1 Notation

Let $d \geq 2$.
The zero vector of $\mathbb{R}^d$ is denoted by $\vec{0}$.
The unit sphere $\left\{x \in \mathbb{R}^d \mid \|x\| = 1\right\}$ is denoted by S^{d-1}.
The closed ball $\left\{x \in \mathbb{R}^d \mid \|x\| \leq R\right\}$ is denoted by $B\left(\vec{0}, R\right)$.
Given a finite set of indices $\mathcal{K}$, $|\mathcal{K}|$ denotes the cardinality of $\mathcal{K}$.

2 Polyhedrals and polytopes

Let us start with some definitions. Our reference is [1].

Definition 2.1 (Affine subspace)**.** An *affine subspace* of $\mathbb{R}^d$ is either the empty set or a translate of a linear subspace, that is, a subset $x + L$ where $x \in \mathbb{R}^d$ and L is a linear subspace of $\mathbb{R}^d$.

Definition 2.2 (*Dimension of a* affine subspace)**.** The dimension of a non-empty affine subspace A is the dimension of the linear subspace L such that $A = x + L$. In other words,

$$\dim A \coloneqq \dim L.$$

Definition 2.3 (*Affine hull*)**.** For any subset M of $\mathbb{R}^d$, there is a smallest affine subspace containing M, namely, the intersection of all affine subspaces containing M. This affine subspace is called the *affine hull* of M, and it is denoted by

$$\operatorname{aff} M.$$

Definition 2.4 (*Dimension of a convex set*)**.** Let C be a convex set of $\mathbb{R}^d$. The dimension of C is defined as the dimension of the affine hull of C, that is

$$\dim C \coloneqq \dim\left(\operatorname{aff} C\right).$$

Definition 2.5 (*Convex hull*)**.** For any subset M of $\mathbb{R}^d$, there is a smallest convex set containing M, namely, the intersection of all convex sets containing M. This convex set is called the *convex hull* of M, and it is denoted by

$$\operatorname{conv} M.$$

Definition 2.6 (Supporting hyperplane)**.** Let C be a non-empty closed convex set in $\mathbb{R}^d$. By a *supporting hyperplane* of C we mean a hyperplane

$$\left\{x \in \mathbb{R}^d \mid x \cdot n = a\right\}, \quad n \in \mathbb{R}^d \backslash \left\{\vec{0}\right\}$$

uch that such that $C \subset \left\{x \in \mathbb{R}^d \mid x \cdot n \leq a\right\}$ and

$$\left\{x \in \mathbb{R}^d \mid x \cdot n = a\right\} \cap C \neq \emptyset.$$

Definition 2.7 (Face)**.** Let C be a non-empty closed convex set in $\mathbb{R}^d$. A convex subset F ($\neq C$ or $\emptyset$) of C is called a *k-face* of C if for any distinct points $y, z \in C$ such that $\left\{x \in \mathbb{R}^d \mid \lambda x + (1-\lambda) z, \lambda \in (0,1)\right\} \cap F \neq \emptyset$, we actually have

$$\left\{x \in \mathbb{R}^d \mid \lambda x + (1-\lambda) z, \lambda \in [0,1]\right\} \subset F.$$

A face F of C is called k*-face* if $\dim F = k$. Every face is closed.

Definition 2.8 (Extreme point)**.** Let C be a non-empty closed convex set of $\mathbb{R}^d$. A point $x \in C$ is called an *extreme point* of C if $\{x\}$ is a face. The set of extreme points of C is denoted by $\operatorname{ext} C$.

Definition 2.9 (Exposed face)**.** Let C be a non-empty closed convex set of $\mathbb{R}^d$. A face F of C of the form

$$F = \left\{x \in \mathbb{R}^d \mid x \cdot n = a\right\} \cap C,$$

where $\left\{x \in \mathbb{R}^d \mid x \cdot n = a\right\}$ is a supporting hyperplane of C, is called *exposed face.*

Here an important result.

Theorem 2.10 (Minkowski's Theorem [1, Thm. 5.10])**.** *Let C be a non-empty compact convex set in $\mathbb{R}^d$. Then*

$$C = \operatorname{conv}(\operatorname{ext} C).$$

Next the definition and properties of a **polyhedral set**.

Definition 2.11 (Polyhedral set)**.** A subset P of $\mathbb{R}^d$ is called *polyhedral set* if Q is the intersection of a finite number of closed half-spaces or $P = \mathbb{R}^d$. Every polyhedral set is closed and convex. A polyhedral set $P \neq \mathbb{R}^d$ has the form

$$P = \bigcap_{k \in \mathcal{K}} \left\{x \in \mathbb{R}^d \mid x \cdot n_k \leq a_k\right\}, \tag{1}$$

where $\mathcal{K}$ is a finite set of indices, $n_k \in \mathbb{R}^d \backslash \left\{\vec{0}\right\}$ and $a_k \in \mathbb{R}$. In (1), it is implicitly assumed that no two closed half-spaces are identical.

Definition 2.12. The representation (1) of P is called *irreducible* if $|\mathcal{K}| = 1$, or $|\mathcal{K}| > 1$ and

$$P \subsetneq \bigcap_{k \in \mathcal{K} \backslash \{j\}} \left\{x \in \mathbb{R}^d \mid x \cdot n_k \leq a_k\right\} \quad \forall\, j \in \mathcal{K}.$$

A representation which is not irreducible is called *reducible.*

Theorem 2.13 ([1, Thm. 8.1])**.** *Let P be a polyhedral set in $\mathbb{R}^d$ with $\dim P = d$ and $P \neq \mathbb{R}^d$. Suppose that $|\mathcal{K}| > 1$ in the representation (1) of P. Then (1) is irreducible if and only if*

$$\left\{x \in \mathbb{R}^d \mid x \cdot n_j = a_j\right\} \cap \bigcap_{k \in \mathcal{K} \backslash \{j\}} \left\{x \in \mathbb{R}^d \mid x \cdot n_k < a_k\right\} \neq \emptyset \quad \forall\, j \in \mathcal{K}.$$

Theorem 2.14 ([1, Thm. 8.2])**.** *Let P be a polyhedral set in $\mathbb{R}^d$ with $\dim P = d$ and $P \neq \mathbb{R}^d$. Then*

$$\partial P = \bigcup_{k \in \mathcal{K}} \left\{ x \in \mathbb{R}^d \mid x \cdot n_k = a_k \right\} \cap P,$$

each $(d-1)$-face of P has the form

$$\left\{ x \in \mathbb{R}^d \,|\, x \cdot n_j = a_j \right\} \cap P$$

(this implies that the number of $(d-1)$-faces of P is finite), and each set

$$\left\{ x \in \mathbb{R}^d \,|\, x \cdot n_j = a_j \right\} \cap P$$

is a $(d-1)$-face if and only if the representation (1) of P is irreducible.

Theorem 2.15 ([1, Thm. 8.3])**.** *Let P be a polyhedral set in $\mathbb{R}^d$ and let F be a face of P. Then there is a $(d-1)$-face of P containing F.*

Next the definition and properties of a **polytope**.

Definition 2.16 (Polytope)**.** A polytope is the convex hull of a non-empty finite set of $\mathbb{R}^d$. Every polytope is compact.

