Part 4 Summative Assessment

1. (40 points)Let B_t be a standard Brownian motion.

   (a) show that

   $$\int_0^t B_s ds = \int_0^t (t-s) dB_s$$

   and then prove that $\int_0^t B_s ds \sim \mathcal{N}\Big(0, \frac{t^3}{3}\Big)$.

   (b) Is

   $$X_t = \begin{cases} 0 & t = 0 \\ \dfrac{\sqrt{3}}{t} \displaystyle\int_0^t B_s ds & t > 0 \end{cases}$$

   a standard Brownian motion? Justify you assertion.

2. (60 points) Let B_t be a standard Brownian motion. Suppose X_t follows the Brownian bridge process with SDE

   $$\begin{cases} dX_t = \frac{y - X_t}{1-t} dt + dB_t \\ X_1 = y \end{cases}$$

   where the end value of X_t at $t = 1$ is y.

   (a) Show that under the condition $X_0 = x$, we have for $0 \leqslant t < 1$,

   $$X_t = yt + (1-t)\left(x + \int_0^t \frac{1}{1-s} dB_s\right)$$

   (b) Using the above expression, find the mean and variance of X_t, given $X_0 = x$.

   (c) Show X_t follows a normal distribution.