$$S=\int d^4x\sqrt{-g}(-R+2(\nabla\phi)^2+e^{-2\phi}F^2)$$

Usando Euler Lagrange, para la densidad lagrangiana:

$$\hat{\mathcal{L}}=-R+2(\nabla\phi)^2+e^{-2\phi}F^2=-R+2g^{ab}\nabla_a\phi\nabla_b\phi+e^{-2\phi}F^2$$

Entonces:

$$0=\frac{\partial\hat{\mathcal{L}}}{\partial\phi}-\nabla_\mu(\frac{\partial\hat{\mathcal{L}}}{\partial(\nabla_\mu\phi)})$$

$$=-2e^{-2\phi}F^2-\nabla_\mu(\frac{\partial}{\partial(\nabla_\mu\phi)}(2g^{ab}\nabla_a\phi\nabla_b\phi))$$

$$=-2e^{-2\phi}F^2-\nabla_\mu(2g^{ab}\delta_{a\mu}\nabla_b\phi+2g^{ab}\nabla_a\phi\delta_{\mu b})=-2e^{-2\phi}F^2-\nabla_\mu(2g^{\mu b}\nabla_b\phi+2g^{a\mu}\nabla_a\phi)$$

$$\Rightarrow e^{-2\phi}F^2+2\nabla_m u(g^{a\mu}\nabla_a\phi)=0$$

$$\Rightarrow \nabla_\mu\nabla^\mu\phi+\frac{1}{2}e^{-2\phi}F^2=0$$

$$\Rightarrow \nabla^2\phi+\frac{1}{2}e^{-2\phi}F^2=0$$

Como la única contribución de S con $A_\mu$es la parte:

$$S_3=\int d^4x\sqrt{-g}e^{-2\phi}F^2$$

Y como $F^2=F_{ab}F^{ab}=F_{ab}g^{ac}g^{bd}F_{cd}=g^{ac}g^{bd}(\partial_aA_b-\partial_bA_a)(\partial_cA_d-\partial_dA_c)=g^{ac}g^{bd}(\nabla_aA_b-\nabla_bA_a)(\nabla_cA_d-\nabla_dA_c)$ porque $\partial_aA_b-\partial_bA_a=(\partial_aA_b-\Gamma^\lambda_{ab}A_\lambda)-(\partial_bA_a-\Gamma^\lambda_{ab}A_\lambda)=\nabla_aA_b-\nabla_bA_a$ Entonces:

$$\hat{\mathcal{L}}_1=e^{-2\phi}g^{ac}g^{bd}(\partial_aA_b-\partial_bA_a)(\partial_cA_d-\partial_dA_c)=e^{-2\phi}g^{ac}g^{bd}(\nabla_aA_b-\nabla_bA_a)(\nabla_cA_d-\nabla_dA_c)$$

Y la ecuación de Euler-Lagrange

$$0=\frac{\partial\hat{\mathcal{L}}_1}{\partial A_e}-\nabla_\mu(\frac{\partial\hat{\mathcal{L}}_1}{\partial(\nabla_\mu A_e)})=$$

$$0=\nabla_\mu(\frac{\partial\hat{\mathcal{L}}_1}{\partial(\nabla_\mu A_e)})=\nabla_\mu(e^{-2\phi}g^{ac}g^{bd}(\delta^\mu_a\delta^e_b-\delta^\mu_b\delta^e_a)(\nabla_cA_d-\nabla_dA_c)+e^{-2\phi}g^{ac}g^{bd}(\nabla_aA_b-\nabla_bA_a)(\delta^\mu_c\delta^e_d-\delta^\mu_d\delta^e_c))$$

$$=\nabla_\mu(e^{-2\phi}(g^{\mu c}g^{ed}-g^{ec}g^{\mu d})(\nabla_cA_d-\nabla_dA_c)+e^{-2\phi}(g^{a\mu}g^{be}-g^{ae}g^{b\mu})(\nabla_aA_b-\nabla_bA_a))$$