Theorem 2.17 ([1, Thm. 7.1])**.** *Let P be a non-empty subset in $\mathbb{R}^d$. P is a polytope if and only if P is a compact convex set with a finite number of extreme points.*

Theorem 2.18. *Let P be a polytope in $\mathbb{R}^d$. Then*

- $P = \operatorname{conv}(\operatorname{ext} P)$*;*
- *every face F of P is also a polytope, and* $\operatorname{ext} F = F \cap \operatorname{ext} P$*;*
- *the number of faces of P is finite;*
- *every face of P is an exposed face.*

Theorem 2.19 ([1, Thm. 10.5])**.** *Let P be a polytope in $\mathbb{R}^d$ such that $\dim P = d$ and let x be a extreme point of P. Then there are at least d 1-faces of P containing x.*

Definition 2.20 (Simple polytope)**.** Let P be a polytope in $\mathbb{R}^d$ such that $\dim P = d$. P is called a *simple* if for $k = 0, \ldots, d-1$ the number of $(d-1)$-faces of P containing any k-face of P is equal to $d - k$.

Theorem 2.21 ([1, Thm. 12.11])**.** *Let P be a polytope in $\mathbb{R}^d$ such that $\dim P = d$. P is simple if and only if each extreme point of P is contained in precisely d $(d-1)$-faces.*

Theorem 2.22 ([1, Thm. 12.12])**.** *Let P be a polytope in $\mathbb{R}^d$ such that $\dim P = d$. P is simple if and only if each extreme point of P is a extreme point of precisely d 1-faces.*

Theorem 2.23 ([1, Thm. 12.14]). *Let P be a polytope in $\mathbb{R}^d$ such that* $\dim P = d$. *Suppose that P is simple. Let $F_1, \ldots, F_{d-k}$ be $d-k$ $(d-1)$-faces of P, where $0 \leq k \leq d-1$. Let*

$$F = \bigcap_{i=1}^{d-k} F_i$$

and assume that $F \neq \emptyset$. Then F is a k-face of P, and $F_1, \ldots, F_{d-k}$ are the only facets of P containg F.

Here the result that connects polyhedral sets and polytopes.

Theorem 2.24 ([1, Thm. 9.2]). *Let P be a non-empty subset of $\mathbb{R}^d$. P is a polytope if and only if P is a bounded polyhedral set.*

Notation. Let C be a non-empty closed convex set in $\mathbb{R}^d$. From now on, the 0-faces, 1-faces, and $(d-1)$-faces are called vertices, edges, and facets of C, respectively.

3 Barycentric coordinates

Let P be a non-empty bounded polyehedral set, or equivalently a polytope, of dimension d. Assume that P is simple. Let $\mathcal{V}$ be the set of vertices P. From [5]§[6]§, there exist a set of smooth functions $b_v : P \to \mathbb{R}$, with $v \in \mathcal{V}$, called barycentric coordinates with respect to P, satisfying

$$b_v(x) \geq 0 \quad \forall\, x \in P \,\text{and}\, \forall\, v \in \mathcal{V}, \tag{2}$$

$$\sum_{v \in \mathcal{V}} b_v(x) = 1 \quad \forall\, x \in P, \tag{3}$$

$$\sum_{v \in \mathcal{V}} b_v(x)\, v = x \quad \forall\, x \in P. \tag{4}$$

Some properties of these functions are the following.

1. Let l be an affine function. Then $l(x) = \sum_{v \in \mathcal{V}} b_v(x)\, l(v)$ for all $x \in P$.

2. Let $v \in \mathcal{V}$ be a vertex of P. Then $b_v(v) = 1$ and $b_v(w) = 0 \;\forall\, w \in \mathcal{V} \setminus \{v\}$. Moreover, $\forall\, x \in P \setminus \{v\} \;\exists\, w \in \mathcal{V} \setminus \{v\}$ such that $b_w(x) > 0$.

   Proof. Let $l : \mathbb{R}^3 \to \mathbb{R}$ be affine function such that $l(v) = 0$ and $l(w) < 0$ for all $w \in \mathcal{V} \setminus \{v\}$. Thus, $l^{-1}(\{0\})$ defines a plane that touches P only at v. We have

   $$0 = l(v) = \sum_{w \in \mathcal{V}} b_w(v)\, l(w) = \sum_{w \in \mathcal{V} \setminus \{v\}} b_w(v)\, l(w)$$

because $v \in l^{-1}(\{0\})$. Now each $b_w(v) \geq 0$ and $l(w) < 0$. Therfore $b_w(v) = 0$ for all $w \in \mathcal{V} \backslash \{v\}$. Since the coordinate functions sum 1, $b_v(v) = 1$. Let $x \in P \backslash \{v\}$. Then

$$0 > l(x) = \sum_{w \in \mathcal{V} \backslash \{v\}} b_w(x)\, l(w).$$

If $b_w(x) = 0$ for all $w \in \mathcal{V} \backslash \{v\}$, then $l(x) = 0$.

3. Let e be a edge of P and v, w be its vertices ($\overline{e} = e \cup \{v, w\}$). Then $b_v(x) + b_w(x) = 1$ and $b_u(x) = 0$ for all $u \in \mathcal{V} \backslash \{v, w\}$, for all $x \in e$. Moreover, for all $x \in P \backslash \overline{e}$ there exists at least one vertex $u \in \mathcal{V} \backslash \{v, w\}$ such that $b_u(x) > 0$. Also

$$b_v(x) = \frac{\langle x - w, v - w \rangle}{|\overline{e}|^2}, \quad b_w(x) = \frac{\langle x - v, w - v \rangle}{|\overline{e}|^2}.$$

Therefore, $b_v(x), b_w(x) > 0$ for all $x \in e$.

Proof. Let $l : \mathbb{R}^3 \to \mathbb{R}$ be affine function such that $l(\overline{e}) = 0$ and $l(u) < 0$ for all $u \in \mathcal{V} \backslash \{v, w\}$. Thus, $l^{-1}(\{0\})$ defines a plane that touches P only at $\overline{e}$. We have

$$0 = l(x) = \sum_{u \in \mathcal{V}} b_u(x)\, l(u) = \sum_{u \in \mathcal{V} \backslash \{v, w\}} b_w(x)\, l(u) \quad \forall x \in e$$

because $\{v, w\} \subset \overline{e} \subset l^{-1}(\{0\})$. Now each $b_u(x) \geq 0$ and $l(u) < 0$. Therfore $b_u(x) = 0$ for all $u \in \mathcal{V} \backslash \{v, w\}$. Since the coordinate functions sum 1, $b_v(x) + b_w(x) = 1$. Let $x \in P \backslash \overline{e}$. Then

$$0 > l(x) = \sum_{u \in \mathcal{V} \backslash \{v, w\}} b_u(x)\, l(u).$$

If $b_u(x) = 0$ for all $u \in \mathcal{V} \backslash \{v, w\}$, then $l(x) = 0$. Note that

$$x - v = \frac{\|x - v\|}{|\overline{e}|}(w - v), \quad x - w = \frac{\|x - w\|}{|\overline{e}|}(v - w) \qquad \forall x \in e.$$

Since $x = b_v(x)\, v + b_w(x)\, w$ for all $x \in e$,

$$x = b_v(x)\, v + (1 - b_v(x))\, w, \quad x = (1 - b_w(x))\, v + b_w(x)\, w$$

and then

$$\frac{\|x - w\|}{|\overline{e}|}(v - w) = b_v(x)(v - w), \quad \frac{\|x - v\|}{|\overline{e}|}(w - v) = b_w(x)(w - v).$$

From this we deduce that

$$b_v(x) = \frac{\|x - w\|}{|\overline{e}|} = \frac{\|x - w\| \|v - w\|}{|\overline{e}|^2} = \frac{|\langle x - w, v - w \rangle|}{|\overline{e}|^2} = \frac{\langle x - w, v - w \rangle}{|\overline{e}|^2}$$

and

$$b_w(x) = \frac{\|x-v\|}{|\overline{e}|} = \frac{\|x-v\|\,\|w-v\|}{|\overline{e}|^2} = \frac{|\langle x-v, w-v\rangle|}{|\overline{e}|^2} = \frac{\langle x-v, w-v\rangle}{|\overline{e}|^2}.$$

4. Let f be a face of P and $\mathcal{W}$ be the set of its vertices ($\overline{f} = f \cup \mathcal{W} \cup \{\text{edges}\}$). Then $\sum_{w\in\mathcal{W}} b_w(x) = 1$ and $b_v(x) = 0$ for all $v \in \mathcal{V}\backslash\mathcal{W}$, for all $x \in f$. Moreover, for all $x \in P\backslash\overline{f}$ there exists at least one vertex $v \in \mathcal{V}\backslash\mathcal{W}$ such that $b_v(x) > 0$.

   Proof. Let $l : \mathbb{R}^3 \to \mathbb{R}$ be affine function such that $l\left(\overline{f}\right) = 0$ and $l(v) < 0$ for all $v \in \mathcal{V}\backslash\mathcal{W}$. Thus, $l^{-1}(\{0\})$ defines a plane that touches P only at $\overline{f}$. We have

   $$0 = l(x) = \sum_{v\in\mathcal{V}} b_v(x)\, l(v) = \sum_{v\in\mathcal{V}\backslash\mathcal{W}} b_v(x)\, l(v) \quad \forall x \in f$$

   because $\mathcal{W} \subset \overline{f} \subset l^{-1}(\{0\})$. Now each $b_v(x) \geq 0$ and $l(v) < 0$. Therfore $b_v(x) = 0$ for all $v \in \mathcal{V}\backslash\mathcal{W}$. Since the coordinate functions sum 1, $\sum_{w\in\mathcal{W}} b_w(x) = 1$. Let $x \in P\backslash\overline{f}$. Then

   $$0 > l(x) = \sum_{v\in\mathcal{V}\backslash\mathcal{W}} b_v(x)\, l(v).$$

   If $b_v(x) = 0$ for all $v \in \mathcal{V}\backslash\mathcal{W}$, then $l(x) = 0$.

5. The coordinate functions $\{b_v\}_{v\in\mathcal{V}}$ are linealy independent.

   Proof. Let $\{\alpha_v\}_{v\in\mathcal{V}}$ be real numbers such that

   $$\sum_{v\in\mathcal{V}} \alpha_v b_v(x) = 0 \quad \forall x \in P.$$

   Setting $x = w \in \mathcal{V}$ it follows that $\alpha_w = \alpha_w b_w(w) = 0$ (see property 1.).

4 Polyhedral domain and deformation function

Let $\mathcal{K}$ be a finite set of indices.

Definition 4.1. For each $k \in \mathcal{K}$, let $\widehat{\phi}_k : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\widehat{\phi}_k(x,t) := x \cdot n_k(t) - a_k(t),$$

where $n_k : \mathbb{R} \to S^{d-1}$ and $a_k : \mathbb{R} \to \mathbb{R}$ are C^∞ functions. Denote $n_k := n_k(0)$, $a_k := a_k(0)$ and $\widehat{\phi}_k(\cdot) := \widehat{\phi}_k(\cdot, 0)$.

Definition 4.2. For each $k \in \mathcal{K}$ and $t \in \mathbb{R}$, let $\omega_k(t)$ be the affine set defined by

$$\omega_k(t) := \left\{x \in \mathbb{R}^d \mid \widehat{\phi}_k(x,t) < 0\right\}.$$

Denote $\omega_k := \omega_k(0)$. Observe that $\omega_k(t)$ is one of the two open half-spaces determined by the hyperplane

$$\left\{x \in \mathbb{R}^d \mid x \cdot n_k(t) = a_k(t)\right\}.$$

Definition 4.3. For each $t \in \mathbb{R}$, let Ω_t be the convex set of $\mathbb{R}^d$ defined by

$$\Omega_t := \bigcap_{k \in \mathcal{K}} \omega_k(t).$$

The set Ω_t is open because it is the finite intersection of open half-spaces. Denote

$$\Omega := \Omega_0 = \bigcap_{k \in \mathcal{K}} \omega_k. \tag{5}$$

As

$$\overline{\Omega} = \bigcap_{k \in \mathcal{K}} \overline{\omega_k} = \bigcap_{k \in \mathcal{K}} \left\{x \in \mathbb{R}^d \mid x \cdot n_k \leq a_k\right\}$$

is a finite intersection of closed half-spaces, $\overline{\Omega}$ is a polyhedral set by definition (see Definition 2.11).

Consider the following four assumptions:

A1. $a_k > 0$ for all $k \in \mathcal{K}$. This assumption implies the following:

(a) $\vec{0} \in \Omega$. Indeed $\widehat{\phi}_k\left(\vec{0}, 0\right) = -a_k < 0$ and hence

$$\vec{0} \in \omega_k = \left\{x \in \mathbb{R}^d \mid \widehat{\phi}_k(x,0) < 0\right\} \quad \forall\, k \in \mathcal{K}.$$

Thus $\overline{\Omega}$ is not-empty.

(b) $\dim \overline{\Omega} = d$. Indeed, since $\vec{0}$ is a interior point of $\overline{\Omega}$, there exists a point $y \in \overline{\Omega} \backslash \left\{\vec{0}\right\}$ in the line segment joining $\vec{0}$ and any point $x \in \mathbb{R}^d$. Thus $x \in \operatorname{aff}\left\{\vec{0}, y\right\}$, and hence $x \in \operatorname{aff} \overline{\Omega}$ (because $\operatorname{aff}\left\{\vec{0}, y\right\} \subset \operatorname{aff} \overline{\Omega}$). Therefore $\mathbb{R}^d \subset \operatorname{aff} \overline{\Omega}$. It follows that $\operatorname{aff} \overline{\Omega} = \mathbb{R}^d$ and $\dim\left(\operatorname{aff} \overline{\Omega}\right) = \dim \mathbb{R}^d = d$.

