setup

```
In [110]: import sympy as sym
          import numpy as np
          %matplotlib notebook
          from matplotlib import pyplot as plt
          from IPython.display import Image
```

Problem 1

```
In [111]: Image("p1.jpg")
```

Out[111]:

Homework #2

1. Verify that if $f(x) = \frac{1}{2} x^T Q x$ & $Q = Q^T$ then $Df(x) = x^T Q$ using the directional deriv.

$f(x) = \frac{1}{2} x^T Q x$

$Df(x) = \frac{d}{d\varepsilon} f(x+\varepsilon\eta)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\left[\frac{1}{2}(x+\varepsilon\eta)^T Q (x+\varepsilon\eta)\right]\Big|_{\varepsilon=0}$

$= \left[\frac{1}{2}\eta^T Q(x+\varepsilon\eta) + \frac{1}{2}(x+\varepsilon\eta)^T Q\eta\right]\Big|_{\varepsilon=0}$

$= \frac{1}{2}\eta^T Q x + \frac{1}{2} x^T Q \eta$

$Df(x) = x^T Q \eta$

* ask how we get rid of η

it is a matter of notation, but $Df(x)\cdot\eta = Df(x)$ for any argument

Problem 2

```
In [2]: def x_init_gen(dim):
            x_inits = (np.zeros((dim,1)) - 10) +  np.random.rand(dim,1) * 20
            return x_inits
```

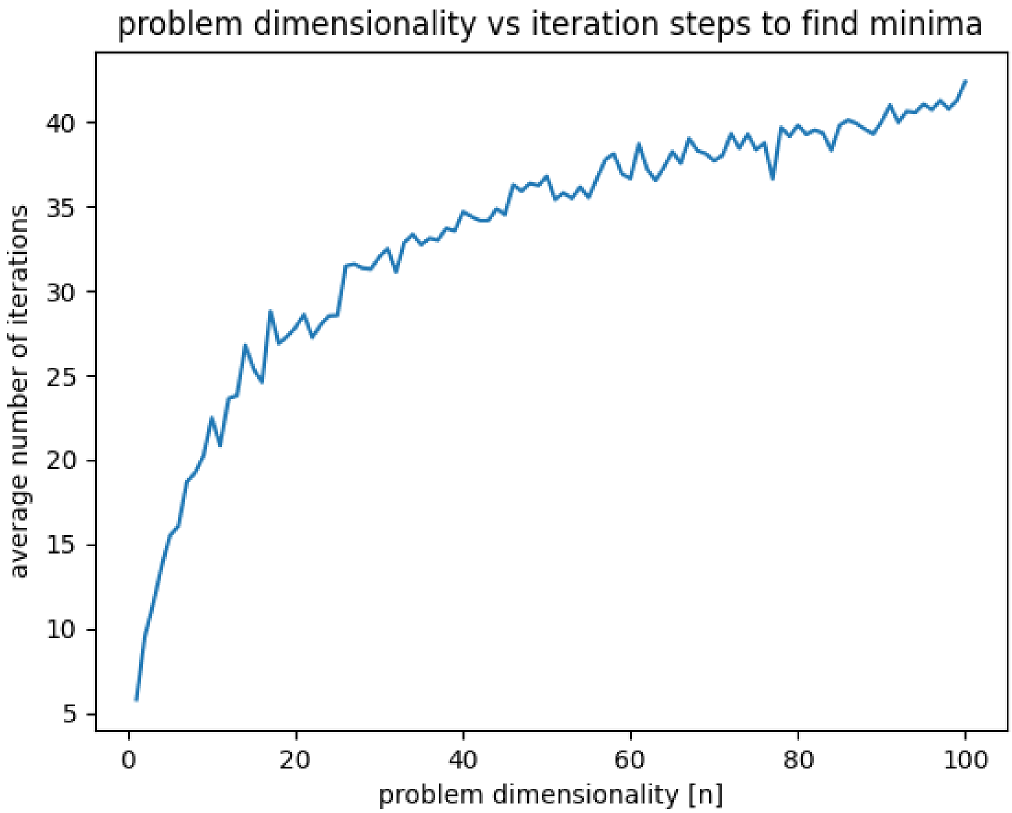
In [3]:
```
#generate random nxn diagonal matrix with elements betwen 0.5 - 1.9
def Q_gen(dim):
    I = np.identity(dim)
    rand_arr = (np.zeros((dim,1)) + 0.5) +  np.random.rand(dim,1) *
1.4
    Q = I * rand_arr
    return Q
```

