Mandatory Assignment 1

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Exercise 1

We consider the ODE problem

$$u'' + \omega^2 u = f(t), \qquad u(0) = I, u'(0) = V, t \in (0, T]$$

a)

The first task is to derive the discretized equation for the first time step (u^1).
We start by discretizing the general problem using central difference:

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + \omega^2 u^n = f \; u^{n+1} = (2 - \omega\Delta t^2)u^n - u^{n-1} + \Delta t^2 f$$

We use a central difference for the initial condition:

$$\frac{u^1 - u^{-1}}{2\Delta t} = V \; u^{-1} = u^1 - 2\Delta t V$$

Inserted into the scheme for n = 0:

$$u^1 = (2 - \omega\Delta t^2)u^0 - u^1 + 2\Delta t V + \Delta t^2 f$$

Giving, after a quick rearrangement:

$$u^1 = (1 - \frac{\omega\Delta t^2}{2})u^0 + \Delta t V + \frac{\Delta t^2 f}{2}$$

b)

We are given the manufactured solution $u_e(x,t) = ct + d$.
Using the initial conditions of the ODE problem, we find the restrictions on c and d:

$u_e(0) = d = I \qquad u_e'(0) = c = V$

We then find f by inserting u_e into the ODE problem, noting that $u_e'' = 0$:

$f = \omega^2(ct + d) = \omega^2(Vt + I)$

We then want to show that $[D_tD_tt]^n = 0$, so we use the central differencing scheme employed earlier, and insert for t:

$\frac{t^{n+1}-2t^n+t^{n-1}}{\Delta t^2} = \frac{(n+1)\Delta t-2n\Delta t+(n-1)\Delta t}{\Delta t^2} = \frac{n\Delta t-\Delta t-2n\Delta t+n\Delta t-\Delta t}{\Delta t^2} = \frac{2n-2n}{\Delta t^2} = 0$

We finally want to show that u_e is a perfect solution, so we use the fact given in the exercise, $[D_tD_t(ct + d)]^n = 0$ and get:

$[D_tD_t(ct + d)]^n + \omega^2(ct + d) = f^n$
$0 + \omega^2(Vt + I) = \omega^2(Vt + I)$

So we see that u_e is a perfect solution of the discrete equations.

c)

See python program vib_undamped_verify_mms.py

d)

See python program vib_undamped_verify_mms.py

e)

As we see from the function cubic in the program listed above, we get a residue for the first term of $c\Delta t^3$, where c is the coefficient of the third-degree term.

A third degree polynomial will therefore not fulfill the discrete equations.

f)

See python program vib_undamped_verify_mms.py

g)

See python program vib_undamped_verify_mms.py

Exercise 21

a)

See python program elastic_pendulum.py

b)

See python program elastic_pendulum.py

c)

For pure vertical motion, we can simplify the ODE problem:

$\frac{\partial^2 x}{\partial t^2} = 0$

$\frac{\partial^2 y}{\partial t^2} = -\frac{\beta}{1-\beta}(1 - \frac{\beta}{\sqrt{(y-1)^2}} - \beta$

From Wolfram Alpha, $\frac{y-1}{\sqrt{(y-1)^2}}$ is the sign of $(y-1)$. Here y must be less than 1, since the pendulum is fixed at scaled location y = 1. We therefore get:

$\frac{\partial^2 y}{\partial t^2} = -\frac{\beta}{1-\beta}(y-1+\beta)-\beta = -\frac{\beta y}{1-\beta}+\frac{\beta}{1-\beta}-\frac{\beta^2}{1-\beta}-\beta = -\frac{\beta y+\beta-\beta^2-\beta(1-\beta)}{1-\beta} = -\frac{\beta}{1-\beta}y = -\omega^2 y$

where $\omega = \sqrt(\frac{\beta}{1-\beta})$

We also get:

$y(0) = \epsilon \qquad y(t) = \epsilon cos(\omega t)$

d)

See python program elastic_pendulum.py