A2. There exists a constant $R > 0$ such that $\overline{\Omega} \subset B(\vec{0}, R)$. This assumption implies the following:

(a) $\overline{\Omega}$ is a polytope. Indeed, as $\overline{\Omega}$ is not-empty polyhedral set, if $\overline{\Omega}$ is bounded, then $\overline{\Omega}$ is a polytope (see Theorem 2.17). Hence, there exists a finite set $\mathcal{V}$ of vertices (0-faces) of $\overline{\Omega}$ and

$$\overline{\Omega} = \operatorname{conv} \mathcal{V}$$

(see Theorems 2.17 and 2.18).

(b) $\partial\Omega = \bigcup_{k\in\mathcal{K}} \partial\omega_k \cap \overline{\Omega}$ (see Theorem 2.14).

(c) Moreover, if $\overline{\Omega}$ is bounded, $|\mathcal{K}|$ must be greater than 1. In fact, since $\overline{\Omega}$ is a polytope and $\dim \overline{\Omega} = d$, $\overline{\Omega}$ has at least $d+1$ $(d-1)$-faces (Corollary 9.4.).

A3. Assume that

$$\left\{ x \in \mathbb{R}^d \mid \widehat{\phi}_j\,(x,0) = 0,\, \widehat{\phi}_k\,(x,0) < 0\; \forall\, k \in \mathcal{K} \backslash \{j\} \right\} \neq \emptyset \quad \forall\, j \in \mathcal{K}.$$

This assumption implies the following:

(a) As $\overline{\Omega}$ is bounded ($\overline{\Omega} \neq \mathbb{R}^d$), $|\mathcal{K}| > 1$, $\dim \overline{\Omega} = d$, this assumption implies that the representation (5) of $\overline{\Omega}$ is irreducible (see Theorem 2.13).

(b) The $(d-1)$-faces of $\overline{\Omega}$ are the sets $\partial\omega_k \cap \overline{\Omega}$, $k \in \mathcal{K}$ (see Theorem 2.14).

Note that

$$\left\{ x \in \mathbb{R}^d \mid \widehat{\phi}_j\,(x,0) = 0,\, \widehat{\phi}_k\,(x,0) < 0\; \forall\, k \in \mathcal{K} \backslash \{j\} \right\} = \partial\omega_j \cap \bigcap_{k\in\mathcal{K}\backslash\{j\}} \omega_k.$$

A4. Each vertex of $\overline{\Omega}$ is contained in precisely d $(d-1)$-faces.

(a) First, recall that for all $v \in \mathcal{V}$ there is $k \in \mathcal{K}$ such that $v \in \partial\omega_k$ (see Theorem 2.15), so, this is a precisely an assumption.

(b) Given $v \in \mathcal{V}$, denote by $\mathcal{K}_v$ the subset of $\mathcal{K}$ such that $\left\{\partial\omega_k \cap \overline{\Omega} \mid k \in \mathcal{K}_v\right\}$ are the d $(d-1)$-faces contaning v. This assumption imposes that $|\mathcal{K}_v| = d$. Thus

$$v \in \bigcap_{k\in\mathcal{K}_v} \partial\omega_k \cap \overline{\Omega}$$

and

$$v \notin \partial\omega_k \quad \forall\, k \in \mathcal{K}\backslash\mathcal{K}_v.$$

(c) Furthermore, as $\overline{\Omega}$ is a polytope and $\dim \overline{\Omega} = d$, this assumption implies that $\overline{\Omega}$ is simple (see Theorem 2.21), and hence $\bigcap_{k\in\mathcal{K}_v} \partial\omega_k \cap \overline{\Omega}$ is a single point (see Theorem 2.23). As $v \in \bigcap_{k\in\mathcal{K}_v} \partial\omega_k \cap \overline{\Omega}$ by assumption, it follows that

$$\{v\} = \bigcap_{k\in\mathcal{K}_v} \partial\omega_k \cap \overline{\Omega}. \tag{6}$$

The non-degeneracy of vertices is a consequence of Assumption **A4**.

Proposition 4.4. $\operatorname{rank}\left(n_k^\top\right)_{k\in\mathcal{K}_v} = d$ *for all* $v \in \mathcal{V}$.

Proof. Let $v \in \mathcal{V}$. Suppose that there exists $w \in \mathbb{R}^d$ such that $w \neq \vec{0}$ and $n_k \cdot w = 0$ for all $k \in \mathcal{K}_v$. Inmediately, we have $n_k \cdot (v + \lambda w) = a_k$ for all $k \in \mathcal{K}_v$ and all $\lambda \in \mathbb{R}$. That is,

$$v + \lambda w \in \partial\omega_k \quad \forall\, k \in \mathcal{K}_v,\ \lambda \in \mathbb{R}.$$

Since $n_k v < a_k$ for all $k \in \mathcal{K}\backslash\mathcal{K}_v$ ($v \notin \partial\omega_k$ for all $k \in \mathcal{K}\backslash\mathcal{K}_v$ but $v \in \overline{\Omega}$), there exists λ such that

$$\lambda < \frac{a_k - n_k v}{|n_k \cdot w|}$$

for all $k \in \mathcal{K}\backslash\mathcal{K}_v$ such that $n_k \cdot w \neq 0$. It follows that

$$\lambda n_k \cdot w \leq \lambda\, |n_k \cdot w| < a_k - n_k v$$

and then

$$n_k \cdot (v + \lambda w) < a_k$$

for all $k \in \mathcal{K}\backslash\mathcal{K}_v$. That is,

$$v + \lambda w \in \omega_k \quad \mathcal{K}\backslash\mathcal{K}_v.$$

Hence $v + \lambda w \in \overline{\Omega}$ and $v + \lambda w \in \partial\omega_k$ for all $k \in \mathcal{K}_v$. This contradicts (6). Therefore $n_k \cdot w = 0$ for all $k \in \mathcal{K}_v$ implies that $w = \vec{0}$. □

Proposition 4.5. $\exists\ \tau > 0$ *such that* $\forall\ v \in \mathcal{V}\ \exists!\ z_v : (-\tau, \tau) \to \mathbb{R}^d$ *satisfying* $z_v(0) = v$ *and*

$$\partial\omega_i(t) \cap \partial\omega_j(t) \cap \partial\omega_k(t) = \{z_v(t)\} \quad \forall\, t \in (-\tau, \tau),$$

where $\mathcal{I}_v = \{i, j, k\}$. *Moreover, each* z_v *is* C^∞ *on* $(-\tau, \tau)$ *and*

$$z_v'(0) = (n_i, n_j, n_k)^{-\top} \left(\left(a_i'(0), a_j'(0), a_k'(0)\right)^\top - \left(n_i'(0), n_j'(0), n_k'(0)\right)^\top v \right).$$

