Homework #1

Josh Cohen

Imports

In [10]:
```
%matplotlib notebook
from matplotlib import pyplot as plt
import numpy as np
import sympy as sym
from IPython.display import Image
```

Problem 3.1

In [24]:
```
Image("p31.jpg")
```

Out[24]:

Homework #1

3.1 a) $g(w) = w\log(w) + (1-w)\log(1-w)$ where $0 < w < 1$

$\nabla g(w) = \log(w) + w\cdot\frac{1}{w} + -1\log(1-w) + (1-w)\cdot\frac{1}{1-w}\cdot -1$

$= \log(w) + 1 + -\log(1-w) + -1$

$= \log(w) - \log(1-w)$

$\nabla g(w) = \log\left(\frac{w}{1-w}\right) \Rightarrow 0$

$\frac{w}{1-w} = 1$

$w = 1-w$

$2w = 1$

$w = \frac{1}{2}$

b) $g(w) = \log(1+e^w) \Rightarrow \nabla g(w) = \frac{1}{1+e^w}\cdot e^w$

$\frac{e^w}{1+e^w} = 0$; $w = -\infty$

c) $g(w) = w\tanh(w) \Rightarrow \nabla g(w) = \tanh(w) + w(1-\tanh^2(w))$

$0 = \tanh(w) + w(1-\tanh^2(w))$; $w = 0$

$(\tanh(w) + w) = \tanh^2(w)\cdot w$

$\frac{1}{w}\tanh(w) + 1 = \tan^2(w)$

$= \tan(w)$

d) $g(\bar{w}) = \frac{1}{2}\bar{w}^T C\bar{w} + b^T\bar{w}$ where $C = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\nabla g(\bar{w}) = C\bar{w} + b = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\bar{w} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$0 = 2w_1 + w_2 + 1 \Rightarrow w_2 = -(2w_1+1)$

$0 = w_1 + 3w_2 + 1$

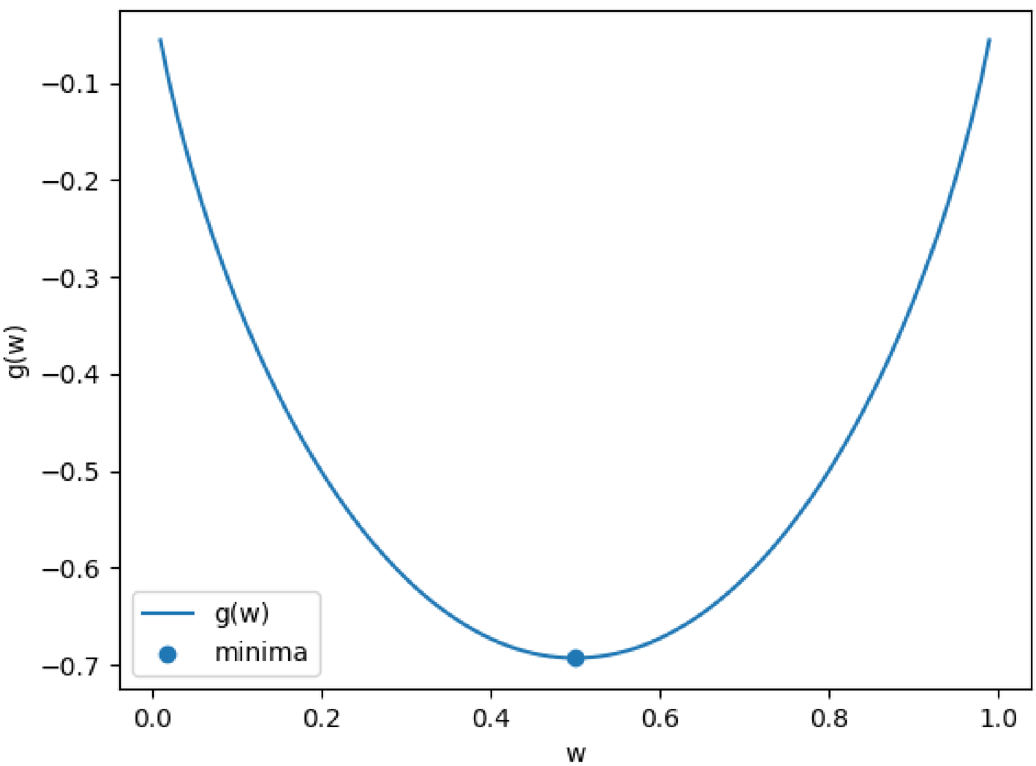
$w_2 = -(-4/5+1) = -1/5$

$0 = w_1 - 6w_1 - 3 + 1$

$5w_1 = -2$ $w_1 = -2/5$

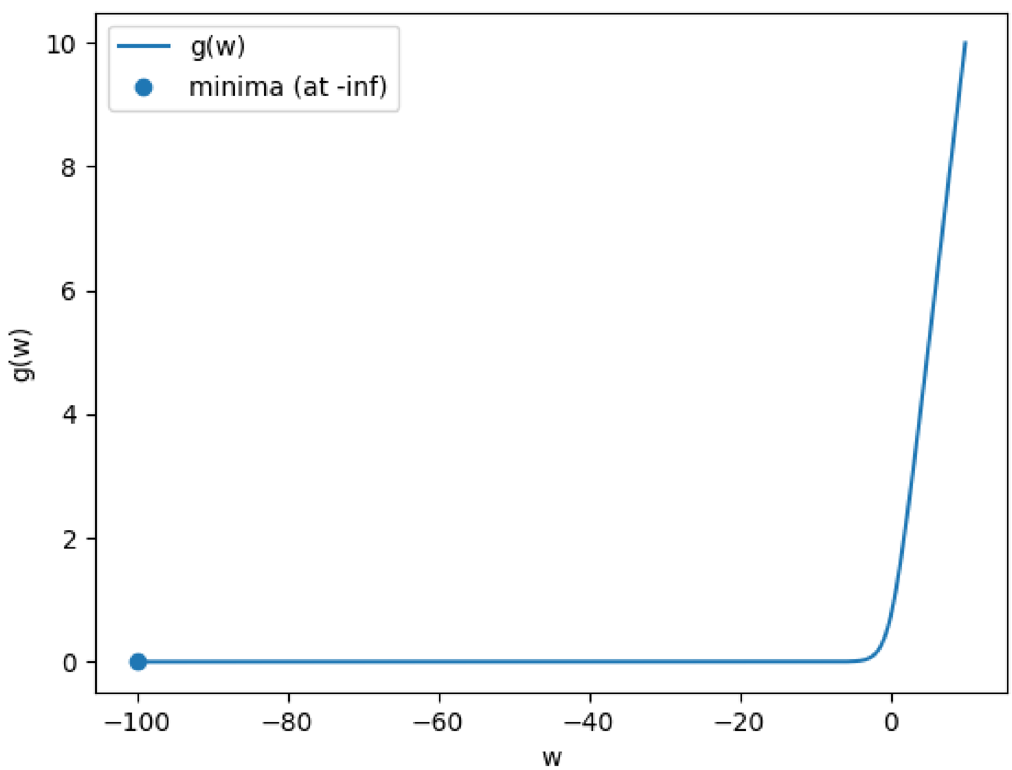
In [5]:

```
#3.1 a)
w = sym.symbols('w')
eq_a = w * sym.ln(w) + (1- w) * sym.log(1-w)
eq_a_lam = sym.lambdify(w, eq_a)
input_vec = np.arange(0.01,1,.01)
output_vec = eq_a_lam(input_vec)
plt.figure()
plt.plot(input_vec, output_vec)
plt.scatter(0.5, eq_a_lam(0.5))
plt.ylabel("g(w)")
plt.xlabel("w")
plt.legend(["g(w)", "minima"])
plt.show()
```



