Homework #2

Josh Cohen Questions: 4.1,4.2,4.5, 5.2,5.9

In [825]:
```
import numpy as np
import sympy as sym
%matplotlib notebook
from matplotlib import pyplot as plt
import pandas as pd
from IPython.display import Image
```

Probelm 4.1

see problems **a)., b)., & d).** below

In [828]:
```
Image("4_1.jpg")
```

Out[828]:

Homework #2 4/24

4.1 a) C is a N×N Symmetric Matrix
Show that if C has all non-negative eigen values then $z^T C z \geq 0$ for all z

We use eigen value decomposition

$C\bar{v} = \lambda \bar{v}$

$CQ = Q\Lambda$

$C = Q\Lambda Q^{-1}$

where Q = N×N Matrix where each column is a $\bar{v}_i$ eigen Vector of C (called V in C.4)

& Λ = N×N diagonal matrix where $\Lambda_{ii} = \lambda_i$ (called D in C.4)

$z^T Q \Lambda Q^T z > 0$

Since $z^T Q \cdot Q^T z$ is positive for all real z, the product $z^T Q \Lambda Q^T z$ is > 0 for Λ of which all λ_i are ≥ 0

b) Show that if $z^T C z > 0$ then all eigen values are non-negative

Noting: $C = VDV^T \Rightarrow z^T V D V^T z > 0$

if this is defined as positive, & we know $z^T V \cdot V^T z$ must be positive for all real z & all square, symmetric C then $z^T C z > 0$ only for a C which D & therefore all λ_i are > 0

d) $C + \tilde{\lambda} I_{N \times N} \geq 0$; $\tilde{\lambda}$ = vector of minimum λ values

$VDV^T + \tilde{\lambda} I_{N \times N} \geq 0 \Rightarrow VDV^T \geq -\tilde{\lambda} I_{N \times N}$

$VD \geq -\tilde{\lambda} V$

$D \geq V^T \tilde{\lambda} V$

$-VDV^T \leq \tilde{\lambda} I_{N \times N}$

$\tilde{\lambda}_i \geq -\lambda_i \cdot \|\bar{v}_i\|^2$ if normalized $\Rightarrow$ $\boxed{\tilde{\lambda}_i > -\lambda_i}$

c).

In [187]:
```
a = sym.MatrixSymbol('a', 1, 1)
b = sym.MatrixSymbol('b', 2, 1)
C = sym.MatrixSymbol('C', 2, 2)
# w1, w2 = sym.symbols('w_1 w_2')
w = sym.MatrixSymbol('w', 2, 1)
g = a + b.T * w + w.T * C * w
grad_g = g.diff(w)
grad_2_g = grad_g.diff(w)
```

In [196]:
```
print("with a g(w) of:")
display(g)
print("gradient g'(w) of:")
display(grad_g)
print("g''(w) of:")
display(sym.factor(grad_2_g.subs({C.T: C})))
```

with a g(w) of:

$$a + b^T w + w^T C w$$

gradient g'(w) of:

$$Cw + b + C^T w$$

g''(w) of:

$$2C$$

In [242]:
```
C_act = sym.Matrix([[1,1],[1,1]])
C_act = sym.Matrix(sym.factor(grad_2_g.subs({C.T: C})).subs({C:C_act
}))
print("therefore the second order derivative of our function is:")
display(C_act)
print("with eigen values of:")
eig_vals = C_act.eigenvals(multiple = True)
print(eig_vals)
print("since the eigen values of C are >= to 0 we can verify that the
quadaratic function is convex")
```

therefore the second order derivative of our function is:

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

with eigen values of:
[0, 4]
since the eigen values of C are >= to 0 we can verify that the quadar
atic function is convex

d).

In [ ]:
```
C = sym.MatrixSymbol()
```

Problem 4.2

In [829]:
```
Image("4_2.jpg")
```

Out[829]:

4.2 a) Show that for $X \in \mathbb{R}^{n \times 1}$
the $n \times n$ outer product has
all non negative eigen values

$\bar{z}^T \bar{X} \cdot \bar{X}^T \bar{z} \geq 0$ $\quad\quad$ $C\bar{v} = \lambda \bar{v}$

$\bar{z}^T X \bar{z}^T \geq 0$ $\quad\quad$ $C = V D V^{-1}$

$\bar{z}^T V D V^{-1} \bar{z}^T \geq 0$

just as in 4.1a $\quad \|\bar{z}^T V\| > 0$

further the matrix is positive definite
because each x_{ii} term is $= x_i \cdot x_i$
So D is non negative
& therefore: $z^T x \cdot x^T z > 0$ ✓

b) repeat the demonstration above
the matrix is still semi positive definite
if all $\delta_p > 0$ then the matrix
is still semi positive definite so the
argument above is true

c) $\sum_{p=1}^{P} \delta_p x_p x_p^T + \lambda I_{n \times n}$ ← Show this has all positive eigen values for $\lambda > 0$