Proof. . $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$

$$F(x,t) = \begin{pmatrix} \widehat{\phi}_i(x,t) \\ \widehat{\phi}_j(x,t) \\ \widehat{\phi}_k(x,t) \end{pmatrix}$$

$$
\begin{aligned}
F(v,0) &= 0\\
D_x F(x,t) &= \begin{pmatrix} \nabla_x \widehat{\phi}_i (x,t)^\top \\ \nabla_x \widehat{\phi}_j (x,t)^\top \\ \nabla_x \widehat{\phi}_k (x,t)^\top \end{pmatrix} = \begin{pmatrix} n_i (t)^\top \\ n_j (t)^\top \\ n_k (t)^\top \end{pmatrix}\\
D_x F(v,0) &= \begin{pmatrix} n_i{}^\top \\ n_j{}^\top \\ n_k{}^\top \end{pmatrix} = (n_k)^\top_{k\in\mathcal{I}_v}\\
\partial_t F(x,t) &= \begin{pmatrix} \partial_t \widehat{\phi}_i (x,t) \\ \partial_t \widehat{\phi}_j (x,t) \\ \partial_t \widehat{\phi}_k (x,t) \end{pmatrix} = \begin{pmatrix} x\cdot n_i'(t) - a_i'(t) \\ x\cdot n_j'(t) - a_j'(t) \\ x\cdot n_k'(t) - a_k'(t) \end{pmatrix}\\
\partial_t F(v,0) &= \begin{pmatrix} v\cdot n_i'(0) - a_i'(0) \\ v\cdot n_j'(0) - a_j'(0) \\ v\cdot n_k'(0) - a_k'(0) \end{pmatrix} = \begin{pmatrix} n_i'(0)^\top \\ n_j'(0)^\top \\ n_k'(0)^\top \end{pmatrix} v - \begin{pmatrix} a_i'(0) \\ a_j'(0) \\ a_k'(0) \end{pmatrix}
\end{aligned}
$$

□

(non-degeneracy of edges) If n_i, n_j are the normal vectors of the faces of $\overline{\Omega}$ containing an edge, then $\operatorname{rank}(n_i, n_j)^\top = 2$ (non-degeneracy of edges). Moreover, given $k \in \mathcal{K}$, $v \cdot n_k < a_k$ for all $v \in \mathcal{V} \backslash \mathcal{W}$, where $\mathcal{W}$ is the subset of vertices contained in $\left\{x \in \mathbb{R}^3 \,|\, x \cdot n_k - a_k = 0\right\}$.

Proposition 4.6. *$\overline{\Omega_t}$ is simple convex polyhedral for small t.*

Proof. Since $a_k(t)$ is smooth, there exists $\tau_0 > 0$ such that $a_k(t) > 0$ for all $t \in (-\tau_0, \tau_0)$. Let n_i, n_j, n_k be the normal vectors of the faces of $\overline{\Omega}$ containing a vertex v. As $\operatorname{rank}(n_i, n_j, n_k)^\top = 3$, we can suppose that

$$\det (n_i(0), n_j(0), n_k(0))^\top > 0.$$

Since the determinant is a continuous functions, there exists $\tau_v > 0$ such that

$$\det (n_i(t), n_j(t), n_k(t))^\top > 0 \quad \forall\, t \in (-\tau_v, \tau_v),$$

and then $\operatorname{rank}(n_i(t), n_j(t), n_k(t))^\top = 3$ for all $t \in (-\tau_v, \tau_v)$. Let n_i, n_j be the normal vectors of the faces of $\overline{\Omega}$ containing an edge e. As $\operatorname{rank}(n_i, n_j)^\top = 2$ if and only if $-1 < n_i \cdot n_j < 1$, we have

$$1 - |n_i(0) \cdot n_j(0)| > 0.$$

Since the function $\lambda \mapsto 1 - |\lambda|$ is continuous, there exists $\tau_e > 0$ such that

$$1 - |n_i(t) \cdot n_j(t)| > 0 \quad \forall\, t \in (-\tau_e, \tau_e),$$

and then $\operatorname{rank}(n_i(t), n_j(t))^\top = 2$ for all $t \in (-\tau_e, \tau_e)$. Let $\tau \le \tau_0, \min_v \tau_v, \min_e \tau_e$. Thus, the domain $\overline{\Omega_t}$ is the intersection of halfspaces $\overline{\omega_k(t)}$ containing to the origen which form non-degenerate vertices and edges for all $t \in (-\tau, \tau)$. Therefore $\overline{\Omega_t}$ is a simple convex polyhedral for all $t \in (-\tau, \tau)$. □

Remark 4.7. The polyhedral Ω_t in Theorem has the same number of vertices, edges and faces that Ω.

In particular,

$$\int_{\partial\Omega} \frac{|\partial_n u|^2}{2} \theta_n = \sum_{k\in\mathcal{K}} \dot{a}_k(0) \int_{\partial\omega_k\cap\overline{\Omega}} \frac{|\partial_n u|^2}{2} - \int_{\partial\omega_k\cap\overline{\Omega}} \frac{|\partial_n u|^2}{2} x\cdot \dot{n}_k(0)$$

Case 2D:

$$n_k(t) = \begin{pmatrix} \cos(\alpha_k + t\delta\alpha_k) \\ \sin(\alpha_k + t\delta\alpha_k) \end{pmatrix},$$

$$a_k(t) = (\lambda_k + t\delta\lambda_k)\, n_k(t)\cdot n_k(0).$$

$$\dot{n}_k(0) = \delta\alpha_k \begin{pmatrix} -\sin(\alpha_k) \\ \cos(\alpha_k) \end{pmatrix}$$

$$\dot{a}_k(0) = \delta\lambda_k$$

$$\int_{\partial\Omega} \frac{|\partial_n u|^2}{2} \theta_n = \sum_{k\in\mathcal{K}} \delta\lambda_k \int_{f_k} \frac{|\partial_n u|^2}{2} - \delta\alpha_k \int_{f_k} \frac{|\partial_n u|^2}{2} x\cdot \begin{pmatrix} -\sin(\alpha_k) \\ \cos(\alpha_k) \end{pmatrix}$$

Caso 3D:

$$n_k(t) = \begin{pmatrix} \cos(\alpha_k + t\delta\alpha_k)\sin(\beta_k + t\delta\beta_k) \\ \sin(\alpha_k + t\delta\alpha_k)\sin(\beta_k + t\delta\beta_k) \\ \cos(\beta_k + t\delta\beta_k) \end{pmatrix},$$

$$a_k(t) = (\lambda_k + t\delta\lambda_k)\, n_k(t)\cdot n_k(0).$$

$$\dot{n}_k(0) = \delta\alpha_k \begin{pmatrix} -\sin(\alpha_k) \\ \cos(\alpha_k) \end{pmatrix}$$

$$\dot{n}_k(0) = \delta\alpha_k \begin{pmatrix} -\sin(\alpha_k)\sin(\beta_k) \\ \cos(\alpha_k)\sin(\beta_k) \\ 0 \end{pmatrix} + \delta\beta_k \begin{pmatrix} \cos(\alpha_k)\cos(\beta_k) \\ \sin(\alpha_k)\cos(\beta_k) \\ -\sin(\beta_k) \end{pmatrix}$$

$$\dot{a}_k(0) = \delta\lambda_k$$

$$\int_{\partial\Omega} \frac{|\partial_n u|^2}{2} \theta_n = \sum_{k\in\mathcal{K}} \delta\lambda_k \int_{f_k} \frac{|\partial_n u|^2}{2} - \delta\alpha_k \int_{f_k} \frac{|\partial_n u|^2}{2} x\cdot \mathbf{i}_k - \delta\beta_k \int_{f_k} \frac{|\partial_n u|^2}{2} x\cdot \mathbf{j}_k$$

$$\delta\lambda_k = -\int_{f_k} \frac{|\partial_n u|^2}{2}$$

$$\delta\alpha_k = \int_{f_k} \frac{|\partial_n u|^2}{2} x\cdot \mathbf{i}_k$$

$$\delta\beta_k = \int_{f_k} \frac{|\partial_n u|^2}{2} x\cdot \mathbf{j}_k$$

$$\int_{\partial\Omega} \frac{|\partial_n u|^2}{2} \theta_n < 0$$

$$\lambda_k^1 = \lambda_k^0 - \tilde{t} \int_{f_k} \frac{|\partial_n u|^2}{2}$$

$$\alpha_k^1 = \alpha_k^0 + \tilde{t} \int_{f_k} \frac{|\partial_n u|^2}{2} x \cdot \mathbf{i}_k$$

$$\beta_k^1 = \beta_k^0 + \tilde{t} \int_{f_k} \frac{|\partial_n u|^2}{2} x \cdot \mathbf{j}_k$$

Assumptions.

