一. Recursive Algorithm.

eg: 测量长度， 第k次测量结果为 z_k

估计真实数据 ⇒ 取平均

$$\hat{x}_k = \frac{1}{k}(z_1 + z_2 + \dots + z_k)$$

$$= \frac{1}{k}(z_1 + z_2 + \dots + z_{k-1}) + \frac{1}{k} z_k$$

$$= \frac{1}{k} \cdot \frac{k-1}{k-1} \underbrace{(z_1 + \dots + z_{k-1})}_{k-1\text{次均值}} + \frac{1}{k} z_k .$$

$$= \frac{k-1}{k} \hat{x}_{k-1} + \frac{1}{k} z_k$$

$$\Rightarrow \hat{x}_k = \hat{x}_{k-1} + \frac{1}{k}(z_k - x_{k-1})$$

令 $\frac{1}{k} = K_k$

※ $k \to \infty \Rightarrow \frac{1}{k} \to 0 \Rightarrow \hat{x}_k \to \hat{x}_{k-1}$

k个，测量结果不再重要.

$$\Rightarrow \hat{x}_k = \hat{x}_{k-1} + K_k (z_k - \hat{x}_{k-1})$$

K_k → Kalman增益.

当前估计 = 上次估计 + 系数 (当前测量 - 上次估计)

引入：

估计误差: e_{EST} 测量误差 e_{MEA}.

则 $$K_k = \frac{e_{EST_{k-1}}}{e_{EST_{k-1}} + e_{MEA_k}}$$

※ 讨论: 在k时刻

① 若 $e_{EST_{k-1}} >> e_{MEA_k}$, $K_k \to 1 \Rightarrow \hat{x}_k = z_k$.

② 若 $e_{EST_{k-1}} >> e_{MEA_k}$, $K_k \to 0 \Rightarrow \hat{x}_k = \hat{x}_{k-1}$

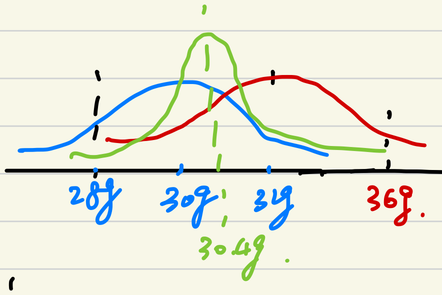
Step 1: 计算 Kalman 增益.

Step 2: 计算 $\hat{x}_k$

Step 3: 更新 $e_{EST_k} = (1 - K_k) e_{EST_{k-1}}$

二. Data Fusion, Cov. Matrix, State Space, Observation

1. Data Fusion.



eg: $z_1 = 30g$ $\quad \sigma_1 = 2g$

$z_2 = 32g$ $\quad \sigma_2 = 4g$, $\quad$ both $\sim N(\mu, \sigma)$,

$\hat{z} = z_1 + k(z_2 - z_1)$, $\quad k \in [0,1]$: $\begin{cases} k=0, & \hat{z} = z_1 \\ k=1, & \hat{z} = z_2. \end{cases}$, z_1, z_2 indep.

求 k s.t. $\min\{\sigma_{\hat{z}}\}$.

$$Var(\hat{z}) = Var(z_1 + k(z_2 - z_1)) = Var(z_1 - kz_1 + kz_2) = Var((1-k)z_1 + kz_2)$$

$$= (1-k)^2 Var(z_1) + k^2 Var(z_2) = (1-k)^2 \sigma_1^2 + k^2 \sigma_2^2$$

$$\frac{d\,Var(\hat{z})}{dk} = 0 \Rightarrow -2\sigma_1^2(1-k) + 2\sigma_2^2 k = 0, \text{ solved } k = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

代入可得: $k = \dfrac{2^2}{2^2+4^2} = 0.2$.

$$\Rightarrow \hat{z} = z_1 + k(z_2 - z_1) = 30 + 0.2(32-30) = \boxed{30.4g}.$$

and $Var(\hat{z}) = (1-0.2)^2 2^2 + 0.2^2 4^2 = 3.2$, $\quad \sigma_{\hat{z}} = 1.79$.

2. Cov. Matrix.

Var. 与 Cov. 在一个矩阵中表现出来.

eg.

x 身高	y 体重	z 年龄.
179	74	33
187	80	31
175	71	28
Avg. 180.3	75	30.7.

