is the condition for a Markov chain

1. $P(X_n = y \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}) = P(X_n = y \mid X_{n-1} = x_{n-1})$

LHS = RHS ✓+5

$$X_n = \sum_{i=0}^{n} \varepsilon_i = \sum_{i=0}^{n-1} \varepsilon_i + \varepsilon_n = X_{n-1} + \varepsilon_n$$

where $\varepsilon_i = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$

Very well done submission.

$$LHS = P(\varepsilon_n = y - x_{n-1} \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1})$$

ε_n is independent of X_i where $i \in \{0, \ldots, n-1\}$ and the history, $X_0 = x_0, \ldots, X_{n-1} = x_{n-1}$, that is being conditioned has a positive probability.

$$\Rightarrow LHS = P(\varepsilon_n = y - x_{n-1}) = \begin{cases} 1/2 & |y - x_{n-1}| = 1 \\ 0 & |y - x_{n-1}| \neq 1 \end{cases}$$

Then, apply the same transformation of the new random variable, epsilon ε.

$$RHS = P(\varepsilon_n = y - x_{n-1} \mid X_{n-1} = x_{n-1})$$

due to the same rule as independence and positive probability of the conditioned history

$$\Rightarrow RHS = P(\varepsilon_n = y - x_{n-1}) = LHS$$

✓28

□

2a) let states $x, y \in \{0, \dots, N\}$

let $x = y \Rightarrow P(X_n = x \mid X_0 = y) = P(X_n = x \mid X_0 = x)$

$= P(X_n = x \mid X_{n-1} = x+1) \cdot P(X_{n-1} = x+1 \mid X_{n-2} = x$

let $n = 2$ $\quad = P(X_2 = x \mid X_1 = x+1) \cdot P(X_1 = x+1 \mid X_0 = x)$

> 0

For the product of positive probabilities is also positive. ✓
This is the case for when $x < y$ and $x > y$.

let $y > x$ $\quad x + n = y$ where $n \in \{1, \dots, N\}$

$$P(X_n = x \mid X_0 = y) = P(X_n = x \mid X_{n-1} = x+1) \cdot P(X_{n-1} = x+1 \mid X_{n-2} = x+2) \cdots P(X_1 = x+n-1 \mid X_0 = x+n)$$

$$= \prod_{k=0}^{n-1} P(X_{n-k} = x+k \mid X_{n-k-1} = x+1+k)$$

✓

> 0

let $x > y$ $\quad x - n = y$ where $n \in \{1, \dots, N\}$

$$P(X_n = x \mid X_0 = y) = P(X_n = x \mid X_{n-1} = x-1) \cdot P(X_{n-1} = x-1 \mid X_{n-2} = x-2) \cdots P(X_1 = x-n+1 \mid X_0 = x-n)$$

$$= \prod_{k=0}^{n-1} P(X_{n-k} = x-k \mid X_{n-k-1} = x-k-1)$$

> 0 $\quad \square$

✓+10

2b) An aperiodic chain is if all states are aperiodic.

state $x \in S$ is aperiodic if the highest common factor of $\Upsilon(x)$ is equal to 1

~~A counter-example will show that the markov chain is not aperiodic.~~

~~let $x \neq T$~~ $\Rightarrow$ where $\Upsilon(x) = \{n \in \mathbb{N} : p_n(x,x) > 0\}$

$$P(X_n = x \mid X_0 = x) = P(X_n = x \mid X_{n-1} = x+1) \cdot P(X_{n-1} = x+1 \mid X_{n-2} = x)$$

$$= \frac{x+1}{N} \cdot \frac{N-x}{N}$$

> 0

or

$$= P(X_n = x \mid X_{n-1} = x-1) \cdot P(X_{n-1} = x-1 \mid X_{n-2} = x)$$

$$= \frac{N-x+1}{N} \cdot \frac{x}{N}$$

> 0

requires an even, $2n$, number of steps $\Rightarrow$ period is equal to 2.