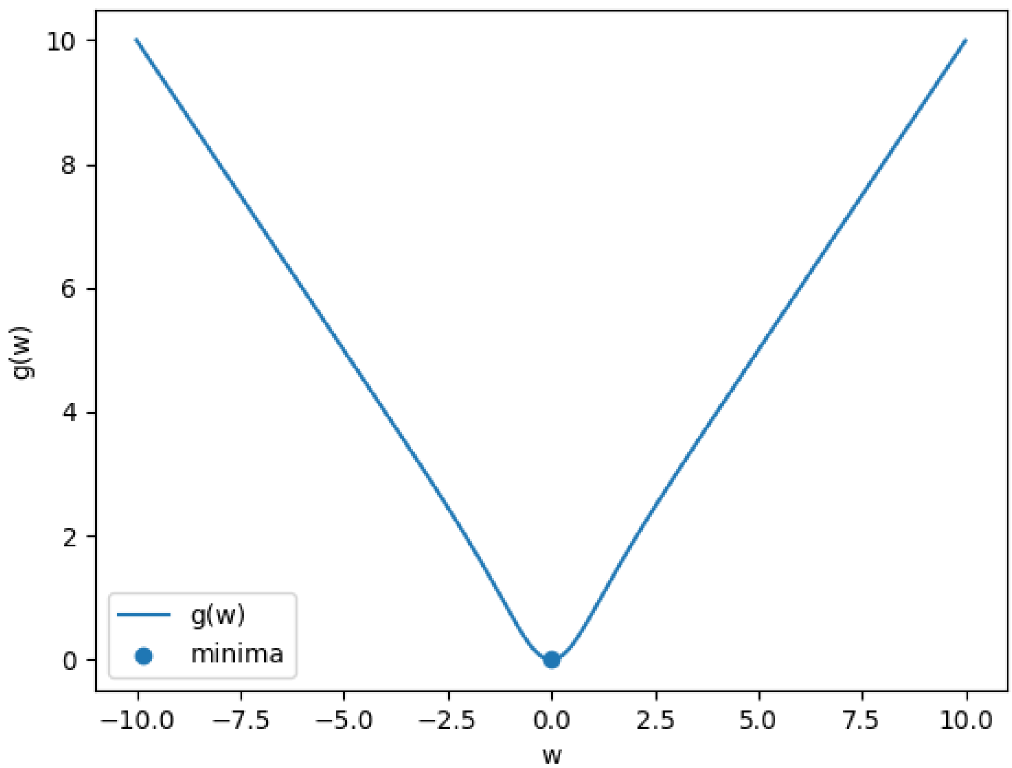
In [7]:

```
#3.1 b)
w = sym.symbols('w')
eq_b = sym.ln(1 + sym.exp(w))
eq_b_lam = sym.lambdify(w, eq_b)
input_vec = np.arange(-100,10,.01)
output_vec = eq_b_lam(input_vec)
plt.figure()
plt.plot(input_vec, output_vec)
plt.scatter(-100, eq_b_lam(-100))
plt.ylabel("g(w)")
plt.xlabel("w")
plt.legend(["g(w)", "minima (at -inf)"])
plt.show()
```