$Var(X) = 24.89$. $\quad Var(Y) = 14$. $\quad Var(Z) = 4.22$.

$Cov(X,Y) = \sigma_X \sigma_Y = 18.7$, $\quad Cov(X,Z) = 4.4$, $\quad Cov(Y,Z) = 3.3$.

定义 Cov. Matrix $P = \begin{pmatrix} \sigma_X^2 & \sigma_X\sigma_Y & \sigma_X\sigma_Z \\ \sigma_Y\sigma_X & \sigma_Y^2 & \sigma_Y\sigma_Z \\ \sigma_Z\sigma_X & \sigma_Z\sigma_Y & \sigma_Z^2 \end{pmatrix}$ $\quad$ 协方差

对角线为 Var.

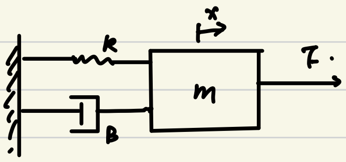
平均值，去均值化.

过滤矩阵： $a = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} - \frac{1}{3}\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}}$

$\Rightarrow p = \frac{1}{3} a^T a. \quad = E[a^T a]$

3. State Space.

eg:



u: input.

$$m\ddot{x} + B\dot{x} + kx = F.$$

$$\Rightarrow m\ddot{x} = u - B\dot{x} - kx.$$

State: $\begin{cases} x_1 = x \\ x_2 = \dot{x}_1 \end{cases}$ $\quad \dot{x}_2 = \ddot{x} = \frac{u}{m} - \frac{B}{m}x_2 - \frac{k}{m}x_1$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{B}{m} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u. \qquad \dot{X}(t) = AX(t) + Bu(t)$$

Measurement

$z_1 = x_1 = x$, position.

$z_2 = x_2 = \dot{x}_1$ velocity.

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad Z(t) = HX(t)$$

离散：后项差分.

$$\begin{cases} X_k = AX_{k-1} + BU_{k-1} \\ Z_k = HX_k \end{cases}$$

引入不确定性： $\begin{cases} W_{k-1} : \text{process noise} \\ V_k : \text{measurement noise.} \end{cases}$

$$\begin{cases} X_k = AX_{k-1} + BU_{k-1} + W_{k-1}. \\ Z_k = HX_k + V_k. \end{cases}$$

问题：估计 $\hat{X}_k = ?$ $\Rightarrow$ Kalman Filter.

三. Kalman Gain.

$$\begin{cases} X_k = AX_{k-1} + BU_{k-1} + \underbrace{w_{k-1}}. & \text{假设其 } P(w) \sim N(0, \underset{\text{协方差阵}}{Q}), \text{ where } Q = \mathbb{E}[ww^T] \\ z_k = HX_k + \underbrace{v_k} & \text{满足 } P(v) \sim (0, \underset{\text{协方差矩阵}}{R}), \text{ where } R = \mathbb{E}[vv^T] \end{cases}$$

先验估计

$$\begin{cases} \hat{x}_k^- = Ax_{k-1} + Bu_{k-1} \quad ① & \text{算得} \\ \hat{z}_k = Hx_k \rightarrow \hat{x}_{k,mea} = H^{-1}z_k \quad ② & \text{测得.} \end{cases} \Rightarrow \text{不具备噪声.}$$

data fusion ⇒

$$\hat{x}_k = \hat{x}_k^- + G(H^{-1}z_k - \hat{x}_k^-) \quad \begin{cases} \hat{x}_k = \hat{x}_k^-, & G=0 \\ \hat{x}_k = H^{-1}z_k, & G=1 \end{cases}$$

定义 $G = K_kH$, ⇒ $\underset{\text{后验估计.}}{\hat{x}_k} = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$, where $K_k \in [0, H^{-1}]$ $\begin{cases} \hat{x}_k = \hat{x}_k^- \\ \hat{x}_k = H^{-1}z_k \end{cases}$

目标: 寻找 K_k, s.t. $\hat{x}_k \rightarrow x_k$.