- Given $k \in \mathcal{K}$, $\overline{\Omega} \backslash \partial\omega_k \subset \omega_k$.

Let $v \in \mathcal{V}$ and n_i, n_j, n_k be the normal vectors of the faces of $\overline{\Omega}$ containing v. Then $\widehat{\phi}_i(v,0) = \widehat{\phi}_j(v,0) = \widehat{\phi}_k(v,0) = 0$. Since $\operatorname{rank}(n_i, n_j, n_k)^\top = 3$, there exist a τ_v and a unique function $z_v : (-\tau_v, \tau_v) \to \mathbb{R}^3$ such that $z_v(0) = v$ and

$$\widehat{\phi}_i(z_v(t), t) = \widehat{\phi}_j(z_v(t), t) = \widehat{\phi}_k(z_v(t), t) = 0 \quad \forall\, t \in (-\tau_v, \tau_v).$$

$$\begin{aligned} e_{i,j}(t) &\coloneqq \left\{ x \in \mathbb{R}^3 \,\middle|\, \widehat{\phi}_i(x,t) = \widehat{\phi}_j(x,t) = 0,\ \widehat{\phi}_k(x,t) < 0\ \forall\, k \in \mathcal{K} \backslash \{i,j\} \right\}, \\ &= \partial\omega_i(t) \cap \partial\omega_j(t) \cap \left(\cap_{k \in \mathcal{K} \backslash \{i,j\}} \omega_k(t) \right) \end{aligned}$$

$$\begin{aligned} f_k(t) &\coloneqq \left\{ x \in \mathbb{R}^3 \,\middle|\, \widehat{\phi}_k(x,t) = 0,\ \widehat{\phi}_j(x,t) < 0\ \forall\, j \in \mathcal{K} \backslash \{k\} \right\}. \\ &= \partial\omega_k(t) \cap \left(\cap_{j \in \mathcal{K} \backslash \{k\}} \omega_j(t) \right) \end{aligned}$$

Denote $\omega_k \coloneqq \omega_k(0)$, $e_{k,k'} \coloneqq e_{k,k'}(0)$, $f_k \coloneqq f_k(0)$ and $\Omega \coloneqq \Omega_0$.

Definition 4.8. D

Theorem 4.9. *There exist $\tau > 0$ and a mapping $T : \overline{\Omega} \times (-\tau, \tau) \to \mathbb{R}^d$ satisfying $T(\Omega, t) = \Omega_t$, $T(\partial\Omega, t) = \partial\Omega_t$ and such that $T(\cdot, t) : \overline{\Omega} \to \overline{\Omega_t}$ is bi-Lipschitz for all $t \in (-\tau, \tau)$. Furthermore, we have*

$$\theta(x) \cdot \nu(x) = \dot{a}_k(0) - x \cdot \dot{n}_k(0) \quad \forall\, x \in \operatorname{int} \partial\omega_k \cap \overline{\Omega}, \tag{7}$$

where $\theta \coloneqq \partial_t T(\cdot, 0)$ and ν is the outward unit normal vector to Ω.

Proof. Let $T : \overline{\Omega} \times (-\tau, \tau) \to \mathbb{R}^3$ be the vector-valued function defined by

$$T(x,t) \coloneqq \sum_{v \in \mathcal{V}} b_v(x)\, z_v(t).$$

Let $v \in \mathcal{V}$. We have

$$T(v,t) = \sum_{w \in \mathcal{V}} b_w(v)\, z_w(t) = \sum_{w \in \mathcal{V}} \delta_{wv} z_w(t) = z_v(t).$$

Let $e_{i,j}$ be an edge and v, w be its vertices. Let $x \in e_{i,j}$. We have

$$T(x,t) = \sum_{u\in\mathcal{V}} b_u(x)\, z_u(t) = b_v(x)\, z_v(t) + b_w(x)\, z_w(t)$$

with $b_v(x), b_w(x) > 0$. Thus, given $k \in \mathcal{K}$ it follows that

$$\hat{\phi}_k(T(x,t),t) = b_v(x)\, z_v(t)\cdot n_k(t) + b_w(x)\, z_w(t)\cdot n_k(t) - a_k(t).$$

If $k \in \{i,j\}$, then

$$\hat{\phi}_k(T(x,t),t) = (b_v(x) + b_w(x))\, a_k(t) - a_k(t) = 0$$

because $z_v(t), z_w(t) \in \partial\omega_k(t)$. Hence $T(e_{i,j},t) \subset \partial\omega_i(t) \cap \partial\omega_j(t)$. If $k \in \mathcal{K}\setminus\{i,j\}$ we have three cases: $\partial\omega_k(t)$ is the third plane containing to $z_v(t)$, $\partial\omega_k(t)$ is the third plane containing to $z_w(t)$ and $\partial\omega_k(t)$ is a plane that does not contain neither $z_v(t)$ and $z_v(t)$. In the first case

$$\begin{aligned}\hat{\phi}_k(T(x,t),t) &= b_v(x)\, a_k(t) + b_w(x)\, z_w(t)\cdot n_k(t) - a_k(t)\\ &< b_v(x)\, a_k(t) + b_w(x)\, a_k(t) - a_k(t)\\ &= (b_v(x) + b_w(x))\, a_k(t) - a_k(t) = 0.\end{aligned}$$

In the second case

$$\begin{aligned}\hat{\phi}_k(T(x,t),t) &= b_v(x)\, z_v(t)\cdot n_k(t) + b_w(x)\, a_k(t) - a_k(t)\\ &< b_v(x)\, a_k(t) + b_w(x)\, a_k(t) - a_k(t)\\ &= (b_v(x) + b_w(x))\, a_k(t) - a_k(t) = 0.\end{aligned}$$

Finally, in the third case,

$$\begin{aligned}\hat{\phi}_k(T(x,t),t) &= b_v(x)\, z_v(t)\cdot n_k(t) + b_w(x)\, z_w(t)\cdot n_k(t) - a_k(t)\\ &< b_v(x)\, a_k(t) + b_w(x)\, a_k(t) - a_k(t)\\ &= (b_v(x) + b_w(x))\, a_k(t) - a_k(t) = 0.\end{aligned}$$