In [4]:
```
#set up symbollic f and grad_f for lambda evals
n = sym.symbols('n')
x = sym.MatrixSymbol('x', n, 1)
Q = sym.MatrixSymbol('Q', n, n)
f = sym.Rational(1,2) * x.T * Q * x
grad_f = f.diff(x)
grad_2_f = grad_f.diff(x)
f_lam = sym.lambdify([x, Q], f)
grad_f_lam = sym.lambdify([x, Q], grad_f)
grad_2_f_lam = sym.lambdify([x,Q], grad_2_f)
```

In [5]:
```
#define our function that performs gradient descent on ambigiously di
mensonal funcs
#creates random x and Q per descent
def ambig_dim_grad_desc(dim):
    eps = 0.1
    num_its = 0
    Q = Q_gen(dim)
    x = x_init_gen(dim)
    grad = grad_f_lam(x, Q)
    mag_grad = np.linalg.norm(grad)
    while (mag_grad > eps):
        x_next = x - grad
        grad = grad_f_lam(x_next, Q)
        mag_grad = np.linalg.norm(grad)
        x = x_next
        num_its = num_its + 1
    return num_its
```

In [6]:
```
n_vals = np.arange(1,101)
ave_it_vals = np.zeros(100)
#iterate over n = 1-100
for n in range(1,101):
    # perform 100 random inits per n and average number of its
    ave = 0
    for i in range(100):
        ave = ave + (1/100 * ambig_dim_grad_desc(n))
    # add averaged val to list
    ave_it_vals[n-1] = ave
```

In [7]:
```
plt.figure()
plt.plot(n_vals, ave_it_vals)
plt.xlabel("problem dimensionality [n]")
plt.ylabel("average number of iterations")
plt.title("problem dimensionality vs iteration steps to find minima")
plt.show()
```



Probelm 3

In [8]:
```
#define new function
x_sym, y, a, b, c, d = sym.symbols('x y a b c d')
f = sym.Matrix([(x_sym -a)**2 + (x_sym -a) * (y-b)+(y-b)**2])
x_0 = c
y_0 = d

g = sym.Matrix([x_sym, y]) - f.jacobian([x_sym,y]).jacobian([x_sym,y
]).inv() * f.jacobian([x_sym,y]).T
```

Symbol

In [9]:
```
#Display Symbolic Equation
display(g)
```

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

why is this type of problem useful to solve?

instead of simple gradient descent this is taking into account the curvature of the function, i.e. the second derivative as well

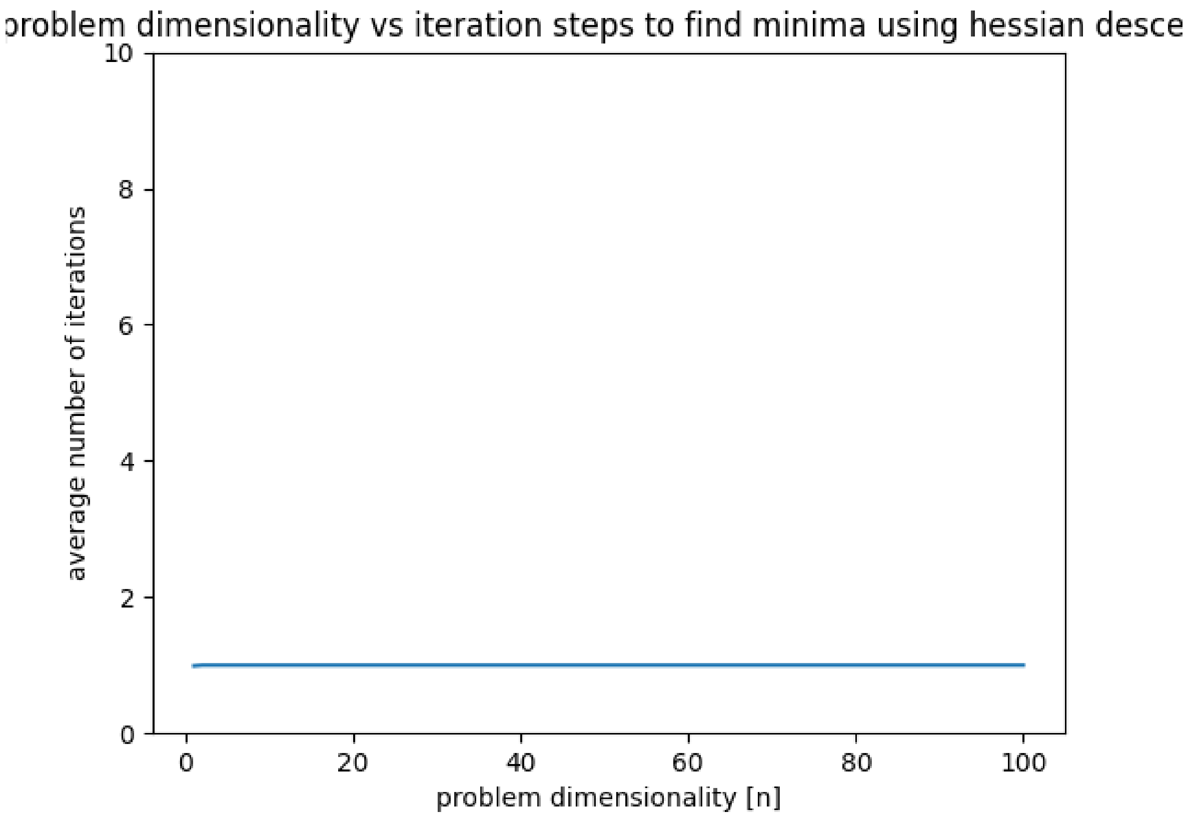
resolve problem 2 with hessian based descent

