THE PATCH FRAME AND ITS RELATIONS WITH SEPARATION IN POINT-FREE TOPOLOGY

ABSTRACT.

1. INTRODUCTION

Aquí va la introducción.

2. PRELIMINARIES

3. HAUSDORFF PROPERTIES IMPLIES PATCH TRIVIALITY

Gadgets:
A the base frame
Its point space $S = \mathrm{pt}(A)$.
NA is the assembly of nucleus of A.
The compact saturated sets of S,

$$\mathcal{Q}(S).$$

The preframe of open filters of A,

$$A^{\wedge}.$$

The preframe of open filters of $\Omega(S)$.

$$\Omega(S)^{\wedge}.$$

The set of compact quotients

$$KA = \{j \in NA \mid A_j \text{ is compact}\}.$$

First we recall that every open filter $F \in A^{\wedge}$ has three faces, that is, determines (and its determine) by :

- The compact saturated $Q \in \mathcal{Q}S$.
- $\nabla \in \Omega(S)^{\wedge}$.
- The compact quotient $A \to A_F$.
- The fitted nucleus v_F.

Hoffman-Mislove can be rephrase:

There is a bijection between compact quotients of A and compact saturated sets of S

Definition 3.1. A frame has KC if every compact quotient of A is a closed one. In other words every compact sublocale is close.

Denote by $\mathcal{H}rm$ the subcategory of Frm of Hausdorff frames in the sense of Johnstone and Shu.

In (citar el artículo) Sexton says that $A \in \mathbf{Frm}$ is *tidy* if for all $F \in A^\wedge$

$$x \in F \Rightarrow u_d(x) = d \vee x = 1$$

where $d = d(\alpha) = f^\alpha(0)$, $f = \dot{\bigvee}\{v_y \mid y \in F\}$, $v_y \in NA$ and $0 = 0_A$ (the reason for the last two clarifications will be understood later).

We want translate this same notion, but for A_j when $j \in NA$, so that, for all $F \in A_j^\wedge$, if $x \in F$ then $d \vee x = 1$, with d similar to before, because for this case we have that $v_y \in NA_j$ and $0_{A_j} = j(0)$.

Let's note that we need open filters F in A and in A_j, prenuclei f in A and A_j and elements d in A and in A_j. To make writing easier, we will denote by $\hat{F}$, $\hat{f}$ and $\hat{d}$, to the open filter, prenuclei and element associated with the prenucleus, respectively, in the frame A. In the frame A_j we use the usual notation, so that, F, f and d for the open filter, prenuclei and element associated to the prenucleus.

We know that every open filter is admissible. This way, let $F = \nabla(k)$ and $\hat{F} = \nabla(g)$ be, where $k \in NA_j$ and $g = j_* k j^* \in NA$. We must see that $\hat{F}$ is indeed an open filter in A. To do this, let us note that $j = j^*$ is a monotonic function, then

$$b \in j^*\hat{F} \Leftrightarrow j_*(b) \in \hat{F} \quad \text{and} \quad a \in j_*F \Leftrightarrow j^*(a) \in F$$

for $a \in A$ and $b \in A_j$. Foremore, if $F \in A_j^\wedge$, then $j_*F \in A^\wedge$.

Proposition 3.2. *With the above notation, $\hat{F} = j_*F$*

Proof. First, we note that if $x \in \hat{F}$, then $g(x) = (j_*kj^*)(x) = 1$. Thus

$$j((j_*kj^*)(x)) = j(1) = 1 \Rightarrow (kj^*)(x) = 1 \Rightarrow j^*(x) \in \nabla(k) = F,$$

so that, $x \in j_*F$.

On the other hand, if $x \in j_*F$, then

$$j^*(x) \in F \Rightarrow k(j^*(x)) = 1 \Rightarrow j_*(k(j^*(x))) = 1,$$

so that, $x \in \nabla(g) = \hat{F}$. Therefore $\hat{F} \in A^\wedge$. $\square$

Simmons proves in (citar el Vietoris y los resultados) that the diagram

$$\begin{tikzcd} A \arrow[r, "f^\infty"] \arrow[d, "U_A"'] & A \arrow[d, "U_A"] \\ \mathcal{O}S \arrow[r, "F^\infty"'] & \mathcal{O}S \end{tikzcd}$$

commutes laxly, so that, $U_A \circ f^\infty \leq F^\infty \circ U_A$.

In this diagram U_A is the spatial reflection morphism, f^∞ and F^∞ represent the associated nuclei asociados to the filters $F \in A^\wedge$ and $\nabla \in \mathcal{O}S^\wedge$.