Again by extending
the argument above, the
semi positive definite matrix remains
semi positive definite when a strictly positive
value is added to it. ∴ the eigen
values of such a matrix will all
be positive

Problem 4.5

a). use 1st order optimality condition to determine the unique stationary point of $g(w) = \log(1 + e^{w^T w})$ where w is 2D

In [245]:
```
w1, w2 = sym.symbols('w_1 w_2')
w = sym.Matrix([w1, w2])
g = sym.log(1 + sym.exp(w.T * w)[0])
grad_g = sym.Matrix([g]).jacobian(w)
ans = sym.solve(grad_g, w)
display(ans)
```

```
[(0, 0)]
```

In [246]:
```
display(g)
display(grad_g)
```

$$\log\left(e^{w_1^2+w_2^2}+1\right)$$

$$\begin{bmatrix} \frac{2w_1e^{w_1^2+w_2^2}}{e^{w_1^2+w_2^2}+1} & \frac{2w_2e^{w_1^2+w_2^2}}{e^{w_1^2+w_2^2}+1} \end{bmatrix}$$

Therefore it can be seen the first order optimality condition is satisfied at $w = (0, 0)$

b). use second order defitionition of convexity to verify that g(w) is convex and thereore the stationary point is the global minimum

In [247]:
```
grad_2_g = grad_g.jacobian(w)
P, D = grad_2_g.diagonalize()
```

In [248]:
```
display(D)
```

$$\begin{bmatrix} \frac{2e^{w_1^2+w_2^2}}{e^{w_1^2+w_2^2}+1} & 0 \\ 0 & \frac{2\left(2w_1^2+2w_2^2+e^{w_1^2+w_2^2}+1\right)e^{w_1^2+w_2^2}}{2e^{w_1^2+w_2^2}+e^{2w_1^2+2w_2^2}+1} \end{bmatrix}$$

Therefore since e^x can never be < 0, from the work done in 4.1 and 4.2 we can show that the NxN symmetric Hessian matrix will never be < 0, therefore the function is convex everywhere and the minima we found in part **a).** is the global minimum

c). Perform Newton's method to find the minimum of g(w) determined in part **a).**, initialize $w_0 = 1_{Nx1}$ plot the cost function over 10 iterations

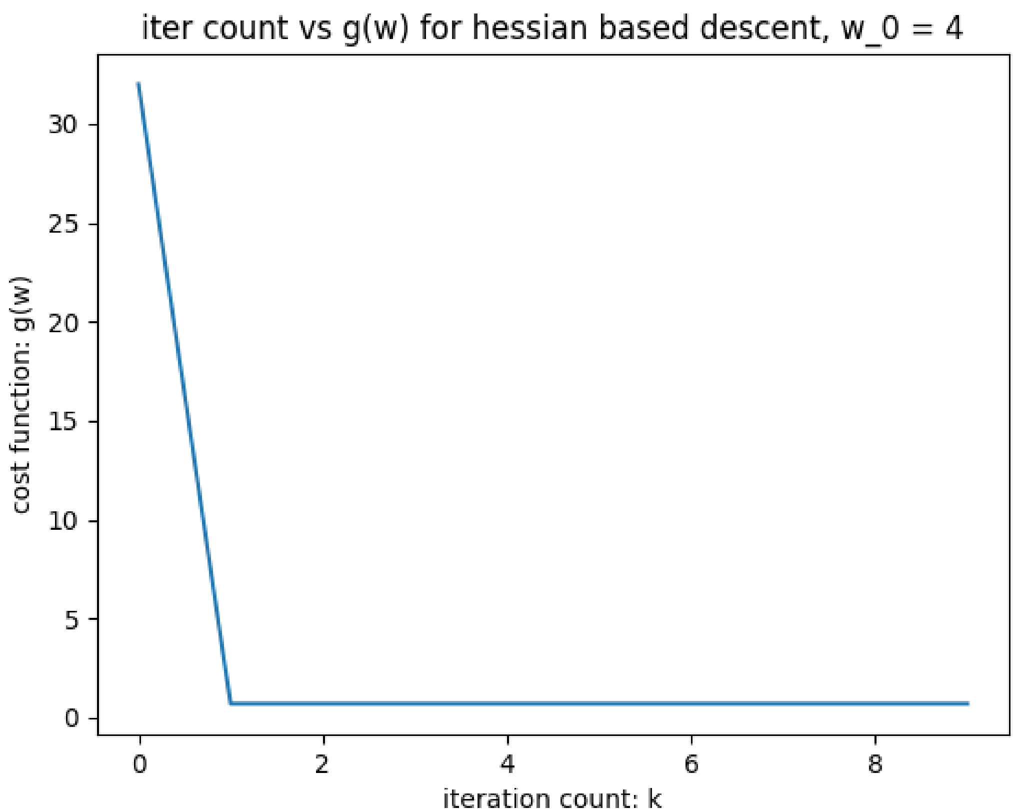
In [249]:
```
cost_func = sym.lambdify([w], g)
grad = sym.lambdify([w], grad_g)
grad_2 = sym.lambdify([w], grad_2_g)
w_0_c = np.ones(2)
```