$$P(X_n = x \mid X_{n-1} = x) = 0 \quad \forall n \in \mathbb{N}.$$

✓ +10

$\square$

2c) stationary distribution is a binomial distribution $Bin(N, \frac{1}{2})$.

$$\Rightarrow \quad \bar{\pi} = P(X_n = N) = \binom{N}{x}\left(\frac{1}{2}\right)^N \quad \text{for } x \in \{0, 1, \dots, N\}$$

$$\Rightarrow \quad \bar{\pi} = \bar{\pi} P$$

where $\bar{\pi} = (\pi_0, \pi_1, \dots, \pi_N)$ describes us reaching state $x \in \{0, \dots, N\}$ from any other state.

$$\bar{\pi} = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \\ \pi_{N-1} \\ \pi_N \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & & & 0 \\ 1/N & 0 & \frac{N-1}{N} & \cdots & & & 0 \\ 0 & 2/N & 0 & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & & & 2/N & 0 \\ \vdots & \vdots & \vdots & & & 0 & 1/N \\ 0 & \cdots & & & & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \\ \pi_{N-1} \\ \pi_N \end{bmatrix} = \begin{bmatrix} \pi_1 \cdot 1/N \\ \pi_0 + \pi_2 \cdot 2/N \\ \pi_1 \cdot \frac{N-1}{N} + \pi_3 \cdot 3/N \\ \vdots \\ \pi_{N-2} \cdot 2/N + \pi_N \\ \pi_{N-1} \cdot 1/N \end{bmatrix}$$

$$\Rightarrow \quad \pi_0 = \pi_1 \cdot 1/N \quad \Rightarrow \quad \pi_1 = N\pi_0$$

$$\pi_1 = \pi_0 + \pi_2 \cdot 2/N \quad \Rightarrow \quad N\pi_0 = \pi_0 + \pi_2 \cdot 2/N$$

$$\Rightarrow \quad \pi_0 (N-1) = \pi_2 \cdot 2/N \quad \Rightarrow \quad \pi_2 = \frac{N(N-1)}{2} \pi_0$$

each Π follows a pattern

$$\Pi_x = \cancel{\frac{N!}{N!(N}} \frac{N!}{(N-x)!\,x!} \Pi_0 \Rightarrow \Pi_x = \binom{N}{x} \Pi_0$$

lastly find Π_0

$$\sum_{x=0}^{N} \Pi_x = 1 \quad : \text{ Law of total probability}$$

$$= \sum_{x=0}^{N} \binom{N}{x} \Pi_0 = 1 \quad : \text{ substitution from above}$$

$\sum_{x=0}^{N} {}^N C_x$ follows the pattern of pascal's triangle and so does $(1+\cancel{x}y)^N$

$$\sum_{x=0}^{N} y^x \, {}^N C_x = {}^N C_0 \, y^0 + {}^N C_{\cancel{1}} \, y^1 + {}^N C_2 \, y^2 + \ldots + {}^N C_N \, y^N$$
$$= (1+y)^N$$

$$\text{if } y=1 \Rightarrow \sum_{x=0}^{N} 1^x \, {}^N C_x = \sum_{x=0}^{N} {}^N C_x = (1+1)^N = (2)^N$$

$$\Rightarrow \text{ substitute } \sum_{x=0}^{N} \binom{N}{x} = 2^N \text{ into } \sum_{\cancel{x}=0}^{N} \binom{N}{x} \Pi_0 = 1$$

$$2^N \cancel{*} \Pi_0 = 1 \Rightarrow \Pi_0 = \frac{1}{2^N} = \left(\frac{1}{2}\right)^N$$

$$\Rightarrow \Pi_x = \binom{N}{x} \left(\frac{1}{2}\right)^N \text{ where } x \in \{0, 1, \ldots, N\}$$

$\square$

✓ +30

3.)

Directed graph: