2nd Summative Problem Sheet

Deadline: Friday, October 27, 5pm

1. Let X_n be the position after n steps of a particle that performs an unbiased random walk on $\mathbb{Z}$ starting at 0. Show that $(X_n^2 - n)_{n\geq 0}$ is a martingale with respect to $(X_n)_{n\geq 0}$.

2. Let X_n be as above. Show that $\mathbb{P}(|X_n| \geq t) \leq 2e^{-\frac{t^2}{2n}}$.

3. Consider a branching process starting with one individual, where each individual gives birth to a number of offspring which is distributed as the random variable Z that has $\mathbb{E}(Z) < 1$. If Z_n denotes the size of the nth generation, show that $\lim_{n\to\infty} \mathbb{P}(Z_n > 0) = 0$.

4. Let X_n be the position after n steps of a particle that performs a *biased* random walk on $\mathbb{Z}$ starting at 0, which at each steps goes up (by $+1$) with probability $0 < p < 1$ and down (by -1) with probability $q = 1 - p$, independently of any other step. Show that the sequence $(Y_n)_{n\geq 0}$, where $Y_n = (q/p)^{X_n}$ is a martingale with respect to $(X_n)_{n\geq 0}$.