The BKM Engine: A GPU-Accelerated Pseudo-Spectral Solver for Classical Blow-Up Scenarios in 3D Incompressible Navier–Stokes

Benchmark simulations with continuous monitoring of Beale–Kato–Majda quantities and vorticity–strain alignment

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Abstract

We introduce a GPU-accelerated pseudo-spectral solver (the **BKM Engine**) designed to probe classical blow-up scenarios in 3D incompressible Navier–Stokes. The code couples adaptive time-stepping with continuous monitoring of Beale–Kato–Majda (BKM) quantities, scale-resolution sufficiency ($k_{\max}\eta > 1$), and vorticity–strain alignment. Benchmark runs (Taylor–Green, anti-parallel tubes, shear layers, isotropic turbulence, Kida–Pelz) at up to 512^3 and $Re = 1600$ exhibit extended stability without hyperviscosity or filtering, with finite $I(t)$ over observed windows and $k_{\max}\eta > 1$ throughout. The results are **numerical**: they certify non-singularity on simulated intervals under strict diagnostics and refinement stability; they do not constitute a PDE proof. The engine and configurations are released for reproducibility and further stress tests.

1 Introduction

The global regularity of 3D incompressible Navier–Stokes remains unresolved. Numerical experiments cannot settle the question, but they can produce a landscape in which singular behaviors might appear. We developed a solver specialized for this purpose: a Fourier pseudo-spectral code with GPU acceleration and a guard/diagnostic layer tied to BKM-type quantities, alignment geometry, and resolution sufficiency. The goal here isn't to enforce stability by modification of the equations themselves, but to monitor whether unmodified dynamics exhibit blow-up signatures under classical stress tests.

2 Mathematical Background

2.1 Energy and Enstrophy

Kinetic energy and enstrophy are

$$E(t) = \frac{1}{2}\int_\Omega \|u(x,t)\|^2\,dx, \tag{1}$$

$$\mathcal{Z}(t) = \frac{1}{2}\int_\Omega \|\omega(x,t)\|^2\,dx, \tag{2}$$

with $\omega = \nabla \times u$.

2.2 Beale–Kato–Majda

For Euler, finite-time singularity at T implies $\int_0^T \|\omega(\cdot,t)\|_{L^\infty}\,dt = \infty$. For Navier–Stokes, viscosity competes with vortex stretching; we therefore track $\|\omega\|_{L^\infty}$ and discrete proxies alongside $\mathcal{Z}(t)$.

2.3 A Posteriori Certification

Definition 1 (Alignment-weighted discrete BKM)**.** For a time grid t_k and solution u_h, define

$$I_A(T) = \sum_{k:t_k \le T} \|\omega(\cdot,t_k)\|_{L^\infty}(1+\alpha f_k)\Delta t_k,$$

where $f_k = \mathrm{volFrac}\{|\cos\theta(\cdot,t_k)| > c_\star\}$, θ is the angle between vorticity and the most-extensional strain eigenvector, and $\alpha \ge 0$, $c_\star \in (0,1)$.

Assumption 1 (Numerical hygiene)**.** (i) $\max_k \|\nabla\cdot u_h(\cdot,t_k)\|_\infty$ is at machine scale; (ii) $\min_{t\le T} k_{\max}\eta(t) > 1$; (iii) budget residual $\varepsilon_{\mathrm{bud}} \ll 1$; (iv) time integrator is L^2-stable/consistent with small a posteriori residual; (v) diagnostics stable under refinement $(N,\Delta t) \mapsto (2N,\Delta t/2)$.

Proposition 1 (A posteriori non-singularity certificate, conditional)**.** *If on* $[0,T]$ *one has* $I_A(T) < \infty$*, the guard ratio* $\varrho(t) < \varrho_{hard}$ *for all* t*, and* $\min_{t\le T} k_{\max}\eta(t) > 1$*, then the computed trajectory is* ***consistent*** *with a regular Navier–Stokes solution up to time* T*. Under the hygiene assumptions, bounded* I_A *together with scale-invariant norms (e.g.* L^3_x*), budget closure, and machine-precision incompressibility yields an a posteriori certificate of no finite-time singularity on* $[0,T]$.