Hence $T(e_{i,j},t) \subset \left(\cap_{k\in\mathcal{K}\setminus\{i,j\}}\omega_k(t)\right)$. We conclude that $T(e_{i,j},t) \subset e_{i,j}(t)$. Let $y \in e_{i,j}(t)$. Then $y = \lambda z_v(t) + (1-\lambda)\, z_w(t)$ for some $0 < \lambda < 1$. Let $x = w + \lambda(v - w)$. We have

$$\begin{aligned}T(x,t) &= b_v(x)\, z_v(t) + b_w(x)\, z_w(t)\\ &= \frac{\langle x-w, v-w\rangle}{|\overline{e_{i,j}}|^2} z_v(t) + \frac{\langle x-v, w-v\rangle}{|\overline{e_{i,j}}|^2} z_w(t)\\ &= \frac{\langle \lambda(v-w), v-w\rangle}{|\overline{e_{i,j}}|^2} z_v(t) + \frac{\langle (1-\lambda)(w-v), w-v\rangle}{|\overline{e_{i,j}}|^2} z_w(t)\\ &= \lambda\frac{\langle v-w, v-w\rangle}{|\overline{e_{i,j}}|^2} z_v(t) + (1-\lambda)\frac{\langle w-v, w-v\rangle}{|\overline{e_{i,j}}|^2} z_w(t)\\ &= y.\end{aligned}$$

Therefore, $T\left(e_{i,j},t\right)=e_{i,j}\left(t\right)$. Moreover, if $T\left(x_1,t\right)=T\left(x_2,t\right)$ for $x_1,x_2\in e_{i,j}$,

$$
\begin{aligned}
b_v\left(x_1\right)z_v\left(t\right)+\left(1-b_v\left(x_1\right)\right)z_w\left(t\right)&=b_v\left(x_2\right)z_v\left(t\right)+\left(1-b_v\left(x_2\right)\right)z_w\left(t\right)\\
b_v\left(x_1\right)\left(z_v\left(t\right)-z_w\left(t\right)\right)+z_w\left(t\right)&=b_v\left(x_2\right)\left(z_v\left(t\right)-z_w\left(t\right)\right)+z_w\left(t\right)\\
b_v\left(x_1\right)\left(z_v\left(t\right)-z_w\left(t\right)\right)&=b_v\left(x_2\right)\left(z_v\left(t\right)-z_w\left(t\right)\right)
\end{aligned}
$$

It follows that $b_v\left(x_1\right)=b_v\left(x_2\right)$, that is

$$
\begin{aligned}
\frac{\langle x_1-w,v-w\rangle}{\left|\overline{e_{i,j}}\right|^2}&=\frac{\langle x_2-w,v-w\rangle}{\left|\overline{e_{i,j}}\right|^2}\\
\langle x_1-w,v-w\rangle&=\langle x_2-w,v-w\rangle
\end{aligned}
$$

and $\langle x_1-x_2,v-w\rangle=0$. Since x_1-x_2 is parallel to $v-w$, $x_1=x_2$. Therefore T is bijective on $e_{i,j}$. Let f_k be a face and $\mathcal{W}$ be the set of its vertices. Let $x\in f_k$. We have

$$
T\left(x,t\right)=\sum_{w\in\mathcal{V}}b_w\left(x\right)z_w\left(t\right)=\sum_{w\in\mathcal{W}}b_w\left(x\right)z_w\left(t\right)
$$

and

$$
\begin{aligned}
\hat{\phi}_k\left(T\left(x,t\right),t\right)&=\sum_{w\in\mathcal{W}}b_w\left(x\right)z_w\left(t\right)\cdot n_k\left(t\right)-a_k\left(t\right)\\
&=\left(\sum_{w\in\mathcal{W}}b_w\left(x\right)\right)a_k\left(t\right)-a_k\left(t\right)=0.
\end{aligned}
$$

Hence $T\left(f_k,t\right)\subset\partial\omega_k\left(t\right)$. Let $j\in\mathcal{K}\backslash\left\{k\right\}$. There are two cases: $\partial\omega_j\left(t\right)$ is a plane touching f_k on the edge with vertices u,v and otherwise. In the first case, there are at least one vertex $w\in\mathcal{W}$ such that $w\in\omega_j\left(t\right)$ (it is because a face has least one vertex out the plane $\partial\omega_j\left(t\right)$). Then

$$
\begin{aligned}
\hat{\phi}_j\left(T\left(x,t\right),t\right)&=\sum_{w\in\mathcal{W}}b_w\left(x\right)z_w\left(t\right)\cdot n_j\left(t\right)-a_j\left(t\right)\\
&=\sum_{w\in\mathcal{W}\backslash\left\{u,v\right\}}b_w\left(x\right)z_w\left(t\right)\cdot n_j\left(t\right)+\left(b_u\left(x\right)+b_v\left(x\right)\right)a_j\left(t\right)-a_j\left(t\right)\\
&=\sum_{w\in\mathcal{W}\backslash\left\{u,v\right\}}b_w\left(x\right)\left(z_w\left(t\right)\cdot n_j\left(t\right)-a_j\left(t\right)\right)+a_j\left(t\right)-a_j\left(t\right)\\
&=\sum_{w\in\mathcal{W}\backslash\left\{u,v\right\}}b_w\left(x\right)\left(z_w\left(t\right)\cdot n_j\left(t\right)-a_j\left(t\right)\right)
\end{aligned}
$$

since $u\cdot n_j\left(t\right)=v\cdot n_j\left(t\right)=a_j\left(t\right)$. We know that $z_w\left(t\right)\cdot n_j\left(t\right)-a_j\left(t\right)<0$ for all $w\in\mathcal{W}\backslash\left\{u,v\right\}$. Is there some $w\in\mathcal{W}$ such that $b_w\left(x\right)>0$? Yes! Indeed, since x is not in the edge of vertices u,v, there exists some $w\in\mathcal{V}\backslash\left\{u,v\right\}$ such that $b_w\left(x\right)>0$. But, since x is in the face f_k, $b_w\left(x\right)=0$ for all $w\in\mathcal{V}\backslash\mathcal{W}$. Then there is some $w\in\mathcal{W}\backslash\left\{u,v\right\}$ such that $b_w\left(x\right)>0$. Thus

$$
\hat{\phi}_j\left(T\left(x,t\right),t\right)=\sum_{w\in\mathcal{W}\backslash\left\{u,v\right\}}b_w\left(x\right)\left(z_w\left(t\right)\cdot n_j\left(t\right)-a_j\left(t\right)\right)<0.
$$

In the second case,

$$\begin{aligned}\hat{\phi}_j\left(T\left(x,t\right),t\right) &= \sum_{w\in\mathcal{W}} b_w\left(x\right) z_w\left(t\right)\cdot n_j\left(t\right) - a_j\left(t\right) \\ &= \sum_{w\in\mathcal{W}} b_w\left(x\right)\left(z_w\left(t\right)\cdot n_j\left(t\right) - a_j\left(t\right)\right)\end{aligned}$$

We know that $z_w\left(t\right)\cdot n_j\left(t\right) - a_j\left(t\right) < 0$. If all $b_w\left(x\right) = 0$ for all $w\in\mathcal{W}$ then $x = 0 \in \partial\omega_k\left(t\right)$. Comtradiction!

