for example, closed compact quotients.

Definition 4.8. Let A be a frame and $F \in A^{\wedge}$. The compact quotient A_F is closed if $A_F = A_{u_d}$ for some $d \in A$.

Proposition 4.9. *If A is a tidy frame, then A_j is tidy.*

Proof. It is easy to prove that $F \subseteq j_*F$. Since A is tidy and $F \in A^{\wedge}$, it is true that

$$x \in F \Rightarrow \hat{d} \vee x = 1,$$

where $\hat{d} = d(\alpha) = f^{\alpha}(0)$.
If $\hat{d} \leq d$, then $d \vee x = 1$, for $d = d(\alpha) = f^{\alpha}(j(0))$.

Thus, for Corollary 4.5

$$\hat{d} = \hat{d}(\alpha) \leq j(\hat{d}(\alpha)) = j(\hat{f}^{\alpha}(0)) \leq f^{\alpha}(j(0)) = d(\alpha) = d.$$

Therefore if $x \in F$, then $d \vee x = 1$ and A_j is tidy. □

Proposition 4.10. *If A has KC, then A_j has KC for every $j \in N(A)$.*

Proof. We consider $k \in NA_j$ such that $(A_j)_k$ is compact. Since any open filter is admissible, we have $\nabla(k) \in A_j^{\wedge}$ and by Proposition 4.3 $j_*\nabla(K) \in A^{\wedge}$.

Let $l = j_* \circ k \circ j^* \in NA$ be, then A_l is a compact quotient of A and exists $a \in A$ such that $l = u_a$. Thus, we have

$$\begin{tikzcd} A \arrow[r, "j^*"'] \arrow[rrr, bend left, "l"] & A_j \arrow[r, "k"'] & (A_j)_k \arrow[r, "j_*"'] & A_j \subseteq A \end{tikzcd}$$

and $a \vee x = k(j(x))$. Therefore, if $x = a$, $k(j(x)) = a$.

We need that $k = u_b$ for some $b \in A_j$. For $x \in A_j$ and $b = j(a)$

$$\begin{aligned} u_b(x) = b \vee x = b \vee j(x) &= j(j(a) \vee j(x)) \\ &= j(k(j(a)) \vee x) \\ &= j(u_a(x)) \\ &= j(k(x)) \\ &= k(x). \end{aligned}$$

Therefore $u_b = k$. □

Proposition 4.11. *If A is a KC frame, the A is a T_1 frame.*

Proof. Let $p \in \mathrm{pt}\, A$ and $a \in A$ be such that $p \leq a \leq 1$. We consider

$$w_p(x) = \begin{cases} 1 & \text{si} \quad x \not\leq p \\ \\ p & \text{si} \quad x \leq p \end{cases}$$

for $x \in A$. $P = \nabla(w_p) = \{x \in A \mid x \not\leq p\}$ is a filter completely prime (in particular, $P \in A^\wedge$). Since A is KC, then A_{w_p} is a closed compact quotient. Thus $u_p = w_p$, futhermore

$$u_p(a) = a \quad \text{and} \quad w_p(a) = 1.$$

that is, $a = 1$. Therefore p is maximal. □

Proposition 4.12. *The following holds:*

(1) *The class of tidy frames is closed under coproducts.*
(2) *The class of KC frames is closed under coprodcuts.*

Proof. □

5. Admissibility intervals

The block structure on a frame is an important problem and its related with some separation properties of frames.

Proposition 5.1. *For $F \in A^\wedge$ and $Q \in \mathcal{Q}S$, if $j \in [v_Q, w_Q]$, then $U_*jU^* \in [v_F, w_F]$, where U^* is the morfism spatial reflection U_* is the right adjoint.*

Proof. Since N is a functor, we have

$$\begin{tikzcd} A \arrow[d, "U"'] \arrow[r, phantom, "\overset{N(\_)}{\longmapsto}"] & NA \arrow[d, "N(U)"] \\ \mathcal{O}S & N\mathcal{O}S \end{tikzcd}$$

and $N(U)_*$ is the right adjoint of $N(U)^\wedge$. Note the following:

(1) $N(U)(j) \leq k \Leftrightarrow j \leq N(U)_*k$.
(2) If $k \in N\mathcal{O}S$ then $N(U)(j) \leq k \Leftrightarrow Uj \leq kU$.
(3) $N(U)_*k = U_*kU^*$ and $UN(U)_*k = k(U)$.

