Just for fun

transport_equation

Equation $\phi_t(x,t) + \theta(x) \cdot \nabla_x \phi(x,t) = f(x,t)$. Applying Backward Euler method,

$$\frac{\phi(x,t+\delta t) - \phi(x,t)}{\delta t} + \theta(x) \cdot \nabla_x \phi(x,t+\delta t) = f(x,t+\delta t).$$

The weak formulation reads

$$\int_\Omega \phi(x,t+\delta t)\psi(x) + \delta t \theta(x) \cdot \nabla_x \phi(x,t+\delta t)\psi(x)\,\mathrm{d}x = \int_\Omega (\phi(x,t) + \delta t f(x,t+\delta t))\psi(x)\,\mathrm{d}x$$

modified_transport_eq

Equation $u_t + V \cdot \nabla u = f$. Change of variable $w = \exp(-\lambda t) u$. Then $u = \exp(\lambda t) w$ and

$$\exp(\lambda t) w_t + \lambda \exp(\lambda t) w + \exp(\lambda t) V \cdot \nabla w = f,$$

$$w_t + V \cdot \nabla w + \lambda w = \exp(-\lambda t) f.$$

Applying the θ-method to

$$w_t = \exp(-\lambda t) f - V \cdot \nabla w - \lambda w$$

we have

$$\begin{aligned}\frac{w_{j+1} - w_j}{\Delta t} &= \theta \exp(-\lambda t_{j+1}) f_{j+1} + (1-\theta)\exp(-\lambda t_j) f_j \\ &\quad - \theta V \cdot \nabla w_{j+1} - (1-\theta) V \cdot \nabla w_j \\ &\quad - \theta \lambda w_{j+1} - (1-\theta)\lambda w_j\end{aligned}$$

that is

$$\begin{aligned}0 =& w_{j+1} - w_j \\ & - \Delta t \theta \exp(-\lambda t_{j+1}) f_{j+1} - \Delta t (1-\theta) \exp(-\lambda t_j) f_j \\ & + \Delta t \theta V \cdot \nabla w_{j+1} + \Delta t (1-\theta) V \cdot \nabla w_j \\ & + \Delta t \theta \lambda w_{j+1} + \Delta t (1-\theta) \lambda w_j\end{aligned}$$

If $\theta = 0$, then

$$w_{j+1} = w_j (1 - \Delta t \lambda) + \Delta t \exp(-\lambda t_j) f_j - \Delta t V \cdot \nabla w_j.$$

If $\theta = 1$, then

$$w_{j+1}(1 + \Delta t \lambda) + \Delta t V \cdot \nabla w_{j+1} = w_j + \Delta t \exp(-\lambda t_{j+1}) f_{j+1}.$$

If $\theta = \frac{1}{2}$, then

$$w_{j+1}\left(1+\frac{\Delta t}{2}\lambda\right)+\frac{\Delta t}{2}V\cdot\nabla w_{j+1}=w_j\left(1-\frac{\Delta t}{2}\lambda\right)-\frac{\Delta t}{2}V\cdot\nabla w_j+\frac{\Delta t}{2}\left(\exp\left(-\lambda t_{j+1}\right)f_{j+1}+\exp\left(-\lambda t_j\right)f_j\right)$$