$$\hat{\phi}_j\left(T\left(x,t\right),t\right) < 0$$

We conclude that $T\left(x,t\right)\in\partial\omega_k\left(t\right)\cap\left(\cap_{j\in\mathcal{K}\setminus\{k\}}\omega_j\left(t\right)\right)$, that is, $T\left(f_k,t\right)\subset f_k\left(t\right)$. Therefore

$$T\left(\overline{f_k},t\right)\subset\overline{f_k\left(t\right)}, T\left(\partial f_{k\partial},t\right) = \partial f_k\left(t\right)$$

and T is injective on $\partial f_{k\partial}$.·

Let $x\in\cap_{k\in\mathcal{K}}\omega_k$. Let $k\in\mathcal{K}$ and $\mathcal{W}$ be the set of vertices of the face f_k. We have

$$\begin{aligned}\hat{\phi}_k\left(T\left(x,t\right),t\right) &= \sum_{v\in\mathcal{V}} b_v\left(x\right) z_v\left(t\right)\cdot n_k\left(t\right) - a_k\left(t\right) \\ &= \left(\sum_{w\in\mathcal{W}} b_w\left(x\right) - 1\right) a_k\left(t\right) + \sum_{v\in\mathcal{V}\setminus\mathcal{W}} b_v\left(x\right) z_v\left(t\right)\cdot n_k\left(t\right) \\ &= -\sum_{v\in\mathcal{V}\setminus\mathcal{W}} b_v\left(x\right) a_k\left(t\right) + \sum_{v\in\mathcal{V}\setminus\mathcal{W}} b_v\left(x\right) z_v\left(t\right)\cdot n_k\left(t\right) \\ &= \sum_{v\in\mathcal{V}\setminus\mathcal{W}} b_v\left(x\right)\left(z_v\left(t\right)\cdot n_k\left(t\right) - a_k\left(t\right)\right) \\ &= \sum_{v\in\mathcal{V}\setminus\mathcal{W}} b_v\left(x\right)\hat{\phi}_k\left(z_v\left(t\right),t\right)\end{aligned}$$

We know that there exists $w\in\mathcal{V}\setminus\mathcal{W}$ such that $b_w\left(x\right) > 0$.
Since $b_v\left(x\right) > 0$ for all v and $\hat{\phi}_k\left(z_v\left(t\right),t\right) < 0$ for all $v\in\mathcal{V}\setminus\mathcal{W}$ (it must be hip.), $\hat{\phi}_k\left(T\left(x,t\right),t\right) < 0$. Hence $T\left(x,t\right)\in\cap_{k\in\mathcal{K}}\omega_k\left(t\right)$. □

5 Injectivity

$$T\left(x,t\right) = x + \sum_{v\in\mathcal{V}} b_v\left(x\right)\left(z_v\left(t\right) - v\right)$$

$$T\left(x,t\right) - T\left(y,t\right) = x - y + \sum_{v\in\mathcal{V}}\left(b_v\left(x\right) - b_v\left(y\right)\right)\left(z_v\left(t\right) - v\right)$$

It holds that

$$\|x-y\|-\left\|\sum_{v\in\mathcal{V}}\left(b_v\left(x\right)-b_v\left(y\right)\right)\left(z_v\left(t\right)-v\right)\right\|\leq\left\|T\left(x,t\right)-T\left(y,t\right)\right\|\leq\|x-y\|+\left\|\sum_{v\in\mathcal{V}}\left(b_v\left(x\right)-b_v\left(y\right)\right)\left(z_v\left(t\right)-v\right)\right\|$$

if

$$\left|b_v\left(x\right)-b_v\left(y\right)\right|\leq\left\|\nabla b_v\left(z\left(x,y,v\right)\right)\right\|\left\|x-y\right\|$$

Then

$$\|x-y\|\left(1-\sum_{v\in\mathcal{V}}\left\|\nabla b_v\left(z\left(x,y,v\right)\right)\right\|\left\|z_v\left(t\right)-v\right\|\right)\leq\left\|T\left(x,t\right)-T\left(y,t\right)\right\|\leq\|x-y\|\left(1+\sum_{v\in\mathcal{V}}\left\|\nabla b_v\left(z\left(x,y,v\right)\right)\right\|\right.$$

Assume that

$$\left\|\nabla b_v\left(z\left(x,y,v\right)\right)\right\|\leq C$$

for all v and all $x,y\in\overline{\Omega}$. Then

$$\|x-y\|\left(1-C\sum_{v\in\mathcal{V}}\left\|z_v\left(t\right)-v\right\|\right)\leq\left\|T\left(x,t\right)-T\left(y,t\right)\right\|\leq\|x-y\|\left(1+C\sum_{v\in\mathcal{V}}\left\|z_v\left(t\right)-v\right\|\right)$$

and if

$$\sum_{v\in\mathcal{V}}\left\|z_v\left(t\right)-v\right\|<\frac{1}{C}$$

From this we deduce that T is bi
Rewriteen T as

$$T\left(x,t\right)=x+\sum_{v\in\mathcal{V}}b_v\left(x\right)\left(z_v\left(t\right)-v\right)$$

it is easy to calculate its Jacobian matrix

$$D_xT\left(x,t\right)=I+\sum_{v\in\mathcal{V}}\left(z_v\left(t\right)-v\right)\otimes\nabla b_v\left(x\right).$$

Let $d:\overline{\Omega}\times\mapsto\mathbb{R}$ be the function defined by

$$d\left(x,t\right)\coloneqq\det\left(I+\sum_{v\in\mathcal{V}}\left(z_v\left(t\right)-v\right)\otimes\nabla b_v\left(x\right)\right).$$

Fix an arbitrary $x\in\overline{\Omega}$. Since $d\left(x,\cdot\right)$ is a continuous function and

$$d\left(x,0\right)=\det\left(I+\sum_{v\in\mathcal{V}}\left(z_v\left(0\right)-v\right)\otimes\nabla b_v\left(x\right)\right)=\det I=1>0,$$

it follows that $\det D_xT\left(x,t\right)=d\left(x,t\right)>0$ for small t. Therefore, one can conclude that $T\left(\cdot,t\right)$ is locally injective for small t.
This idea was taken from [4]§. Recall that non zero determinant of $D_xT\left(x,t\right)$ implies local injectivity. To obtain global injectivity consider [2]§ [3]§.

6 Surjectivity

h1: $U, V \subset \mathbb{R}^n$ are closed, bounded and convex
h2: $f : U \to V$ is continuous
h2: $f\,(\mathrm{int}U) \subset \mathrm{int}V$
h3: $f : \partial U \to \partial V$ is an homeomorphism
Then f is surjective.
Proof.
Let $p \in \mathrm{int}U$.
Let $H : I \times \partial V \to V$ defined by
$H\,(t, y) = f\,\big((1-t)\,f^{-1}\,(y) + tp\big)$, which is well-defined, continuous, $H\,(0, y) = y$ and $H\,(1, y) = f\,(p)$. The homotopy H continuously contracts the boundary ∂V to a $f\,(p)$. Then, for all $z \in V$, is there some x such that $H\,(t, y) = z$?

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