In 3), if $j = k$, $N(U)_*(j) = U_* j U^*$ and $UN(U)_* j = jU$. For $x \in F$

$$x \in A \xrightarrow{U^*} \mathcal{O}S \xrightarrow{j} \mathcal{O}S \xrightarrow{U_*} A$$

and $U_*(j(U(x)) = \bigwedge(S \setminus j(U(x)))$. Note that $U_*(j(U^*(x))) \subseteq \mathrm{pt}\, A$. Thus

$$\begin{aligned} x \in F \Leftrightarrow Q \subseteq U(x) \Leftrightarrow U(x) \in \nabla(j) = \nabla(Q) \Leftrightarrow S \setminus j(U(x)) = \emptyset \\ \Leftrightarrow \bigwedge(S \setminus j(U(x))) = 1 = (U_* j U^*)(x) \\ \Leftrightarrow x \in \nabla(U_* j U^*) \end{aligned}$$

Therefor $F = \nabla(U_* j U^*)$. □

In this way we have a function

$$\mho \colon [V_Q, W_Q] \to [V_F, W_F]$$

Theorem 5.2. *Let $A \in \mathcal{H}rm$ then for every $F \in A^\wedge$ with corresponding $\mathcal{Q}$ compact saturated we have*

$$\mathcal{O}\mathcal{Q} \cong \uparrow \mathcal{Q}'$$

, that is, the frame of opens of the point space of A_F is isomorphic to a compact closed quotient of a Hausdorff space.

Proof. □

EJEMPLOS DE marcos pt que no sean KC

HAY que COMENTAR LAS COSAS QUE ESTAN MAL comentar me refiero a ponerlas entre

Trivially KC implies patch trivial (or equivalently tidy) we want some converse of this fact.

Following articulo de igor.,

Definition 5.3. A frame A has *fitted points* (p-fit for short) if for every point $p \in \mathrm{pt}(A)$ the nucleus

$$\mathrm{w}_p \text{ is fitted}$$

that is, to said for every point p the nucleus w_p is alone in its block.

In general for each $p \in \mathrm{pt}(A)$, the nucleus w_p is the largest memeber of his block, that is,

$$[v_{\mathcal{P}}, \mathrm{w}_p]$$

the corresponding block, here $\mathcal{P} = \{x \in A \mid x \not\leq p\}$ in this case we know how to calculate

$$v_{\mathcal{P}}.$$

using the prenucleus $f_{\mathcal{P}}$ we know that

$$v_{\mathcal{P}} = f_{\mathcal{P}}^{\infty} = (\dot{\bigvee}\{\mathrm{v}_x \mid x \in \mathcal{P}\})^{\infty}$$

moreover:

$$f_{\mathcal{P}}(x) = \begin{cases} 1 & \text{si} \quad x \nleq p \\ \\ \leq p & \text{si} \quad x \leq p \end{cases}$$

for $x \in A$.

and in fact $\mathrm{w_p} = u_p \vee v_{\mathcal{P}} = f_{\mathcal{P}} \circ u_p$. If $\mathrm{w_p}$ is fitted, that is,

$$\mathrm{w_p} = v_{\mathcal{P}}$$

then one need to have $u_p \leq v_{\mathcal{P}}$ then

$$p \leq v_{\mathcal{P}}(0)$$

by the equation of $f_{\mathcal{P}}$ we have

$$0 \leq \cdots \leq f^{\alpha}_{\mathcal{P}}(0) \leq \cdots \leq$$

Proposition 5.4. *Let A be a frame for each $p \in \mathrm{pt}(A)$ the following are equivalent:*

(i) w_p *is fitted.*
(ii) w_p *is alone in its block.*
(iii) $u_p \leq v_{\mathcal{P}}$.
(iv) $u_p \leq f_{\mathcal{P}}$.
(v) $f_{\mathcal{P}} \circ u_p = v_{\mathcal{P}}$.
(vi) *aqui debe de ir una formula de primer de orden.*

Proposition 5.5. *In a* p-fit *frame for each $p \in \mathrm{pt}(A)$ the nucleus w_p is a maximal element in $\mathrm{p}A$.*

Proof. First we dealing with the basics v_F for $F \in A^{\wedge}$ of the patch frame, given any w_p suppose that $\mathrm{w}_p \leq v_F$ then by (propiedades generales de los w) $v_F = \mathrm{w}_b$ where $b = v_F(0)$ thus

$$\mathrm{w}_p \leq \mathrm{w}_b \Leftrightarrow \mathrm{w}_p(b) = b$$

since w_p is two valuated we have $b = 1$ or $b = p$ if the first case occur then we are done, for the case $b = p$ we have $v_f(p) = p$ that is, to say, $p \notin F$, then by the Birkhoff's separation lemma we can find a completely prime filter D such that

$$F \subseteq G \not\ni p$$

let q the corresponding point associated to G, then $p \leq q$ since A is p-fit $v_G = \mathrm{w}_q$ and thus $\mathrm{w}_p \leq \mathrm{w}_q$ wich is equivalent to $\mathrm{w}_p(q) = q$ again since we are dealing with points one neccesary has $p = q$.