In [10]:
```
#define our function that performs gradient descent on ambigiously di
mensonal funcs
#creates random x and Q per descent
def ambig_dim_grad_desc_3(dim):
    eps = 0.1
    num_its = 0
    Q = Q_gen(dim)
    x = x_init_gen(dim)
    grad = grad_f_lam(x, Q)
    grad_2 = grad_2_f_lam(x, Q)
    mag_grad = np.linalg.norm(grad)
    while (mag_grad > eps):
        x_next = x - np.matmul(np.linalg.inv(grad_2), grad)
        grad = grad_f_lam(x_next, Q)
        grad_2 = grad_2_f_lam(x_next, Q)
        mag_grad = np.linalg.norm(grad)
        x = x_next
        num_its = num_its + 1
    return num_its
```

In [11]:
```
x = x_init_gen(3)
Q = Q_gen(3)
grad = grad_f_lam(x, Q)
grad_2 = grad_2_f_lam(x, Q)
```

In [12]:
```
n_vals = np.arange(1,101)
ave_it_vals = np.zeros(100)
#iterate over n = 1-100
for n in range(1,101):
    # perform 100 random inits per n and average number of its
    ave = 0
    for i in range(100):
        ave = ave + (1/100 * ambig_dim_grad_desc_3(n))
    ave_it_vals[n-1] = ave
```

In [13]:
```
plt.figure()
plt.plot(n_vals, ave_it_vals)
plt.xlabel("problem dimensionality [n]")
plt.ylabel("average number of iterations")
plt.ylim([0, 10])
plt.title("problem dimensionality vs iteration steps to find minima u
sing hessian descent")
plt.show()
```



Noting the difference:

Because the equation is quadratic, applying a quadratic fit always finds the minima within 1 iteration regardless of the dimension

Probelm 4

In [24]:
```
#define symbols
theta, u1, u2, x, y  = sym.symbols('theta u_1 u_2 x y')
q = sym.Matrix([x, y, theta])
u = np.array([1, -1/2])
#get lambda eq for update
qdot = sym.lambdify([q], sym.Matrix([sym.cos(theta)*u[0], sym.sin(the
ta)*u[0], u[1]]))
#get init conditions and controls
q_init = np.array([0,0,np.pi/2])

T = 2 * np.pi
```