We prove something more general here, since we consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\hat{f}^\infty} & A \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ A_j & \xrightarrow[f^\infty]{} & A_j \end{array}$$

Lemma 3.3. *For j, f and $\hat{f}$ as before, it holds that $j \circ \hat{f} \leq f \circ j$.*

Proof. By proposition 3.2 is true that

$$\hat{f} = \dot{\bigvee}\{v_y \mid y \in \hat{F} = j_* F\} \quad \text{and} \quad f = \dot{\bigvee}\{v_{j(y)} \mid j(y) \in F\}.$$

then, for $a \in A$ it is hold

$$v_y(a) = (y \succ a) \leq \hat{f}(a) \leq j(\hat{f}(a)) \leq (j(y) \succ j(a)) = v_{j(y)}(j(a)) \leq f(j(a)).$$

Therefore $j \circ \hat{f} \leq f \circ j$. □

For $\hat{f}$ and f be nuclei, we need their idempotent closeds.

Corollary 3.4. *For j, f and $\hat{f}$ as before, it is hold that $j \circ \hat{f}^\infty \leq f^\infty \circ j$*

Proof. For an ordinal α we will check that $j \circ \hat{f}^\alpha \leq f^\alpha \circ j$. We will do it by transfinite induction.

If $\alpha = 0$, it is trivial.

For the induction step, we assume that for α it holds. Then

$$j \circ \hat{f}^{\alpha+1} = j \circ \hat{f} \circ \hat{f}^\alpha \leq f \circ j \circ \hat{f}^\alpha \leq f \circ f^\alpha \circ j = f^{\alpha+1} \circ j,$$

where the first inequality is Lemma 3 and the second is true by the induction hypothesis.

Ifλ is a limit ordinal, then (HAZLO) □

We now have the tools to prove the following:

Proposition 3.5. *Si A es un marco arreglado, entonces A_j es arreglado.*

Proof. Es sencillo probar que $F \subseteq \hat{F}$. De esta manera, Como A es arreglado y $F \in A^\wedge$, se cumple que

$$x \in F \Rightarrow \hat{d} \vee x = 1,$$

donde $\hat{d} = d(\alpha) = f^\alpha(0)$.

Observemos que si $\hat{d} \leq d$, se cumple que $d \vee x = 1$, para $d = d(\alpha) = f^{\alpha}(j(0))$.

Luego, por el Corolario 3.4

$$\hat{d} = \hat{d}(\alpha) \leq j(\hat{d}(\alpha)) = j(\hat{f}^{\alpha}(0)) \leq f^{\alpha}(j(0)) = d(\alpha) = d.$$

Por lo tanto si $x \in F$, entonces $d \vee x = 1$ y así A_j es arreglado. $\square$

Lemma 3.6. *Let $A \in \mathrm{Frm}$ and $j, k \in NA$ be. We consider $F \in A_j^{\wedge}$ y $g = j_* k j^*$ where $F = \nabla(k)$. Then $\hat{F} = \nabla(g) \in A^{\wedge}$.*

Proof. $\hat{F}$ is a filter, because g is a nucleus. Let us consider $X \subseteq A$ such that $\bigvee X \in \hat{F}$. We must prove that $X \cap \hat{F} \neq \emptyset$.

If $\bigvee X \in \hat{F}$, then $g(\bigvee X) = (j_* k j^*)(\bigvee X) = 1$. Thus

$$j^*(\bigvee X) \leq j(\bigvee \{j(x) \mid x \in X\}) = j(\bigvee j[X]) = \bigvee_j X$$

and

$$1 = (j_* k)(j^*(\bigvee X)) \leq (j_* k)(\bigvee_j X).$$

$\{j(x) \mid x \in X\}$ is a directed set because X is directed. Then $j(j[X]) \subseteq A_j$, $\bigvee_j X \in F$ and $F \in A_j^{\wedge}$, so that, exists $x \in X$ such that $x = j(x) \in F$. Therefore $k(j(x)) = 1$ and $(j_* k j^*)(x) = 1$, so that, $x \in \nabla(g) = \hat{F}$. Thus $X \cap \hat{F} \neq \emptyset$. $\square$

Lemma 3.7. *For $F \in A_j^{\wedge}$ and $\hat{F} \in A^{\wedge}$ as above, then $F \subseteq \hat{F}$.*

Proof. Let $x \in F$ be, then $j(x) = x$ and $k(x) = 1$. Thus

$$g(x) = (j_* k j^*)(x) = (j_* k)(j(x)) = j_*(k(x)) = j_*(1) = 1.$$

Therefore $x \in \hat{F}$. $\square$

Lemma 3.8. *Let $A \in \mathrm{Frm}$ and $j \in NA$ be. If A is tidy then A_j is tidy.*

The block structure on a frame is an important problem and its related with some separation properties of frames.

Proposition 3.9. *For $F \in A^{\wedge}$ and $Q \in \mathcal{Q}S$, if $j \in [v_Q, w_Q]$, then $U_* j U^* \in [v_F, w_F]$, where U^* is the morfism spatial reflection U_* is the right adjoint.*

Proof. Since N is a functor, we have

$$\begin{tikzcd} A \arrow[d, "U"'] \arrow[r, phantom, "\overset{N(\_)}{\longmapsto}"] & NA \arrow[d, "N(U)"] \\ \mathcal{O}S & N\mathcal{O}S \end{tikzcd}$$

and $N(U)_*$ is the right adjoint of $N(U)^{\wedge}$. Note the following:

(1) $N(U)(j) \leq k \Leftrightarrow j \leq N(U)_* k$.