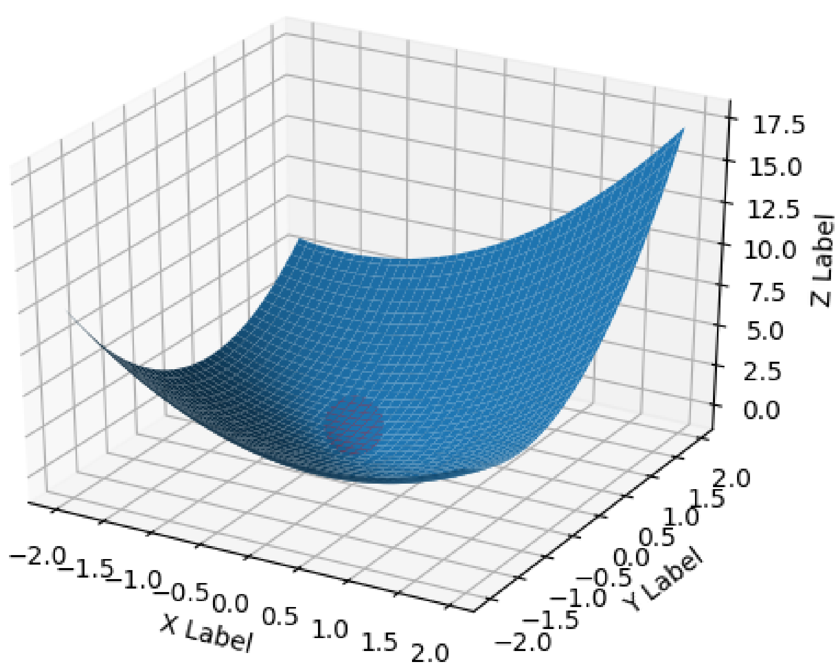


In [8]:

```
#3.1 c)
w = sym.symbols('w')
eq_c = w * sym.tanh(w)
eq_c_lam = sym.lambdify(w, eq_c)
input_vec = np.arange(-10,10,.01)
output_vec = eq_c_lam(input_vec)
plt.figure()
plt.plot(input_vec, output_vec)
plt.scatter(0, eq_c_lam(0))
plt.ylabel("g(w)")
plt.xlabel("w")
plt.legend(["g(w)", "minima"])
plt.show()
```

