Jacob's PhD

Master Equation Model

Started May 13th, 2020

1 Markov Chains

A Markov chain is a stochastic model which describes a sequence of states with the condition that the probability of entering a particular state is only dependent on the previous state. For instance, we may have a two state model of a light switch, on or off, or a three state model of a person's moves in the board game "Ticket to Ride" (draw trains, draw destinations, or build).

The Markov chain is represented as a probability vector that evolves in time. Let's take a simple example. Consider we have a system that has three states as described in Figure **??**. The player of Ticket to Ride has three possible actions: draw trains, draw destinations, or build. We assume that the choice of picking any given action is dependent only on the prior action (ignoring the stage of the game).

Consider an arbitrary three state Markov chain with states 1, 2, and 3 as shown in Figure **??**. Then, we can construct a matrix of transition probabilities as:

$$\mathbf{Q} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

Using the notation p_{ij} to indicate to a transition to state i from state j (i.e. j–>i).

Assume an arbitrary initial state $\mathbf{p}(0)$ where the entries of $\mathbf{p}$ are the probabilities of being at a particular state (e.g. one might say the system starts in state 1 and $\mathbf{p}(0) = [1, 0, 0]$). Then, the probability updates according to:

$$\mathbf{p}(t + \tau) = \mathbf{Q}^T \mathbf{p}(t) \tag{1}$$

for some discrete timestep τ.

2 The Master Equation

The master equation is a set of first-order differential equations that describes the time evolution of the probability for the system to occupy a certain discrete state. In our case, the states are the different stations in the TFL network. The system then is the passengers on the network. This can be thought of as a random walk with a large number of walkers on the graph. The probability of occupying station 'k' at a given time is given by the probability of transitioning to station 'k' from station 'j' given the occupation probability of 'j' minus the probability of transitioning to station 'j' from station 'k' given the occupation probability of 'k'; summed over all possible neighbours 'j' of station 'k', i.e.

$$\frac{dP_k}{dt} = \sum_{j \neq k} (A_{kj} P_j - A_{jk} P_k) \tag{2}$$

or, in matrix form.

$$\frac{d\vec{P}}{dt} = \mathbf{A}(t)\vec{P} \tag{3}$$

In the simplest terms, the probability of occupying 'k' is a function of the probability flux into 'k' and the probability flux out of 'k'.

2.1 Developing the Model – Simplest Case

Because our network is connected (there is only one component), we can initialize the model by randomly generating the occupation probability. Our transition matrix will be based off of TFL link load data (NUMBAT 2018MTT Link Load) which estimates the number of passengers travelling along various sections of the tube and rail network. We use the convention A_{ij} indicating the transition probability to station 'i' from station 'j', i.e. the probability to transition to station 'i' given the passenger is at station 'j', i.e. from 'j' to 'i'. Saying this fifty different ways helps me implement it properly the first time.