Just for fun

eit_continuous_md

Usual weak formulation: $u \in H^1(\Omega)$ with $\int_{\partial\Omega} \gamma u \,\mathrm{ds} = 0$ such that

$$\int_\Omega \sigma \nabla u \cdot \nabla v \,\mathrm{dx} = \int_\Omega S v \,\mathrm{dx} + \langle f, \gamma v \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \quad \forall\, v \in H^1(\Omega)$$

Multiplier formulation: find $(u, \lambda) \in H^1(\Omega) \times \mathbb{R}$ such that

$$\begin{aligned} \int_\Omega \sigma \nabla u \cdot \nabla v \,\mathrm{dx} + \lambda \int_{\partial\Omega} v \,\mathrm{ds} &= \int_\Omega S v \,\mathrm{dx} + \langle f, \gamma v \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} && \forall\, v \in H^1(\Omega) \\ \mu \int_{\partial\Omega} \gamma u \,\mathrm{ds} &= 0 && \forall\, \mu \in \mathbb{R} \end{aligned}$$

These equations are the optimality conditions of the problem

$$\min_{(u,\lambda) \in H^1(\Omega) \times \mathbb{R}} \frac{1}{2} \int_\Omega \sigma \left|\nabla u\right|^2 \mathrm{dx} - \int_\Omega S u \,\mathrm{dx} - \langle f, \gamma u \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} + \lambda \int_{\partial\Omega} \gamma u \,\mathrm{ds}$$

transport_equation

Equation $\phi_t(x, t) + \theta(x) \cdot \nabla_x \phi(x, t) = f(x, t)$. Applying Backward Euler method,

$$\frac{\phi(x, t + \delta t) - \phi(x, t)}{\delta t} + \theta(x) \cdot \nabla_x \phi(x, t + \delta t) = f(x, t + \delta t).$$

The weak formulation reads

$$\int_\Omega \phi(x, t + \delta t) \psi(x) + \delta t \theta(x) \cdot \nabla_x \phi(x, t + \delta t) \psi(x) \,\mathrm{dx} = \int_\Omega (\phi(x, t) + \delta t f(x, t + \delta t)) \psi(x) \,\mathrm{dx}$$

modified_transport_eq

Equation $u_t + V \cdot \nabla u = f$. Change of variable $w = \exp(-\lambda t) u$. Then $u = \exp(\lambda t) w$ and

$$\exp(\lambda t) w_t + \lambda \exp(\lambda t) w + \exp(\lambda t) V \cdot \nabla w = f,$$

$$w_t + V \cdot \nabla w + \lambda w = \exp(-\lambda t) f.$$

Applying the θ-method to

$$w_t = \exp(-\lambda t) f - V \cdot \nabla w - \lambda w$$

we have

$$\begin{aligned} \frac{w_{j+1} - w_j}{\Delta t} &= \theta \exp(-\lambda t_{j+1}) f_{j+1} + (1 - \theta) \exp(-\lambda t_j) f_j \\ &\quad - \theta V \cdot \nabla w_{j+1} - (1 - \theta) V \cdot \nabla w_j \\ &\quad - \theta \lambda w_{j+1} - (1 - \theta) \lambda w_j \end{aligned}$$

that is

$$\begin{aligned} 0 = &w_{j+1} - w_j \\ &- \Delta t \theta \exp(-\lambda t_{j+1}) f_{j+1} - \Delta t (1 - \theta) \exp(-\lambda t_j) f_j \\ &+ \Delta t \theta V \cdot \nabla w_{j+1} + \Delta t (1 - \theta) V \cdot \nabla w_j \\ &+ \Delta t \theta \lambda w_{j+1} + \Delta t (1 - \theta) \lambda w_j \end{aligned}$$

If $\theta = 0$, then

$$w_{j+1} = w_j \left(1 - \Delta t \lambda\right) + \Delta t \exp\left(-\lambda t_j\right) f_j - \Delta t V \cdot \nabla w_j.$$

If $\theta = 1$, then

$$w_{j+1} \left(1 + \Delta t \lambda\right) + \Delta t V \cdot \nabla w_{j+1} = w_j + \Delta t \exp\left(-\lambda t_{j+1}\right) f_{j+1}.$$

If $\theta = \frac{1}{2}$, then

$$w_{j+1} \left(1 + \frac{\Delta t}{2} \lambda\right) + \frac{\Delta t}{2} V \cdot \nabla w_{j+1} = w_j \left(1 - \frac{\Delta t}{2} \lambda\right) - \frac{\Delta t}{2} V \cdot \nabla w_j + \frac{\Delta t}{2} \left(\exp\left(-\lambda t_{j+1}\right) f_{j+1} + \exp\left(-\lambda t_j\right) f_j\right)$$