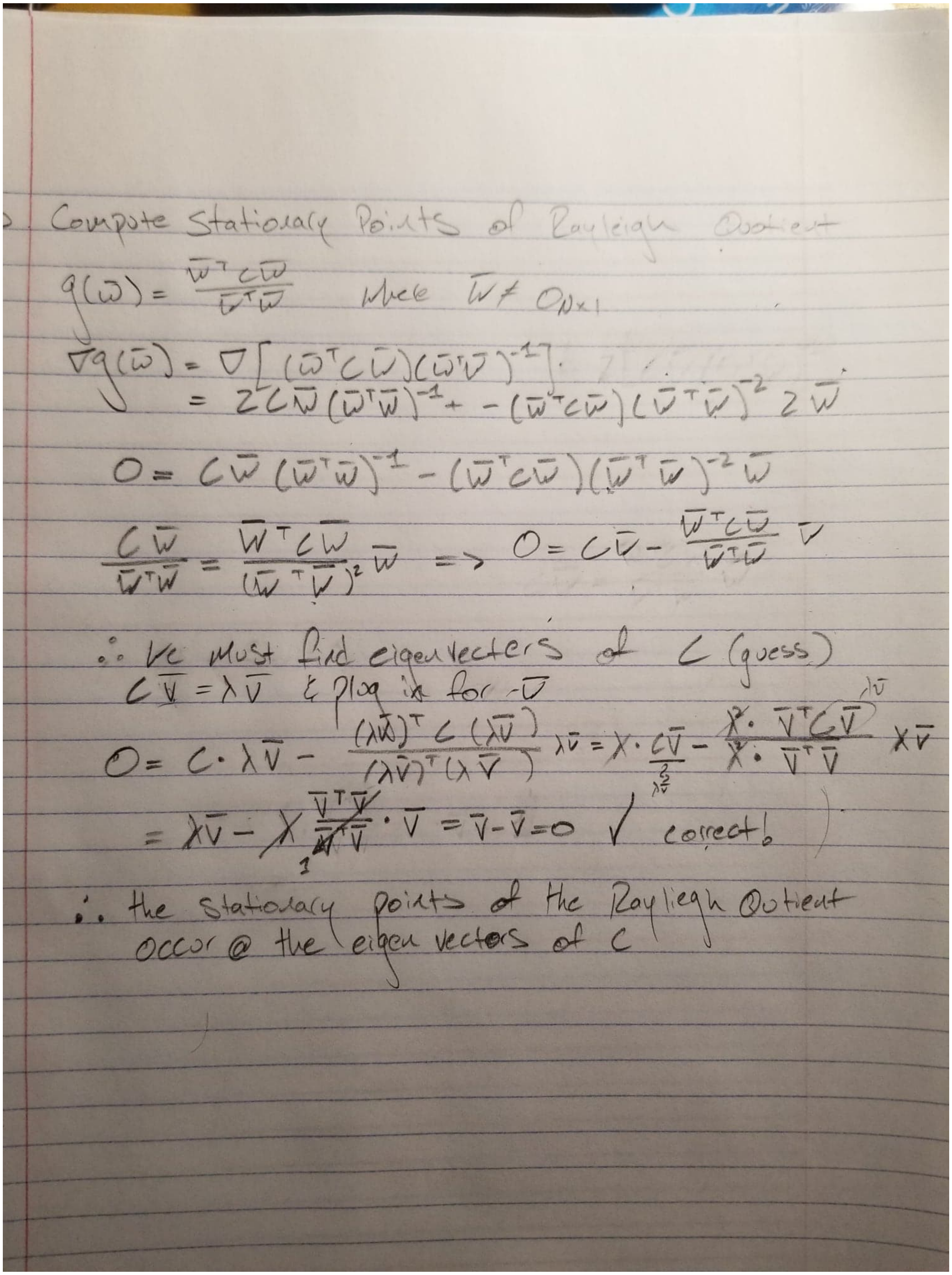


In [13]:
```
#3.1 d)
w_1, w_2 = sym.symbols('w_1, w_2')
w = sym.Matrix([w_1, w_2])
C = sym.Matrix([[2,1],[1,3]])
b = sym.Matrix([1,1])

from mpl_toolkits.mplot3d import Axes3D
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')

eq_d = sym.Rational(1,2) * w.T * C * w + b.T * w
eq_d_lam = sym.lambdify([w_1,w_2], eq_d)
w_1_inp = w_2_inp = np.arange(-2,2,.05)
X, Y = np.meshgrid(w_1_inp, w_2_inp)
zs = np.array(eq_d_lam(np.ravel(X), np.ravel(Y)))
Z = zs.reshape(X.shape)
surface = ax.plot_surface(X, Y, Z)
#serves to mark the minima!
scatter = ax.scatter(-.4, .2, eq_d_lam(-.4, .2), c='r', s = 500)
ax.set_xlabel('X Label')
ax.set_ylabel('Y Label')
ax.set_zlabel('Z Label')
plt.show()
```



Problem 3.3

In [14]:
```
Image("3_3.jpg")
```
Out[14]:



Compute Stationary Points of Rayleigh Quotient

$q(\bar{w}) = \frac{\bar{w}^T C \bar{w}}{\bar{w}^T \bar{w}}$ where $\bar{w} \neq 0_{N\times 1}$

$\nabla q(\bar{w}) = \nabla\left[(\bar{w}^T C \bar{w})(\bar{w}^T \bar{v})^{-1}\right]$

$= 2C\bar{w}(\bar{w}^T\bar{w})^{-1} + -(\bar{w}^T C \bar{w})(\bar{w}^T\bar{w})^{-2}\, 2\bar{w}$

$0 = C\bar{w}(\bar{w}^T\bar{w})^{-1} - (\bar{w}^T C\bar{w})(\bar{w}^T\bar{w})^{-2}\bar{w}$

$\frac{C\bar{w}}{\bar{w}^T\bar{w}} = \frac{\bar{w}^T C\bar{w}}{(\bar{w}^T\bar{w})^2}\bar{w} \Rightarrow 0 = C\bar{v} - \frac{\bar{w}^T C\bar{w}}{\bar{w}^T\bar{w}}\bar{v}$