In [250]:
```
#define function that we will use for gradient descent
def grad_desc(w_last):
    num_its = 10
    cost_vals = np.zeros(num_its)
    grad_g_last = grad(w_last)
    grad_2_g_last = grad_2(w_last)
    cost_vals[0] = cost_func(w_last)
#     print(cost_func(w_last))
#     print(grad_2_g_last.shape)
#     print(grad_g_last.T.shape)
    for x in range(1, num_its):
        w_next = w_last - np.matmul(np.linalg.inv(grad_2_g_last), gra
d_g_last.T)[0]
        cost_vals[x] = cost_func(w_next)
        w_last = w_next
        grad_g_last = grad(w_last)
        grad_2_g_last = grad_2(w_last)
    return cost_vals
```

In [255]:
```
cost_vals = grad_desc(w_0_c)
plt.figure()
it_vec = np.arange(10)
plt.plot(it_vec, cost_vals)
plt.xlabel("iteration count: k")
plt.ylabel("cost function: g(w)")
plt.ylim([-0.1,2.5])
plt.title("iter count vs g(w) for hessian based descent, w_0 = 1")
plt.show()
```

d). Run code from **c).** again with intiial guess at $w_0 = 4 * 1_{Nx1}$, explain behavior

In [244]:
```python
w_0_d = np.ones(2) * 4
cost_vals = grad_desc(w_0_d)
plt.figure()
it_vec = np.arange(10)
plt.plot(it_vec, cost_vals)
plt.xlabel("iteration count: k")
plt.ylabel("cost function: g(w)")
plt.title("iter count vs g(w) for hessian based descent, w_0 = 4")
plt.show()
```



why does this result make sense for our $g(w)$? Since our g(w) is a quadratic it makes sense that our second order update rule converges quickly, the one with the greater initial error will then have (due to its globally convex nature) a steeper initial descent which will allow for it to converge quicker and closer to the actual solution

Problem 5.2

a). fit linear model to dataset: *kleibers_law_data.csv*

In [470]:
```
#read in data
kliebers_df = pd.read_csv("kleibers_law_data.csv", header=None, index
_col=None)
P = kliebers_df.shape[1]
x_vals = kliebers_df.iloc[0,:]
y_vals = kliebers_df.iloc[1,:]
#define symbolic cost function
b, w, xp, yp, p = sym.symbols('b, w xp yp p')
w_tilda = sym.Matrix([b, w])
xp_gen = sym.lambdify(x_sym,sym.Matrix([1, x_sym]))
yp_gen = sym.lambdify(y_sym, sym.Matrix([y_sym]))

#to perform gradient descent calculate A and b_vec
Ap = sym.lambdify(xp, (xp_gen(xp) * xp_gen(xp).T))
bp = sym.lambdify((xp, yp), xp_gen(xp) * yp_gen(yp))
A = np.zeros((2,2))
b = np.zeros((2,1))
for (xp,yp) in zip (np.log(x_vals.iloc[1:]), np.log(y_vals.iloc[1
:])):
    A = A + Ap(xp)
    b = b + bp(xp,yp)
```

In [471]:
```
display(A)
display(b)
```

```
array([[ 1497.        , -2644.10450903],
       [-2644.10450903, 14575.33617811]])

array([[ 8474.33295994],
       [-8512.73814967]])
```

In [472]:
```
A_inv = np.linalg.inv(A)
w_star = np.matmul(A_inv,b)
display(w_star)
```

```
array([[6.8119607 ],
       [0.65170352]])
```