$$=\nabla_\mu(e^{-2\phi}(g^{\mu c}g^{ed}-g^{ec}g^{\mu d})(\delta_cA_d-\delta_dA_c)+e^{-2\phi}(g^{a\mu}g^{be}-g^{ae}g^{b\mu})(\delta_aA_b-\delta_bA_a))$$

$$=\nabla_\mu(e^{-2\phi}(g^{\mu c}g^{ed}-g^{ec}g^{\mu d})F_{cd}+e^{-2\phi}(g^{a\mu}g^{be}-g^{ae}g^{b\mu})F_{ab})$$

$$=\nabla_\mu(e^{-2\phi}(g^{\mu a}g^{eb}-g^{ea}g^{\mu b})F_{ab}+e^{-2\phi}(g^{a\mu}g^{be}-g^{ae}g^{b\mu})F_{ab})=2\nabla_\mu(e^{-2\phi}(g^{\mu a}g^{eb}-g^{ea}g^{\mu b})F_{ab})$$

$$\Rightarrow 0=\nabla_\mu(e^{-2\phi}(g^{\mu a}g^{eb}-g^{ea}g^{\mu b})F_{ab})=\nabla_\mu(e^{-2\phi}(F^{\mu e}-F^{e\mu})=2\nabla_\mu(e^{-2\phi}F^{\mu e})$$

$$\Rightarrow \nabla_\mu(e^{-2\phi}F^{\mu e})=0$$

Para hacer la variación respecto a la métrica, separamos la acción en 3 partes:

$$S_1=\int d^4x\sqrt{-g}\,(-R)$$

$$S_2 = \int d^4x\sqrt{-g}\,2(\nabla\phi)^2$$

$$S_3 = \int d^4x\sqrt{-g}\,e^{-2\phi}F^2$$

La variación del primer termino es la variación de la acción de Einstein-Hilbert que da:

$$\frac{\delta S_1}{\delta g^{\mu\nu}} = -\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = \sqrt{-g}(\frac{1}{2}Rg_{\mu\nu} - R_{\mu\nu})$$

Además $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$, entonces:

$$\delta S_2 = \int d^4x\frac{\delta S_2}{\delta g^{\mu\nu}}\delta g^{\mu\nu}$$

$$\delta S_2 = \int d^4x(\delta(\sqrt{-g}2(\nabla\phi)^2) = \int d^4x\delta(\sqrt{-g})2(\nabla\phi)^2 + \sqrt{-g}2\delta((\nabla\phi)^2))$$

$$= \int d^4x(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}2(\nabla\phi)^2 + \sqrt{-g}2\nabla_\mu\phi\nabla_\nu\phi\delta g^{\mu\nu})$$

$$= \int d^4x\sqrt{-g}(-g_{\mu\nu}\delta g^{\mu\nu}(\nabla\phi)^2 + 2\nabla_\mu\phi\nabla_\nu\phi)\delta g^{\mu\nu}$$

Entonces:

$$\frac{\delta S_2}{\delta g^{\mu\nu}} = \sqrt{-g}(-g_{\mu\nu}\delta g^{\mu\nu}(\nabla\phi)^2 + 2\nabla_\mu\phi\nabla_\nu\phi)$$

Finalmente δS_3

$$\delta S_3 = \int d^4x\delta(\sqrt{-g}e^{-2\phi}F^2) = \int d^4xF_{ab}F_{cd}e^{-2\phi}\delta(\sqrt{-g}g^{ac}g^{bd})$$

$$= \int d^4xF_{ab}F_{cd}e^{-2\phi}(\delta\sqrt{-g}g^{ac}g^{bd} + \sqrt{-g}\delta g^{ac}g^{bd} + \sqrt{-g}g^{ac}\delta g^{bd}$$

$$= \int d^4xF_{ab}F_{cd}e^{-2\phi}(\delta\sqrt{-g}g^{ac}g^{bd} + \sqrt{-g}\delta g^{ac}g^{bd} + \sqrt{-g}g^{ac}\delta g^{bd}$$