引入误差 $e_k = x_k - \hat{x}_k$, $P(e_k) \sim N(0, P)$, where $P = \mathbb{E}[ee^T]$

*目标: $\min\{\text{trace}(P)\}$. 迹最小.

$$\Rightarrow P_k = \mathbb{E}[ee^T] = \mathbb{E}[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T]$$

$$\Rightarrow x_k - \hat{x}_k = x_k - (\hat{x}_k^- + K_k(z_k - H\hat{x}_k^-))$$

$$= x_k - \hat{x}_k^- - K_k\overset{= Hx_k + v_k}{z_k} + K_kH\hat{x}_k^-$$

$$= x_k - \hat{x}_k^- - K_kHx_k - K_kv_k + K_kH\hat{x}_k^-$$

$$= (x_k - \hat{x}_k^-) - K_kH(x_k - \hat{x}_k^-) - K_kv_k$$

$$= (I - K_kH)\underbrace{(x_k - \hat{x}_k^-)}_{= e_k^-} - K_kv_k$$

$$\Rightarrow P_k = \mathbb{E}[((I - K_kH)e_k^- - K_kv_k)\underbrace{((I-K_kH)e_k^- - K_kv_k)^T}_{= e_k^{-T}(I-K_kH)^T - v_k^TK_k^T}]$$

$$= \mathbb{E}[(I-K_kH)e_k^-e_k^{-T}(I-K_kH)^T - \underbrace{(I-K_kH)e_k^-v_k^TK_k^T}_{\downarrow} - \underbrace{K_kv_ke_k^{-T}(I-K_kH)^T}_{= 0.} + K_kv_kv_k^TK_k^T]$$

$$(I-K_kH)\mathbb{E}(e_k^-v_k^T)K_k^T$$

$$= (I-K_kH)\mathbb{E}(e_k^-)\mathbb{E}(v_k^T)K_k^T = 0$$

$$P_k^- $$

$$= (I-K_kH)\,\underbrace{E(e_k^- e_k^{-T})}_{}\,(I-K_kH)^T + K_k E(v_k v_k^T) K_k^T$$

$$= (P_k^- - K_k H P_k^-)(I - H^T K_k^T) + K_k R K_k^T$$

$$= P_k^- - K_k H P_k^- - \underbrace{P_k^- H^T K_k^T}_{= K_k H P_k^{-T}} + K_k H P_k^- H^T K_k^T + K_k R K_k^T$$

状态更新.

$$\Rightarrow tr(P_k) = tr(P_k^-) - 2tr(K_k H P_k^-) + tr(\underset{A}{K_k}\,\underset{B}{H P_k^- H^T}\,\underset{A^T}{K_k^T}) + tr(K_k \underset{对称}{R} K_k^T)$$

对称 对称

$$\frac{\partial tr(P_k)}{\partial k_k} = 0 - 2(HP_k^-)^T + 2K_k H P_k^- H^T + 2K_k R = 0$$

$$\Rightarrow P_k^{-T} H^T + K_k (H P_k^- H^T + R) = 0$$

$$\Rightarrow K_k (H P_k^- H^T + R) = P_k^- H^T.$$

$$\Rightarrow K_k = \frac{P_k^- H^T}{H P_k^- H^T + R}.$$

Kalman Gain

$R\uparrow \Rightarrow K_k \to 0,\quad R\downarrow \Rightarrow K_k = H^{-1} \Rightarrow \hat{x}_k = H^{-1} z_k.$

四. Priori / Posteriori Error Cov. Matrix.

$$\begin{cases} x_k = A x_{k-1} + B u_{k-1} + w_{k-1}, & w \sim N(0, Q) \\ z_k = H x_k + v_k, & v \sim N(0, R). \end{cases}$$

先验估计.

$$\hat{x}_k^- = A \hat{x}_{k-1} + B u_{k-1}$$

已知.

后验估计

$$\hat{x}_k = \hat{x}_k^- + \boxed{K_k}(z_k - H\hat{x}_k^-)$$

Kalman Gain ?

$$K_k = \frac{P_k^- H^T}{H P_k^- H^T + R}.$$

求 P_k^-.

$$P_k^- = E[e_k^- e_k^{-T}], \quad \text{where } \boxed{e_k^- = x_k - \hat{x}_k^-}$$

$$= A x_{k-1} + \cancel{B u_{k-1}} + w_{k-1} - A\hat{x}_{k-1} - \cancel{B u_{k-1}}$$

$$= A(x_{k-1} - \hat{x}_{k-1}) + w_{k-1} = A e_{k-1} + w_{k-1}$$

$$\Rightarrow P_k^- = E[(Ae_{k-1}+w_{k-1})(e_{k-1}^T A^T + w_{k-1}^T)]$$

indep. $\to 0$ $\quad \to 0.$

$$= E[Ae_{k-1}e_{k-1}^T A^T + Ae_{k-1}w_{k-1}^T + w_{k-1}e_{k-1}^T A^T + w_{k-1}w_{k-1}^T]$$