In [473]:
```
w_star[1][0]
```

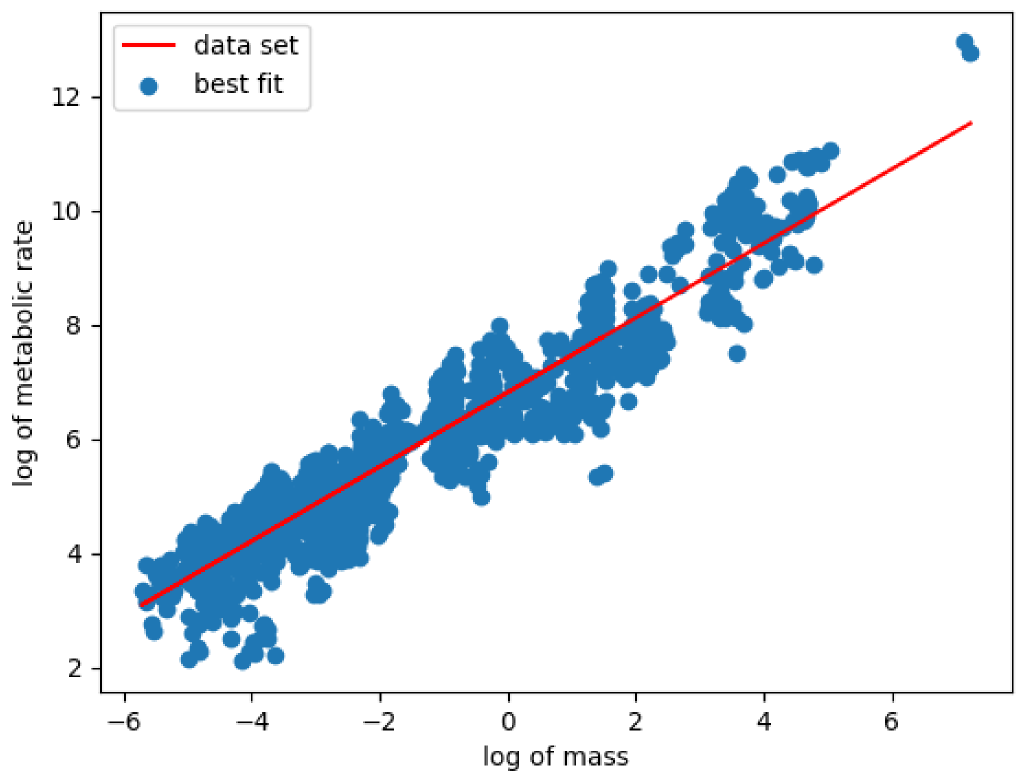
Out[473]:
```
0.6517035173670688
```

In [477]:
```
plt.figure()
ax = plt.gca()
plt.ylabel("log of metabolic rate")
plt.xlabel("log of mass")
plt.scatter(np.log(x_vals), np.log(y_vals))

#plot our fitted line
log_xs = np.log(x_vals)
log_ys = w_star[1][0] * log_xxs + w_star[0][0]
plt.plot(log_xs, log_ys, 'r')

plt.legend(["data set", "best fit"])

plt.show()
```



b).. right out nonlinear relationship between mass x and metabolic rate y

In [480]:
```
slope = w_star[1][0]
intercept = w_star[0][0]
print("best fit parameters for:\n\rm: {}\r\nb: {}".format(slope, inte
rcept))
```

```
best fit parameters for:
m: 0.6517035173670688
b: 6.811960700573648
```

writing this out as a non-linearized equation:

$$y_p = e^{w_0 + \log x_p w_1}$$

or

$$y_p = e^{w_0} x_p^{w_1}$$

c). plug in a amass of 10kg to find out how many calories a 10kg animal needs daily

In [513]:
```
animal_mass = 10 #in kg
daily_cals = np.exp(intercept) * animal_mass**slope * 1000/4.18 #in c
al/day
```

In [521]:
```
print("There for the 10kg animal needs: {:.2f} calories daily;\nHowev
er, in our 'calorie' which is infact a kilocalorie this number is:
{:.2f} 'calories' a day".format(daily_cals, daily_cals/1000))
```

```
There for the 10kg animal needs: 974819.43 calories daily;
However, in our 'calorie' which is infact a kilocalorie this number i
s: 974.82 'calories' a day
```

Problem 5.9

Standardize and normalize quality metrics, verify example 5.5 and 5.6

In [526]:
```
#read in data
boston_housing_df = pd.read_csv("boston_housing.csv", header=None, in
dex_col=None).dropna(axis=1)
auto_data_df = pd.read_csv("auto_data.csv", header=None, index_col=No
ne).dropna(axis=1)
```