∴ we must find eigenvectors of C (guess)

$C\bar{v} = \lambda\bar{v}$ & plug in for $\bar{v}$ $\lambda\bar{v}$

$0 = C\cdot\lambda\bar{v} - \frac{(\lambda\bar{w})^T C(\lambda\bar{v})}{(\lambda\bar{v})^T(\lambda\bar{v})}\lambda\bar{v} = \lambda\cdot \underbrace{C\bar{v}}_{\lambda\bar{v}} - \frac{\cancel{\lambda^2}\cdot\bar{v}^T C\bar{v}}{\cancel{\lambda^2}\cdot\bar{v}^T\bar{v}}\lambda\bar{v}$

$= \lambda\bar{v} - \lambda\,\cancel{\frac{\bar{v}^T\bar{v}}{\bar{v}^T\bar{v}}}_{1}\cdot\bar{v} = \bar{v} - \bar{v} = 0$ ✓ correct!

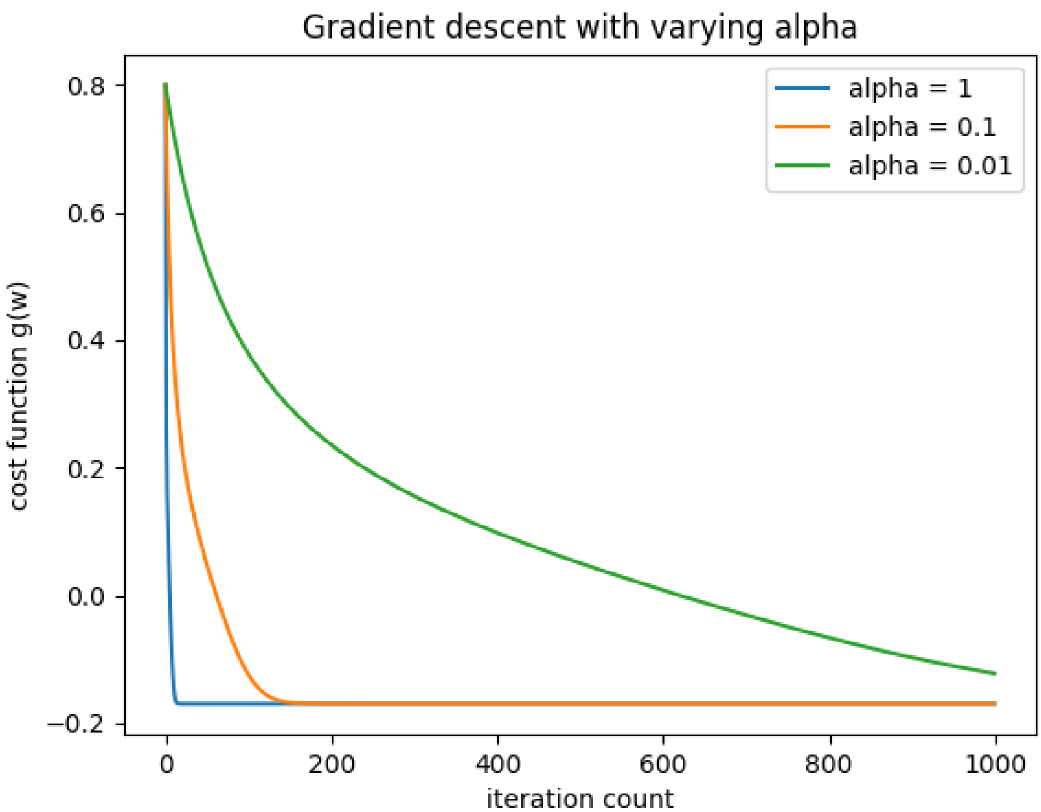
∴ the stationary points of the Rayleigh Quotient occur @ the eigen vectors of C