Now consider any closed u_c such that, $\mathrm{w}_p \leq \mathrm{u}_c$ then $\mathrm{w}_p(c) = 1$ and thus $1 = c$.

Therefore in basics of the patch the nuclei w_p are maximal, now consider any $k \in \mathrm{p}A$ such that $k \in \mathfrak{K}A$ □

Proposition 5.6. *Let A be a frame then if*

$$v_F \neq v_G$$

Definition 5.7. A frame A is *tame* if does not have wild points.

Proposition 5.8. *In a tame* p-fit *frame the patch frame $\mathrm{p}A$ is T_1.*

Since every hausdorff frame is tame and p-fit we have:

Corollary 5.9. *If* $A \in \mathcal{H}rm$ *then, the patch frame* $\mathrm{p}A$ *is* T_1.

Definition 5.10. Let A be a frame a nucleus k on A it said to be *kq* if A_j is a compact frame.

Denote by

$$\mathfrak{K}A = \{j \in NA \mid j \text{ is } kq\}.$$

Definition 5.11. A frame A is *compact closed Hausdorff* (KCH for short) if every compact quotient of A is closed and Hausdorff.

Denote by $\mathfrak{f}A = \{kq \text{ fitted nuclei }\} = \{v_F \mid F \in A^\wedge\}$
denote by $\mathfrak{C}A = \{a \in A \mid \mathrm{u}_a \in \mathfrak{K}A\}$

6. The pro-compact fit completion of a frame

Here we are going to construct the pro-compact fit completion of a frame A.
This construction is similar to the pro-compact closed completion of a frame, but here we consider only compact fitted quotients.
Firs we consider the family

$$\{A \to A_F \mid v_F \in \mathfrak{f}A\}$$

of compact fitted quotients of A.

Denote by $F^*\colon A \to A_F$ the canonical morphism to the compact fitted quotient, that is,

$$F^*(a) = v_F(a) \ \ \forall a \in A$$

If $v_F \le v_G$

$$\begin{tikzcd} & A \arrow[dl, "v_F^*"'] \arrow[dr] & \\ A_F \arrow[rr, "(FG)"'] & & A_G \end{tikzcd}$$

where $(FG)(v_F(a)) = v_G(a)$ for all $a \in A$.
this cones have a limit in the category of frames, denote by $\hat{A}$ this limit and by $\pi_F\colon \hat{A} \to A_F$ the canonical projection for each F.
thus in particular we have a unique morphism $\gamma\colon A \to \hat{A}$ such that the follwing diagram commutes
$\pi_F \circ \gamma = v_F^*$ for all F.

$$\begin{tikzcd} A \arrow[rr, "\gamma"] \arrow[dr, "v_F^*"'] & & \hat{A} \arrow[dl, "\pi_F"] \\ & A_F & \end{tikzcd}$$

The morphism $\gamma\colon A \to \hat{A}$ is given by

$$\gamma(a)\colon \mathfrak{f}A \to \bigcup_{F\in\mathfrak{f}A} A_F$$

therefore $\gamma(a)(F) = v_F(a)$

thus for $f \in \hat{A}$ and each F we have the set $\{x \in A \mid v_F(x) = f(F)\}$ the fiber of f in F.

Now for the the left adjoint of γ is given by

$$\gamma_*(f) = \bigvee\{a \in A \mid \gamma(a) \leq f\}$$

$$\gamma(a) \leq f \Leftrightarrow a \leq \gamma_*(f)$$

note that if $f \in \hat{A}$ then for a v_F we have $f(v_F) \in A_F$ thus $f(v_F) = v_F(x)$ for some $x \in A$, therefore

$$f(v_F) = v_F(x) = \gamma(x)(v_F)$$

then $\{x \in A \mid v_F(x) = f(F)\}_F \subseteq \{x \in A \mid \gamma(x) \leq f\}$ for each F

As shown in [Esc01]

References

[Esc01] Martın Hötzel Escardó, *The regular-locally compact coreflection of a stably locally compact locale*, Journal of Pure and Applied Algebra **157** (2001), no. 1, 41–55.

[Sex03] Rosemary A Sexton, *A point-free and point-sensitive analysis of the patch assembly*, The University of Manchester (United Kingdom), 2003.

[Sim04] Harold Simmons, *The vietoris modifications of a frame*, Unpublished manuscript, 79pp., available online at http://www. cs. man. ac. uk/hsimmons (2004).