In [609]:
```
#standardize and normalize
bh_mean = boston_housing_df.mean(axis = 1)
ad_mean = auto_data_df.mean(axis=1)
bh_std = boston_housing_df.std(axis=1)
ad_std = auto_data_df.std(axis=1)

bh_std_norm = boston_housing_df.sub(bh_mean, axis=0).div(bh_std, axis
=0)
ad_std_norm = auto_data_df.sub(ad_mean, axis=0).div(ad_std, axis=0)
```

In [740]:
```
def get_best_fit(df):
    inputs = df.iloc[:-1,:]
    output = df.iloc[-1,:]
    N, P = inputs.shape
    b, yp_sym = sym.symbols('b yp')
    w = sym.MatrixSymbol('w', N, 1).as_explicit() #first element is b
    w_tilda = sym.Matrix([b, w])
    xp_sym = sym.MatrixSymbol('x', N,1)
    xp_gen = sym.lambdify([xp_sym],sym.Matrix([sym.Matrix([1]), xp_sy
m]))
    yp_gen = sym.lambdify(yp_sym, sym.Matrix([yp_sym]))
    #to perform gradient descent calculate A and b_vec
#     Ap = sym.lambdify([xp_sym], (sym.MatMul(xp_gen(xp_sym),xp_gen(x
p_sym).T)))
    bp = sym.lambdify(([xp_sym], yp_sym), xp_gen(xp_sym) * yp_gen(yp_
sym))
    A = np.zeros((N+1,N+1))
    b = np.zeros((N+1,1))
    for i in range(1,P):
        xp = xp_gen(np.expand_dims(inputs.iloc[:,i].to_numpy(),1))
        yp = output.iloc[i]
        A = A + np.matmul(xp,xp.T)
        b = b + xp * yp

    A_inv = np.linalg.inv(A)
    w_star = np.matmul(A_inv, b)
    x_real = np.concatenate((np.ones((P,1)).T, inputs.to_numpy()))
    y_real = np.expand_dims(output.to_numpy(),1)
    return (w_star, x_real, y_real)
```

In [795]:
```
def get_errors(w_star, x_real, y_real):
    y_theoro = np.matmul(x_real.T, w_star)
    y_diff = np.abs(np.subtract(y_theoro,y_real))
    y_diff_sq = np.multiply(y_diff,y_diff)
    RMSE = np.sqrt(np.mean(y_diff_sq))
    MAD = np.mean(y_diff)

    return (RMSE, MAD)
```

In [797]:
```
w_star, x_real, y_real = get_best_fit(bh_std_norm)
bh_RMSE, bh_MAD = get_errors(w_star, x_real, y_real)
w_star, x_real, y_real = get_best_fit(ad_std_norm)
ad_RMSE, ad_MAD = get_errors(w_star, x_real, y_real)
```

In [824]:
```
print("Therefore we have the following errors for our best fits:\r\n"
)
print("Boston Housing:\r\nRMSE: {:.5f}\r\nMAD: {:.5f}".format(bh_RMSE
, bh_MAD))
print("Actual RMSE/MAD: {:.5f}\r\nTarget RMSE/MAD: {:.5f}".format(bh_
RMSE/bh_MAD, 4500/3000 ))
print("Actual RMSE/Theoretical RMSE: {:.5f}".format(4500/bh_RMSE))
print("Actual MAD/Theoretical MAD: {:.5f}".format(3000/bh_MAD))
print("")
print("Auto Data:\r\nRMSE: {:.5f}\r\nMAD: {:.5f}".format(ad_RMSE, ad_
MAD))
print("Actual RMSE/MAD: {:.5f}\r\nTarget RMSE/MAD: {:.5f}".format(ad_
RMSE/ad_MAD, 3.3/2.5))
print("Actual RMSE/Theoretical RMSE: {:.5f}".format(3.3/ad_RMSE))
print("Actual MAD/Theoretical MAD: {:.5f}".format(2.5/ad_MAD))
```

```
Therefore we have the following errors for our best fits:

Boston Housing:
RMSE: 0.50882
MAD: 0.35624
Actual RMSE/MAD: 1.42832
Target RMSE/MAD: 1.50000
Actual RMSE/Theoretical RMSE: 8844.05012
Actual MAD/Theoretical MAD: 8421.39522

Auto Data:
RMSE: 0.42199
MAD: 0.32036
Actual RMSE/MAD: 1.31724
Target RMSE/MAD: 1.32000
Actual RMSE/Theoretical RMSE: 7.82009
Actual MAD/Theoretical MAD: 7.80373
```

note

the results above should be satisfactory evidence that the best fits were found

In [ ]:
```
```