$$= \int d^4xe^{-2\phi}\sqrt{-g}F_{ab}F_{cd}(g^{ac}g^{bd} + \sqrt{-g}\delta g^{ac}g^{bd} + \sqrt{-g}g^{ac}\delta g^{bd}$$

$$= \int d^4xe^{-2\phi}\sqrt{-g}\frac{-F^{cd}F_{cd}}{2}g_{\mu\nu}\delta g^{\mu\nu} + F_{\mu b}F_{\nu d}e^{-2\phi}\sqrt{-g}g^{bd}\delta g^{\mu\nu} + F_{\mu b}F_{\nu d}e^{-2\phi}\sqrt{-g}g^{\mu\nu}\delta g^{bd}$$

$$\int d^4x(2F_{\mu b}F_{\nu d}e^{-2\phi}\sqrt{-g}g^{bd} - \frac{1}{2}F^2e^{-2\phi}\sqrt{-g}g_{\mu\nu})\delta g^{\mu\nu}$$

$$\Rightarrow \frac{\delta S_3}{\delta g^{\mu\nu}} = \sqrt{-g}e^{-2\phi}(2F_{\mu b}F_{\nu d}g^{bd} - \frac{1}{2}F^2g_{\mu\nu})$$

Finalmente sumando las acciones

$$\Rightarrow 0 = \frac{\delta S}{\delta g^{\mu\nu}} = \frac{\delta S_1}{\delta g^{\mu\nu}} + \frac{\delta S_2}{\delta g^{\mu\nu}} + \frac{\delta S_3}{\delta g^{\mu\nu}}$$

$$= \sqrt{-g}(\frac{1}{2}Rg_{\mu\nu} - R_{\mu\nu} - g_{\mu\nu}(\nabla\phi)^2 + 2\nabla_\mu\phi\nabla_\nu\phi + 2e^{-2\phi}F_{\mu b}F_{\nu d}g^{bd} - \frac{1}{2}g_{\mu\nu}e^{-2\phi}F^2)$$

$$\Rightarrow R_{\mu\nu} = 2\nabla_\mu\phi\nabla_\nu\phi + 2e^{-2\phi}F_{\mu\rho}F_\nu^\rho - \frac{1}{2}g_{\mu\nu}e^{-2\phi}F^2 + \frac{1}{2}Rg_{\mu\nu} - g_{\mu\nu}(\nabla\phi)^2$$

Si contraemos tenemos:

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}(2\nabla_\mu\phi\nabla_\nu\phi + 2e^{-2\phi}F_{\mu\rho}F_\nu^\rho - \frac{1}{2}g_{\mu\nu}e^{-2\phi}F^2 + \frac{1}{2}Rg_{\mu\nu} - g_{\mu\nu}(\nabla\phi)^2)$$

$$= 2(\nabla\phi)^2 + 2e^{-2\phi}F_{\mu\rho}F^{\rho\mu} - 2e^{-2\phi}F^2 + 2R - 4(\nabla\phi)^2$$

$$= -2(\nabla\phi)^2 + 2e^{-2\phi}F^2 - 2e^{-2\phi}F^2 + 2R$$

$$= -2(\nabla\phi)^2 + 2R$$

$$\Rightarrow R = 2(\nabla\phi)^2$$

Entonces sustituyendo:

$$\Rightarrow R_{\mu\nu} = 2\nabla_\mu\phi\nabla_\nu\phi + 2e^{-2\phi}F_{\mu\rho}F_\nu^\rho - \frac{1}{2}g_{\mu\nu}e^{-2\phi}F^2 + (\frac{1}{2}R - (\nabla\phi)^2)g_{\mu\nu}$$

$$\Rightarrow R_{\mu\nu} = 2\nabla_\mu\phi\nabla_\nu\phi + 2e^{-2\phi}F_{\mu\rho}F_\nu^\rho - \frac{1}{2}g_{\mu\nu}e^{-2\phi}F^2$$