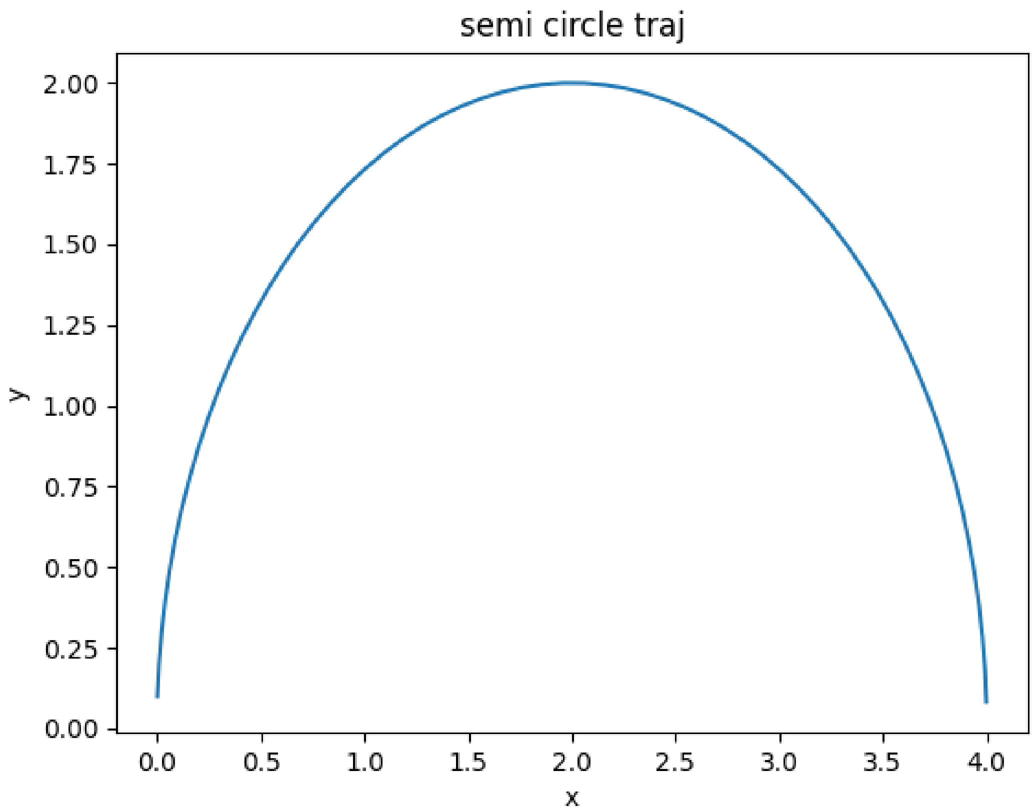
In [25]:
```
#tommy's rk4
def integrate(f,x0,dt):
    k1=dt*np.squeeze(f(x0))
    k2=dt*np.squeeze(f(x0+k1/2.))
    k3=dt*np.squeeze(f(x0+k2/2.))
    k4=dt*np.squeeze(f(x0+k3))
    xnew=x0+(1/6.)*(k1+2.*k2+2.*k3+k4)
    return xnew
```

In [26]:
```
def simulate(f,x0, tspan,dt):
    N = int((max(tspan)-min(tspan))/dt)
    x = np.copy(x0)
    tvec = np.linspace(min(tspan),max(tspan),N)
    xtraj = np.zeros((len(x0),N))
    for i in range(N):
        xtraj[:,i]=integrate(f,x,dt)
        x = np.copy(xtraj[:,i])
    return xtraj
```

In [27]:
```
def dynamics(q):
    q_dot = qdot(q)
    return q_dot
```

In [28]:
```
tspan = [0, T]
dt = 0.1
# N = int((max(tspan)-min(tspan))/dt)
# tvec = np.linspace(min(tspan),max(tspan),N)
xvec = simulate(dynamics, q_init,tspan, dt)
# tvec = np.linspace(min(tspan),max(tspan),N)
```

In [29]:
```
plt.figure()
plt.plot(xvec[0,:], xvec[1,:])
plt.title("semi circle traj")
plt.xlabel("x")
plt.ylabel("y")
plt.show()
```



In [30]:
```
tspan = [0, T]
dt = 0.1
N = int((max(tspan)-min(tspan))/dt)
tvec = np.linspace(min(tspan),max(tspan),N)
```

Problem 5

If desired trajectory is: $(x_d, y_d, \theta_d) = (\frac{4}{2\pi}, 0, \frac{\pi}{2})$ How is the movement constrained?

The movement is constrained from 3 dimensions to 1.