Scope. Numerical certification under well-posed diagnostics; not a PDE theorem. Stability under refinement is required.

2.4 Spectral Perspective

In Fourier variables, incompressibility is $k\cdot\hat{u}_k = 0$. Resolution sufficiency is measured by $k_{\max}\eta$; values $k_{\max}\eta \gtrsim 1$ indicate resolved dissipation scales. We report $k_{\max}\eta$ alongside spectra in all cases.

2.5 Motivation

Numerical experiments themeslves cannot prove regularity, but they can falsify naive blow-up narratives for specific data and parameter ranges. The engine enforces incompressibility to machine precision and tracks $I(t)$, alignment, budgets, and $k_{\max}\eta$. What follows are computations that pass these checks.

3 Engine Design

We use a Fourier pseudo-spectral discretization on $(2\pi)^3$ with 2/3 de-aliasing and orthogonal projection onto divergence-free modes. Time integration is variable-step Runge–Kutta with a CFL controller and a guard metric ϱ that reduces Δt near critical growth. All heavy kernels run on the GPU; please reference Appendix A for implementation details.

3.1 Spectral Discretization

The velocity field $u(x,t)$ is represented as

$$u(x,t) = \sum_{k\in\mathbb{Z}^3} \hat{u}(k,t)e^{ik\cdot x},$$

with incompressibility enforced by orthogonal projection. Nonlinear terms use 2/3 de-aliasing to suppress aliasing errors. The skew-symmetric form preserves energy to machine precision in the inviscid limit.

3.2 Adaptive Time-Stepping

Time integration uses variable-step Runge–Kutta with timestep Δt constrained by:

$$\Delta t \leq \frac{C}{\max_x \|u(x,t)\|/\Delta x},$$

where $C < 1$. Additional adaptive controls reduce Δt when ϱ approaches critical values.

3.3 Guard Mechanisms

The guards here are diagnostic, not corrective. The stability metric is

$$\varrho = \frac{1 + \Delta t \cdot L}{1 + \nu \cdot \Delta t \cdot \kappa_h},$$

where L estimates the Lipschitz constant from maximum vorticity, and κ_h is the maximum discrete Laplacian eigenvalue. Two levels trigger step reductions when metrics approach critical values:

- **Soft guard:** $\varrho_{\text{soft}} = 0.985$ triggers timestep reduction
- **Hard guard:** $\varrho_{\text{hard}} = 0.995$ initiates enhanced projection

3.4 BKM and Alignment Monitoring

Beyond the core BKM integral $I(t) = \int_0^t \|\omega(\tau)\|_\infty \, d\tau$, the engine tracks Lipschitz constants, energy budget errors, divergence metrics, and vorticity-strain alignment statistics with thresholds $c_\star = 0.95$, $f_\star = 0.01$.

4 Benchmark Problems

To stress the solver we use five canonical cases: Taylor–Green (TG), anti-parallel vortex tubes (APVT), Kelvin–Helmholtz shear (KH), decaying isotropic turbulence (ISO-D), forced isotropic turbulence (ISO-F), and Kida–Pelz (KP). These span cascade, reconnection, transient enstrophy surges, and statistical stationarity.

4.1 Convergence Studies

TG runs at 128^3, 256^3, 512^3 with $Re = 800$ show peak dissipation timing and $\|\omega\|_\infty$ converging to within 2% between the two highest resolutions. This is our baseline for grid sufficiency.

4.2 Taylor–Green Vortex

Initialization on the $(2\pi)^3$ cube: $u(x,0) = (\sin x \cos y \cos z, -\cos x \sin y \cos z, 0)$. At $Re = 1600$ and 512^3, the cascade and dissipation peak match literature timing ($t_p \approx 9.0$) to within 1.5%. The BKM integral grows steadily, bounded over the window; $k_{\max}\eta \approx 3.4$ at peak dissipation.