上次. 本次, $E[e_{k-1}]=0$, $E[w_{k-1}^T]=0$

$$= AE[e_{k-1}e_{k-1}^T]A^T + E[w_{k-1}w_{k-1}^T]$$

$$= AP_{k-1}A^T + Q.$$

预测:

先验: $\hat{x}_k^- = Ax_{k-1} + Bu_{k-1}$ ①

先验误差协方差: $P_k^- = AP_{k-1}A^T + Q.$ ②

校正 Kalman Gain: $K_k = \dfrac{P_k^- H}{HP_k^- H^T + R}$ ③

后验估计: $\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$ ④

更新协方差:

$$P_k = P_k^- - K_kHP_k^- - P_k^-H^TK_k^T + \underbrace{K_kHP_k^-H^TK_k^T + K_kRK_k^T}_{代入K_k.}$$

$$= P_k^- - K_kHP_k^- - \cancel{P_k^-H^TK_k^T} + \cancel{P_k^-H^TK_k^T}$$

$$= P_k^- - K_kHP_k^- = (I - K_kH)P_k^-$$ ⑤

Recursive.

五. Extended KF. (非线性 → 线性)

Non-linear.

$$\begin{cases} x_k = f(x_{k-1}, u_{k-1}, w_{k-1}) & W \sim N(0, Q) \\ z_k = h(x_k, v_k) & V \sim N(0, R) \end{cases}$$

ND 的随机变量通过 NLS 后就不再是 ND 了.

线性化: Taylor Series.

$$f(x) = f(x_0) + \frac{\partial f}{\partial x}(x - x_0).$$

系统有误差, 无法在真实点处线性化.

k-1 时的后验估计

$f(x_k)$ 在 $\hat{x}_{k-1}$ 处线性化. z_k 在 $\tilde{x}_k$ 处线性化.

误差假设为0. $f(\hat{x}_{k-1}, u_{k-1}, 0) = \tilde{x}_k$

$$\Rightarrow \begin{cases} x_k = f(\hat{x}_{k-1}, u_{k-1}, w_{k-1}) + A(x_{k-1} - \hat{x}_{k-1}) + W_k w_{k-1} \\ z_k = \underbrace{h(\tilde{x}_k, \cancel{v_k})}_{=\tilde{z}_k} + H(x_k - \tilde{x}_k) + V v_k \end{cases}$$

Jacobian, $A = \dfrac{\partial f}{\partial x}\Big|_{\hat{x}_{k-1}, u_{k-1}}$ $\quad W_k = \dfrac{\partial f}{\partial w}\Big|_{\hat{x}_{k-1}, u_{k-1}}$

$H = \dfrac{\partial h}{\partial x}\Big|_{\tilde{x}_k}$ $\quad V = \dfrac{\partial h}{\partial v}\Big|_{\tilde{x}_k}$

eg:

$$\begin{cases} x_1 = x_1 + \sin x_2 & = f_1 \\ x_2 = x_1^2 & = f_2 \end{cases}$$

$$\Rightarrow A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & \cos x_2 \\ 2x_1 & 0 \end{bmatrix} \Bigg|_{\substack{\hat{x}_{1\,k-1} \\ \hat{x}_{2\,k-1}}} = \begin{bmatrix} 1 & \cos \hat{x}_{2\,k-1} \\ 2\hat{x}_{1\,k-1} & 0 \end{bmatrix} = A_k.$$

$$\Rightarrow \begin{cases} x_k = \tilde{x}_k + A(x_{k-1} - \hat{x}_{k-1}) + W w_{k-1} & W w_{k-1} \sim N(0, WQW^T) \\ z_k = \tilde{z}_k + H(x_k - \tilde{x}_k) + V v_k & V v_k \sim N(0, VRV^T) \end{cases}$$

先验：

$$\hat{x}_k^- = f(\hat{x}_{k-1}, u_{k-1}, 0)$$

$$P_k^- = A P_{k-1} A^T + WQW^T$$

$$K_k = \frac{P_k^- H^T}{H P_k^- H^T + VRV^T}$$

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - h(\hat{x}_k^-, 0))$$

$$P_k = (I - K_k H) P_k^- .$$