Multiple View Geometry: Solution Exercise Sheet 6

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Part I: Theory

1. (a) E is essential matrix $\quad\Rightarrow\quad \Sigma = \text{diag}\{\sigma, \sigma, 0\}$:

$$R_z(\pm\frac{\pi}{2})\Sigma = \begin{pmatrix} 0 & \mp 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mp\sigma & 0 \\ \pm\sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -(R_z(\pm\frac{\pi}{2})\Sigma)^\top$$

$$\begin{aligned} -\hat{T}^\top &= -(UR_z\Sigma U^\top)^\top \\ &= U(-R_z\Sigma)^\top U^\top \\ &= UR_z\Sigma U^\top \\ &= \hat{T} \end{aligned}$$

(b) i. U, V are orthogonal matrices $\Rightarrow U^\top U = Id$ and $VV^\top = Id$
R_z is a rotation matrix $\Rightarrow R_z R_z^\top = Id$

$$\begin{aligned} R^\top R &= (UR_z^\top V^\top)^\top (UR_z^\top V^\top) \\ &= VR_z U^\top U R_z^\top V^\top \\ &= VR_z R_z^\top V^\top \\ &= VV^\top \\ &= Id \end{aligned}$$

ii. U and V are special orthogonal matrices with $\det(U) = \det(V^\top) = 1$ (Slide 9, Chapter 5).

$$\det(R) = \det(UR_z^\top V^\top) = \underbrace{\det(U)}_{1} \cdot \underbrace{\det(R_z^\top)}_{1} \cdot \underbrace{\det(V^\top)}_{1} = 1$$

2. (a) $H = R + Tu^\top \Leftrightarrow R = H - Tu^\top$.

$$\begin{aligned} E &= \hat{T}R \\ &= \hat{T}(H - Tu^\top) \\ &= \hat{T}H - \underbrace{\hat{T}T}_{=T\times T=0} u^\top \\ &= \hat{T}H \end{aligned}$$

(b)

$$
\begin{aligned}
H^\top E + E^\top H &= H^\top(\hat{T}H) + (\hat{T}H)^\top H \\
&= H^\top(\hat{T}H) + H^\top\hat{T}^\top H \\
&= H^\top\hat{T}H - H^\top\hat{T}H \quad \text{(because } \hat{T} \text{ is skew-symmetric, i.e. } \hat{T}^\top = -\hat{T}\text{)} \\
&= 0
\end{aligned}
$$

3. (Refer to Slide 6 Chapter 5 for the notations below)

Solution 1:
T : $\overrightarrow{o_2 o_1}$ seen in coordinate system o_2
$-R^\top T$: $\overrightarrow{o_1 o_2}$ seen in coordinate system o_1

e_1 is the image of o_2 in coordiante system o_1
$\Rightarrow e_1 = K\overrightarrow{o_1 o_2} = K(-R^\top T)$ (K is the intrinsic parameter matrix)
e_2 is the image of o_1 in coordiante system o_2
$\Rightarrow e_2 = K\overrightarrow{o_2 o_1} = KT$

$$
\begin{aligned}
Fe_1 &= (\underbrace{K^{-\top}\hat{T}RK^{-1}}_{F})(\underbrace{K(-R^\top T)}_{e_1}) \\
&= K^{-\top}\hat{T}R\underbrace{K^{-1}K}_{I}(-R^\top T) \\
&= K^{-\top}\hat{T}\underbrace{R(-R^\top)}_{-I}T \\
&= -K^{-\top}\underbrace{\hat{T}T}_{=T\times T=0} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
e_2^\top F &= (\underbrace{KT}_{e_2})^\top(\underbrace{K^{-\top}\hat{T}RK^{-1}}_{F}) \\
&= T^\top\underbrace{K^\top K^{-\top}}_{I}\hat{T}RK^{-1} \\
&= T^\top\hat{T}RK^{-1} \\
&= T^\top(T\times R)K^{-1} \quad (T\times R \text{ is a vector perpendicular to } T) \\
&= 0K^{-1} \\
&= 0
\end{aligned}
$$

Solution 2:
$\forall\, x_2: \quad l_1 = \{x_1 \mid x_2^\top F x_1 = 0\}$
In particular: $e_1 \in l_1 \quad \Rightarrow \quad x_2^\top F e_1 = 0 \quad \forall x_2$
$\qquad\qquad\qquad\qquad \Rightarrow \quad Fe_1 = 0$

The camera center o_2 is in the preimage of every $x_2 \Rightarrow$ The epipol e_1 (which is the projection of o_2 to the image plane of image 1) lies on all epipolar lines l_1

Analogous: $e_2^\top F = 0$.

Part II: Practical Exercises

Remarks

1. **Are there two or four possible solutions for R and T?**
   The answer is four. You may find it confusing since in Slide 9 you are told to have two sets of R and T, while in Slide 14 it says there are four possible solutions. Recall that the essential matrix E is calculated by solving the equation $\chi E^s = 0$ (at the bottom of Slide 11). Ideally the E^s you get should lie in the nullspace of χ. However, since there are always erros in the point pairs you've chosen to make χ, in practice it is very difficult to get an E^s that makes χE^s exactily 0. Instead we use the SVD of χ to get the E^s which minimizes $||\chi E^s||$. In other words, the E^s we get will give us $\chi E^s = \sigma$ with σ being some very small vector. Now think about this: what will happen if we turn the sign of the E^s? We will get $\chi(-E^s) = -\sigma$ which still gives us the smallest $||\chi E^s||$ ($||\chi E^s|| = ||\sigma|| = ||-\sigma|| = ||\chi(-E^s)||$).

   Now we know that solving $\chi E^s = 0$ for E^s will always give us two possible solutions E and $-E$. From each of them you can get two possible sets of R and T using the equations in Slide 9. Altogether we get four possible solutions. In practice we do the calculation in a little different way. We usually calculate according to Slide 14 to get four solutions out of E. The two extra solutions we get are nothing but the ones we should have got from $-E$.

2. **How to get the correct R and T from the possible solutions?**
   The criterion you should use to rule out the incorrect solutions is that all the reconstructed 3D points should have positive depth seen from **both of the camera coordinate systems**. In other words, bothλ_1^j and λ_2^j in Slide 17 need to be positive.