Problem 3.5

In [95]:
```
###
# Problem 3.5
###
#Note: problem says to implement functions with numpy, I will impleme
nt them symbollically
#        with sympy then use lambdify to get a lambda function that p
roduces numpy arrays
import numpy as np
import sympy as sym
#define symbol and symbollic equations
w = sym.symbols('w')
g = sym.Rational(1,50) * (w**4 + w**2 + 10 * w)
grad_g = sym.Rational(1,25) * (2 * w**3 + w + 5)#obtained by hand cal
c
#obtain lambda functions from symbolic equations
lam_g = sym.lambdify(w, g)
lam_grad_g = sym.lambdify(w, grad_g)
#define init condition and alpha sizes
w_naught = 2
alpha_vals = [1, 10**-1, 10**-2]

#define our cost function
epsilon = 10**-10
#returns true if stationary point reached
def cost(grad_g):
    return (grad_g < epsilon)
```

In [112]:
```
#define function that we will use for gradient descent
def grad_desc(g, grad_g, w_last, alpha):
    num_its = 1000
    cost_vals = np.zeros(num_its)
    grad_g_last = grad_g(w_last)
    cost_vals[0] = g(w_last)
    for x in range(1, num_its):
        w_next = w_last - alpha * grad_g(w_last)
        cost_vals[x] = g(w_next)
        w_last = w_next
    return cost_vals

x_vals = np.arange(1000)
```

In [129]:
```
plt.figure()
strings = []
for alph in alpha_vals:
    cost_vals = grad_desc(lam_g, lam_grad_g, w_naught, alph)
    plt.plot(x_vals, cost_vals)
    strings.append("alpha = {}".format(alph))
plt.xlabel("iteration count")
plt.ylabel("cost function g(w)")
plt.title("Gradient descent with varying alpha")
plt.legend(strings)
plt.show()
```



Probelm 3.6

In [15]:
```
import sympy as sym
import numpy as np
w = sym.symbols('w')
g = sym.lambdify(w, sym.Abs(w))
num_its = 20
w_naught = 1.75
alpha_fixed = 0.5
#define piecewise gradient function grad_g
grad_g = sym.lambdify(w, sym.Piecewise((1, w > 0), (-1, w <0)))
```

In [16]:

```
#define function that we will use for fixed alpha gradient descent
def grad_desc(g, grad_g, w_last, alpha):
    num_its = 20
    cost_vals = np.zeros(num_its)
    grad_g_last = grad_g(w_last)
    cost_vals[0] = g(w_last)
    for x in range(1, num_its):
        w_next = w_last - alpha * grad_g(w_last)
        cost_vals[x] = g(w_next)
        w_last = w_next
    return cost_vals
```