In [96]:
```
from scipy.optimize import minimize
tspan = [0, T]
dt = 0.1
N = int((max(tspan)-min(tspan))/dt)
#symbolic objective function
Q = sym.Matrix([[1, 0, 0],[0,.1,0],[0, 0, .01]])
R = sym.Matrix([[1, 0],[0,.1]]) #u1 matters more than u2
P = sym.Matrix([[10000, 0, 0], [0, 1000, 0], [0, 0, 10]])
#these will be replaced with actual xi and ui vals per it
x, y, theta, u1, u2, t = sym.symbols('x y theta u_1 u_2 t')
x_sym = sym.Matrix([x, y, theta])
u_sym = sym.Matrix([u1, u2])
x_d = sym.Matrix([4/(2*sym.pi) *t, 0, sym.pi/2])

x_0 = np.array([0,0,np.pi/2])
del_x = x_sym - x_d
#construct l
l = sym.Rational(1,2) * del_x.T * Q * del_x + sym.Rational(1,2) * u_s
ym.T * R * u_sym
l = sym.lambdify([x_sym, u_sym, t], l)
#construct m
m = sym.Rational(1,2) * del_x.T * P * del_x
m = sym.lambdify([x_sym, t], m)
#construct f, our update function given above as qdot
f = sym.lambdify([x_sym, u_sym], sym.Matrix([sym.cos(theta)*u1, sym.s
in(theta)*u1, u2]))

#define our discretized cost function
def h(xu):
    x_sym = xu.reshape(5,N)[:3, :]
    u_sym = xu.reshape(5,N)[3:,:]
    val = 0
    #perform discrete integration
    for i in range(N):
        val = val + l(np.expand_dims(x_sym[:,i],1), np.expand_dims(u_
sym[:,i],1), i)
    val = val + m(np.expand_dims(x_sym[:,-1],1), N)
    return val

#return a list of all constraints
def g(xu):
#     xu = xu.reshape(5, N)
    constraints = np.empty(3 * N)
    #first constraint is x0
    constraints[0] = xu[0] - x_0[0]
    constraints[N] = xu[N] - x_0[1]
    constraints[2*N] = xu[2*N] - x_0[2]
    #TODO: unflatten to use matrix math to make sexier
    for i in range(1,N):
        #get state at i-1
        x_prev = np.array([xu[i-1], xu[i + N - 1], xu[i + 2*N - 1]])
        #get controls
        conrtols = np.array([xu[i+3*N], xu[i+4*N]])
        #get update
        update = f(x_prev,conrtols)
        #constraint equation from lecture 3
```

```
        for itr in range(3):
            constraints[i + itr * N] = xu[i + itr * N] - x_prev[itr]
- dt * update[itr]
    return constraints


# make sure inputs vectors of form: np.array([[el,el,el]]).T
```

In [97]:
```
#flatten x to: x, y, theta and append u
xvec.shape
uvec = np.zeros((u.shape[0], N)) + np.expand_dims(u,1)
```

In [98]:
```
print(xvec.shape)
print(uvec.shape)
```

```
(3, 62)
(2, 62)
```

In [99]:
```
xu = np.concatenate((xvec,uvec))
print(xu.shape)
```

```
(5, 62)
```

