Multiple View Geometry: Exercise Sheet 10

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Exercise: July 20th, 2017

Part I: Theory

1. **Variational Calculus and Euler-Lagrange**

   (a) Under the assumption that h vanishes at the boundary of Ω, prove that

   $$\frac{dE(u)}{du} = \frac{\partial \mathcal{L}(u, \nabla u)}{\partial u} - \text{div}\left(\frac{\partial \mathcal{L}(u, \nabla u)}{\partial(\nabla u)}\right).$$

   We can expand $\mathcal{L}(u + \epsilon h, \nabla u + \epsilon \nabla h)$ in terms of ϵ:

   $$\mathcal{L}(u + \epsilon h, \nabla u + \epsilon \nabla h) = \mathcal{L}(u, \nabla u) + \epsilon h \frac{\partial \mathcal{L}}{\partial u} + \epsilon \nabla h \frac{\partial \mathcal{L}}{\partial(\nabla u)} + \mathcal{O}(\epsilon^2)$$

   Inserting into $\left.\frac{\delta E(u)}{\delta u}\right|_h$ gives

   $$\left.\frac{\delta E(u)}{\delta u}\right|_h = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_\Omega \left(\epsilon h(x) \left.\frac{\partial \mathcal{L}}{\partial u}\right|_{u(x)} + \epsilon \nabla h(x) \left.\frac{\partial \mathcal{L}}{\partial(\nabla u)}\right|_{\nabla u(x)} + \mathcal{O}(\epsilon^2)\right) dx$$

   The ϵ in the first two terms cancels, and in the $\mathcal{O}(\epsilon^2)$ it will go to zero for $\epsilon \to 0$. Integration by parts of the second term yields

   $$\int_\Omega \nabla h(x) \frac{\partial \mathcal{L}}{\partial(\nabla u)} dx = \int_{\partial\Omega} h(x) \frac{\partial \mathcal{L}}{\partial(\nabla u)} ds - \int_\Omega h(x) \text{div}\left(\frac{\partial \mathcal{L}}{\partial(\nabla u)}\right) dx =$$

   $$= -\int_\Omega h(x) \text{div}\left(\frac{\partial \mathcal{L}}{\partial(\nabla u)}\right) dx.$$

   Thus,
   $$\left.\frac{\delta E(u)}{\delta u}\right|_h = \int_\Omega h(x) \left(\frac{\partial \mathcal{L}}{\partial u} - \text{div}\left(\frac{\partial \mathcal{L}}{\partial(\nabla u)}\right)\right) dx \quad \Rightarrow \quad \text{claim}.$$

   (b) Which condition must hold true for a minimizer u_0 of $E(u)$ ...

   - ... in general?

   $$\frac{dE(u)}{du} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{L}(u, \nabla u)}{\partial u} = \text{div}\left(\frac{\partial \mathcal{L}(u, \nabla u)}{\partial(\nabla u)}\right).$$

   - ... if $\mathcal{L}(u, \nabla u) = \mathcal{L}(u)$?

   $$\frac{\partial \mathcal{L}(u)}{\partial u} = 0.$$

- ... if $\mathcal{L}(u, \nabla u) = \mathcal{L}(\nabla u)$?

$$\text{div}\left(\frac{\partial \mathcal{L}(\nabla u)}{\partial (\nabla u)}\right) = 0 \, .$$

2. **Multiview Reconstruction as Shape Optimization**

(a) Write down the Euler-Lagrange equation for the given energy $E(u)$.

The E-L equations are

$$\frac{\mathrm{d}E(u)}{\mathrm{d}u} = -\text{div}\left(\frac{\partial \mathcal{L}(\nabla u)}{\partial (\nabla u)}\right) = 0 \quad \text{with} \quad \mathcal{L}(\nabla u) = \rho |\nabla u|$$

Taking the derivative w.r.t. ∇u gives

$$0 = -\text{div}\left(\rho(x)\frac{\nabla u(x)}{|\nabla u(x)|}\right) = -\left[\nabla \rho(x)^{\top}\frac{\nabla u(x)}{|\nabla u(x)|} + \rho(x)\frac{\nabla^2 u(x) - 1}{|\nabla u(x)|}\right]$$

(b) Write down one gradient descent iteration for $E(u)$.

$$u^{(k+1)}(x) = u^{(k)}(x) - \tau \frac{\mathrm{d}E(u)}{\mathrm{d}u} = u^{(k)}(x) + \tau \left[\nabla \rho(x)^{\top}\frac{\nabla u(x)}{|\nabla u(x)|} + \rho(x)\frac{\nabla^2 u(x) - 1}{|\nabla u(x)|}\right]$$