4.3 Anti-Parallel Tubes

At $Re = 800$–1500, $N = 256^3$–512^3, sheets form and reconnect; alignment spikes near reconnection, but $k_{\max}\eta > 1$ and $I(t)$ finite. Guard activations coincide with reconnection but do not terminate runs.

4.4 Isotropic Turbulence (Decaying)

Random divergence-free initial field. Spectra flatten near $k^{-5/3}$ during peak. $k_{\max}\eta \gtrsim 1.2$. Statistics computed over 5 turnover times.

4.5 Isotropic Turbulence (Forced)

Low-k stochastic forcing yields stationarity. Dissipation and structure functions stable; ensemble averages taken over 3 seeds. $k_{\max}\eta > 1$ throughout.

4.6 Kida–Pelz

Symmetric initial data at $Re = 800$–1200, $N = 256^3$. Sharp enstrophy surge but $I(t)$ finite. Guards reduce Δt near spikes but no artificial damping added.

5 Diagnostics and Results

5.1 Energy and Dissipation

TG: canonical decay, peak ε after cascade onset. KH: slower decay, coherent rolls persist. APVT: intermittent bursts of ε at reconnection.

5.2 Spectral Resolution

Across runs $k_{\max}\eta > 1$; TG (512^3, $Re = 1600$) peaked at 3.4. CFL stayed conservative (≈ 0.01–0.02).

5.3 BKM Integral

All cases show $I(t)$ growth but no divergence. APVT grows fastest; TG and KH steadier; ISO-D/ISO-F finite.

5.4 Alignment Statistics

Vorticity tends to align with the intermediate strain eigenvector. Reconnection sites show enhanced alignment; guard events correlate with spikes in fraction $f(t)$ above threshold.

Observation 1 (Empirical alignment-enstrophy coupling). *When $f(t) > f_\star$ ($\approx 10^{-2}$), $d\mathcal{Z}/dt$ accelerates. Suggests geometry influences enstrophy growth beyond scaling laws.*

5.5 Verification

TG energy decay and dissipation timing match prior DNS within 1–2%. Confirms solver fidelity.

5.6 Summary Table

Table 1: Peak dissipation values, Re_λ, $\mathrm{CFL}_{\min}$, max $k_{\max}\eta$, max $I(t)$. All cases resolved ($k_{\max}\eta > 1$), finite I, controlled CFL.

Case	Grid	Re	Re_λ	$\mathrm{CFL}_{\min}$	$\max(k_{\max}\eta)$	$\max I(t)$
Taylor–Green	512^3	1600	13.3	0.010	3.40	24.7
KH Shear	128^3	800	8.1	0.012	1.8	18.2
Anti-parallel	256^3	1000	9.8	0.011	1.2	31.5
Isotropic-D	256^3	1200	11.2	0.011	1.5	22.1
Isotropic-F	256^3	1200	11.2	0.011	1.5	28.9

6 Discussion

6.1 Implications

Finite but bounded growth. $I(t)$ grows substantially but not to blow-up over simulated windows.

Resolution sufficiency. $k_{\max}\eta > 1$ and conservative CFL eliminate under-resolution as an explanation.

Geometric diagnostics. Alignment fraction correlates with enstrophy surges; supports geometric heuristics.

6.2 Limits of Numerics

Windows limited ($T < 50$ turnover times). Resolutions max at 512^3. $Re > 2000$ not credible on current hardware. Extrapolation of I plateaus is speculative.

6.3 Novel Contributions

Guards are diagnostic only, not stabilizers. Alignment statistics embedded in solver loop. Sustained runs at $Re = 1600$ without hyperviscosity.

6.4 Statistical Robustness

Error bars from ensemble averages (3–5 seeds). Grid convergence checked. Turnover-time averaging ensures stationarity.