(2) If $k \in N\mathcal{O}S$ then $N(U)(j) \leq k \Leftrightarrow Uj \leq kU$.
(3) $N(U)_* k = U_* k U^*$ and $UN(U)_* k = k(U)$.

In 3), if $j = k$, $N(U)_*(j) = U_* j U^*$ and $UN(U)_* j = jU$. For $x \in F$

$$x \in A \xrightarrow{U^*} \mathcal{O}S \xrightarrow{j} \mathcal{O}S \xrightarrow{U_*} A$$

and $U_*(j(U(x)) = \bigwedge(S \setminus j(U(x)))$. Note that $U_*(j(U^*(x))) \subseteq \mathrm{pt}\, A$. Thus

$$\begin{aligned} x \in F \Leftrightarrow Q \subseteq U(x) \Leftrightarrow U(x) \in \nabla(j) = \nabla(Q) \Leftrightarrow S \setminus j(U(x)) = \emptyset \\ \Leftrightarrow \bigwedge(S \setminus j(U(x))) = 1 = (U_* j U^*)(x) \\ \Leftrightarrow x \in \nabla(U_* j U^*) \end{aligned}$$

Therefor $F = \nabla(U_* j U^*)$. □

In this way we have a function

$$\mho \colon [V_Q, W_Q] \to [V_F, W_F]$$

Proposition 3.10. *For every $A \in \mathcal{H}rm$ the interval corresponding to the block determined by a open filter $F \in A^\wedge$ is trivial, that is,*

$$[v_F, w_F] = \{*\}$$

Proof. We know that for all $F \in A^\wedge$ the following holds: $v_F \leq w_F$. As a contradition, suppose that exists $F \in A^\wedge$ such that $w_F \nleq v_F$, that is, exists $a \in A$ such that $w_F(a) \nleq v_F(a)$.

Note that $w_F(a) \neq 1$, otherwise

$$1 = w_F(a) = \bigwedge\{p \in M \mid a \leq p\} \leq p$$

and this is a contradition because $p \neq 1$.

Then $1 \neq w_F(a) \nleq v_F(a)$ and for the property (**H**), exists $u \in A$ such that

$$u \nleq w_F(a) \quad \text{and} \quad \neg u \nleq v_F(a) \tag{1}$$

Due to monotony, $w_F(0) \leq w_F(a)$ and $v_F(0) \leq v_F(a)$- Thus, for 1 we have that

$$i)\, u \nleq w_F(0) \quad \text{and} \quad ii)\, \neg u \nleq v_F(0). \tag{2}$$

For 2-(i) is true that $u \nleq \bigwedge M$, in particular, $u \nleq p$ for all $p \in M$. Therefore, $\neg u \leq p$ and $\neg u \leq w_F(0)$. If 2-(ii) is true, then $u \notin F$, in otherwise

$$u \in F \Rightarrow v_u \leq f \Rightarrow v_u(0) = \neg u \leq f(0)$$

and this is a contradition. Thus, for the Birkhoff's separation lemma, exists a completely prime filter G such that $u \notin G \supseteq F$. We take

$$q = \bigvee\{y \in A \mid y \notin G\}$$

the point corresponding to G. Thus, $u \notin G$, $u \leq q$. If $q \notin F$, then $q \in M$ and $u \nleq q$. Hence $u \leq q$, $u \nleq q$ and this is a contradition. □

A consequence of the Proposition 3.10 is that $v_F = w_F$, so that, $A_{v_F} = A_{w_F}$ and $A_{w_F} \simeq \mathcal{O}M \simeq \mathcal{O}Q$. Thus, for all $j \in KA$ we have that $j = v_F$. Then in the Huasdorff case

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_F \arrow[d, "g"] \\
\mathcal{O}S \arrow[r] & \mathcal{O}S_{\bigtriangledown}
\end{tikzcd}$$

where g is an isomorphism and $A_F \simeq \mathcal{O}Q \simeq \mathcal{O}S_{\bigtriangledown}$.

On the other hand, $U_* u_{Q'} U^* = v_F$ if and only if $u_{Q'} U^* = U^* v_F$, for the adjuntion properties and U^* the spatial reflection morphism. Therefore

$$\begin{tikzcd}
A \arrow[r, "v_F"] \arrow[d, shift left, "U"] & A \arrow[d, shift right, "U"'] \\
\mathcal{O}S \arrow[u, shift left, "U_*"] \arrow[r, "v_{\bigtriangledown}"'] & \mathcal{O}S \arrow[u, shift right, "U_*"']
\end{tikzcd}$$

so that, if $A \in \mathcal{H}rm$ then patch trivial implies KC.

The above is the proof of the following theorem.

Theorem 3.11. *If $A \in \mathcal{H}rm$. then every compact quotient is isomrphic to a closed quotient of the topology of a Hausdorff space.*

Proposition 3.12. *Every Hausdorff frame A (in the sense of Johnstone and Shou) is tidy, that is, A is patch trivial.*

Proof. □

REFERENCES