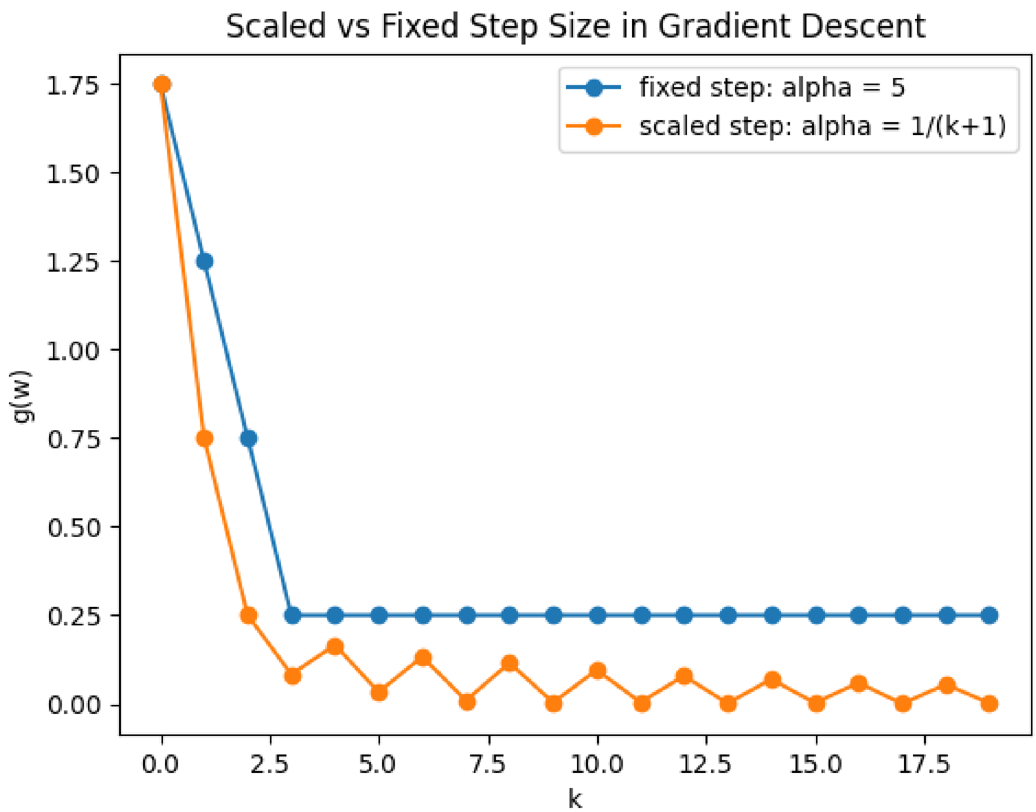
In [17]:

```
#define function that we will use for scaled alpha gradient descent
def grad_desc_scale_alpha(g, grad_g, w_last, num_its):
    alpha = 1/1
    cost_vals = np.zeros(num_its)
    grad_g_last = grad_g(w_last)
    cost_vals[0] = g(w_last)
    for x in range(1, num_its):
        w_next = w_last - alpha * grad_g(w_last)
        alpha = 1/(x+1)
        cost_vals[x] = g(w_next)
        w_last = w_next
    return cost_vals
```

In [18]:

```
cost_fixed_alph = grad_desc(g, grad_g, w_naught, alpha_fixed)
cost_scaled_alph = grad_desc_scale_alpha(g, grad_g, w_naught, 20)
```

In [19]:
```
plt.figure()
x_vals = np.arange(20)
plt.plot(x_vals,cost_fixed_alph, '-o')
plt.plot(x_vals,cost_scaled_alph, '-o')
plt.ylabel('g(w)')
plt.xlabel('k')
plt.title("Scaled vs Fixed Step Size in Gradient Descent")
plt.legend(["fixed step: alpha = 5", "scaled step: alpha = 1/(k+1)"])
plt.show()
```



Probelm 3.8

In [20]:
```
w = sym.MatrixSymbol('w', 10, 1)
g = w.T * w
grad_g = g.diff(w)

g_lam = sym.lambdify([w], g)
grad_g_lam = sym.lambdify([w], grad_g)
```

In [21]:
```
#init
w_naught = np.zeros(10) + 1.0
alpha_vals = [0.001, 0.1, 1]
```

In [22]:
```python
#define function that we will use for fixed alpha gradient descent
def grad_desc(g, grad_g, w_last, alpha):
    num_its = 20
    cost_vals = np.zeros(num_its)
    grad_g_last = grad_g(w_last)
    cost_vals[0] = g(w_last)
    for x in range(1, num_its):
        w_next = w_last - alpha * grad_g(w_last)
        cost_vals[x] = g(w_next)
        w_last = w_next
    return cost_vals
```

In [23]:
```python
plt.figure()
x_vals = np.arange(20);
strings = []
for alph in alpha_vals:
    strings.append("alpha val: {}".format(alph))
    cost_vals = grad_desc(g_lam, grad_g_lam, w_naught, alph)
    plt.plot(x_vals, cost_vals)
plt.legend(strings)
plt.xlabel("iter number, k")
plt.ylabel("g(w)")
plt.title("Gradient descent of g(w) for varying alpha")
plt.show()
```