In [100]:
```
my_cons = {'type':'eq', 'fun':g}
```

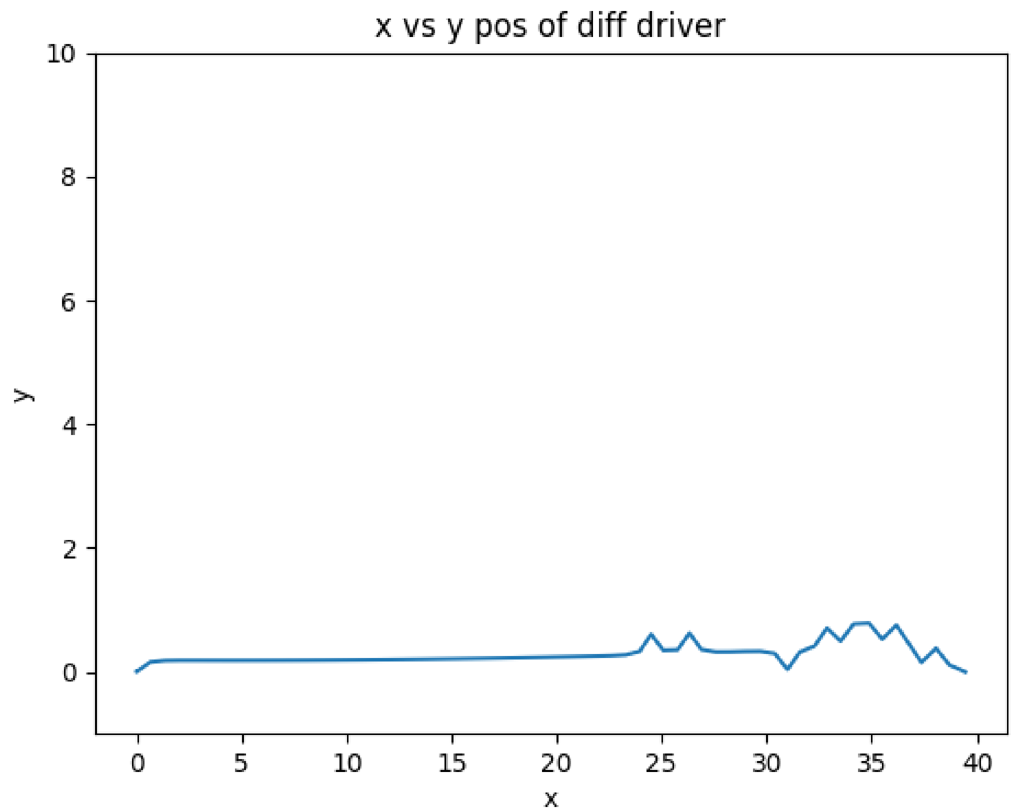
In [101]:
```
xu.flatten().shape
```

Out[101]:
```
(310,)
```

In [102]:
```
my_min = minimize(h, xu.flatten(), constraints=my_cons)
```

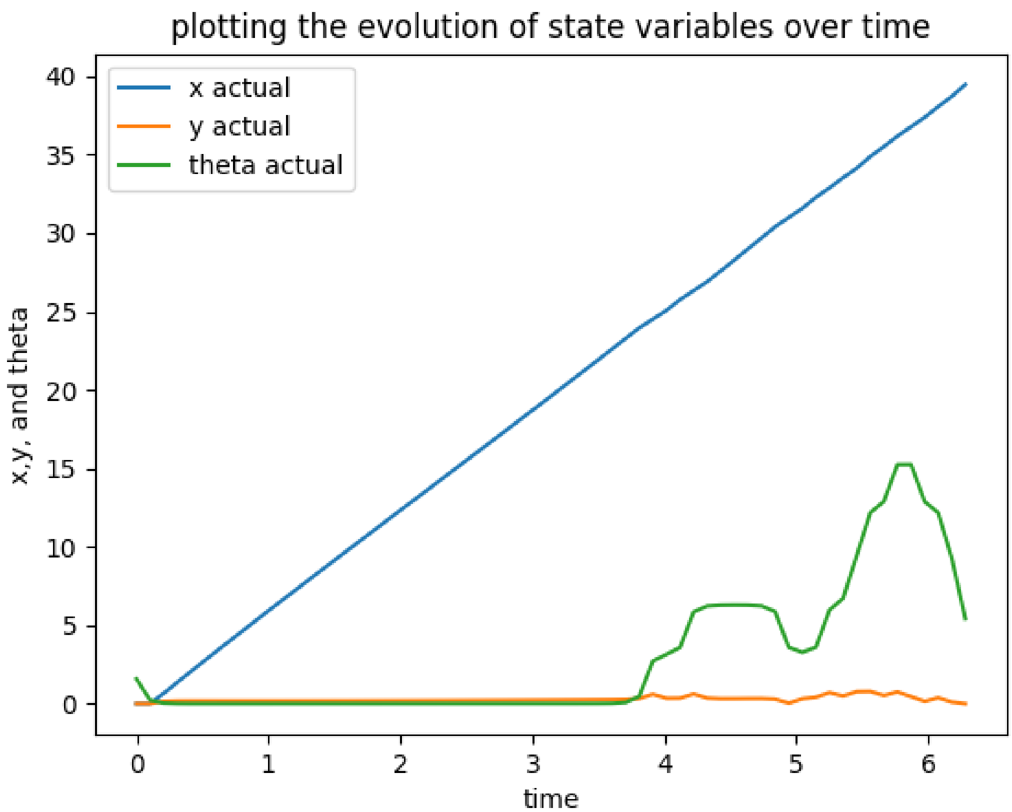
In [104]:
```
soln = my_min.x.reshape(5,N)
```

In [116]:
```
plt.figure()
plt.plot(soln[0,:], soln[1,:])
plt.ylim([-1, 10])
plt.xlabel("x")
plt.ylabel("y")
plt.title("x vs y pos of diff driver")
plt.show()
```



In [113]:
```
plt.figure()
plt.plot(tvec, soln[0,:])
plt.plot(tvec, soln[1,:])
plt.plot(tvec, soln[2,:])

plt.xlabel("time")
plt.ylabel("x,y, and theta")
plt.legend(["x actual", "y actual", "theta actual"])
plt.title("plotting the evolution of state variables over time")
plt.show()
```



note

while this is far from perfect, the infrastructure is sound and just requires tuning of the P, R, and Q parameters