6.5 Future Work

Extend to higher Re and longer T via multi-GPU. Add wall-bounded and rotating/stratified cases. Explore adaptive mesh.

7 Conclusion

The BKM Engine is a GPU-accelerated spectral solver with guard-based diagnostics for singularity testing. Across five benchmarks at up to 512^3, it preserved incompressibility and energy budgets, produced finite I, and correlated alignment spikes with enstrophy growth. No blow-up observed under tested conditions. The solver provides an auditable platform for further stress testing; it isn't a proof of regularity.

A Implementation Notes

Core routines written in Python/CuPy. FFTs and spectral multiplications executed on GPU. Guard triggers $< 0.1\%$ steps. Production version: v2.0.0-unified (dated 2025-01-23). Source code and configs included in repository.

B Benchmark Configuration

Standard initializations reproduced; full parameters in Table 2.

TG: periodic box, 512^3, $Re = 1600$. KH: 128^3, $Re = 800$, perturbation added. APVT: 256^3, $Re = 1000$, symmetry-breaking perturbation. ISO-D/F: 256^3, $Re = 1200$, random divergence-free initial data (decay) or stochastic forcing (forced). KP: 256^3, $Re = 800$–1200, symmetric alignment.

Table 2: Complete benchmark parameters

Case	Grid	Re	ν	CFL Target	Guard Thresholds
Taylor–Green	512^3	1600	6.25×10^{-4}	0.25	$\varrho_s = 0.985$
KH Shear	128^3	800	1.25×10^{-3}	0.25	$\varrho_s = 0.985$
Anti-parallel	256^3	1000	1.0×10^{-3}	0.25	$\varrho_s = 0.985$
Isotropic-D	256^3	1200	8.33×10^{-4}	0.25	$\varrho_s = 0.985$
Isotropic-F	256^3	1200	8.33×10^{-4}	0.25	$\varrho_s = 0.985$

C Reproducibility

All runs checkpointed at fixed intervals. Diagnostics logged to CSV for post-processing. Figures in main text generated from stitched outputs. Engine portable to CUDA GPUs; dependencies limited to NumPy/CuPy. Configs and scripts in repository for full replication.

D Correspondence: Formulation $\leftrightarrow$ Code

- BKM integral: cumulative sum of $\|\omega\|_\infty \Delta t$
- Guard parameter ϱ: `compute_stability_metric()`
- Alignment statistics: strain tensor eigen-decomposition
- Energy conservation: `energy_balance_error()`
- Divergence monitoring: spectral $\nabla \cdot u$

E Limitations

Engine sustains high-resolution DNS, but isn't a proof of regularity. Evidence is bounded by finite resolution and window. Numerical certification depends on diagnostic sufficiency and refinement stability. Interpret results within these constraints.

References

[1] Beale, J.T., Kato, T., Majda, A. (1984). Remarks on the breakdown of smooth solutions for the 3D Euler equations. *Comm. Math. Phys.*, 94(1), 61–66.

[2] Hou, T.Y., Li, R. (2006). Dynamic depletion of vortex stretching and non-blowup of the 3-D incompressible Euler equations. *J. Nonlinear Sci.*, 16(6), 639–664.

[3] Kerr, R.M. (1993). Evidence for a singularity of the three-dimensional, incompressible Euler equations. *Phys. Fluids A*, 5(7), 1725–1746.

[4] Donzis, D.A., Yeung, P.K., Sreenivasan, K.R. (2008). Dissipation and enstrophy in isotropic turbulence: resolution effects and scaling in direct numerical simulations. *Phys. Fluids*, 20(4), 045108.

[5] Gibbon, J.D. (2008). The three-dimensional Euler equations: where do we stand? *Physica D*, 237(14–17), 1894–1904.

[6] Brachet, M.E. (1991). Direct simulation of three-dimensional turbulence in the Taylor–Green vortex. *Fluid Dyn. Res.*, 8(1–4), 1–8.