Stochastic Processes : Part 4 Summative Assessment

Q1. Let B_t be a standard Brownian motion

a) Show that

$$\int_0^t B_s \, ds = \int_0^t (t-s) \, dB_s$$

Integration by parts:

$$\int_0^t v \, du = uv \Big|_0^t - \int_0^t u \, dv$$

where $u = s$ $\quad v = B_s$
$du = ds$ $\quad dv = dB_s$

Substitute:

$$\int_0^t B_s \, ds = [s \cdot B_s]_0^t - \int_0^t s \, dB_s$$

$$= [t \cdot B_t - 0 \cdot B_0] - \int_0^t s \, dB_s$$

$$= t \cdot B_t - \int_0^t s \, dB_s$$

$$= t \cdot [B_t - B_0] - \int_0^t s \, dB_s$$

Since $B_t - B_0 = B_t$ where $B_0 = 0$ due to B_t being a Standard Brownian motion.

now using the below calculus theorem

$$\int_a^b df(x) = f(b) - f(a)$$

where in this specific case $b = t$, $a = 0$ and $f(x) = B_s$

$$\int_0^t dB_s = B_t - B_0$$

Substituting this result in this previous equation

$$\int_0^t B_s \, ds = t \cdot \int_0^t dB_s - \int_0^t s \, dB_s$$

$$= \int_0^t t \cdot dB_s - \int_0^t s \, dB_s$$

$$= \int_0^t (t - s) \, dB_s$$

$\square$

as required

Now, prove that

$$\int_0^t B_s \, ds \sim N(0, t^3/3)$$

$$E\left[\int_0^t B_s \, ds\right] = E\left[\int_0^t (t-s) \, dB_s\right]$$

The integral within the expectation is an Itó integral

An Itó integral takes the form

$$I(t) = \int_0^t M(s)\, dW(s)$$

where $dW(s)$ is a Brownian motion with a filtration $\{F(s), s \geq 0\}$.

Also, $M(s)$ is the integrand. It can be an adapted stochastic process, but in this case $M(s)$ is a deterministic integrand.

$$M(s) = t - s$$

Furthermore $dW(s) = dB_s$ in this case

Theorem 1.21 in the lecture notes states that with these above definitions, the Itó integral, $I(t)$, is normally distributed with expected value zero and variance $\int_0^t M^2(s)\, ds$.

$$\Rightarrow E\left[\int_0^t B_s\, ds\right] = E\left[\int_0^t (t-s)\, dB_s\right]$$

$$= E[\,I(t)\,]$$

$$= \cancel{t}\,0$$

the variance :

$$Var\left[\int_0^t B_s\, ds\right] = Var\left[\int_0^t (t-s)\, dB_s\right]$$

$$= Var[\, I(t)\,]$$

$$= E[\, I^2(t)\,] \quad \text{since } E[I(t)]=0$$

$$= \int_0^t M^2(s)\, ds$$

$$\int_0^t M^2(s)\, ds = \int_0^t (t-s)^2\, ds$$

$$= \int_0^t (t^2 - 2ts + s^2)\, ds$$

$$= \left[\, t^2 s - t s^2 + s^3/3 \,\right]_0^t$$

$$= \left(t^3 - t^3 + t^3/3 \right) - 0$$

$$Var\left[\int_0^t B_s ds\right] = t^3/3$$

and so $\int_0^t B_s\, ds \sim N(0, t^3/3)$ □

as required

✓+20

b) Is $X_t = \begin{cases} 0 & t=0 \\ \sqrt{3}/t \int_0^t B_s ds & t>0 \end{cases}$

a standard Brownian motion?

I will use the Martingale Characterisation of a Brownian motion to see whether X_t is a B.M.

Martingale Characterisation of a Brownian motion:

Let X_t be a stochastic process. X_t is a Brownian motion iff:

$\tilde{a}$) X_t is a martingale

$\tilde{b}$) $\forall t \geq 0$, $\langle X \rangle_t = \langle X, X \rangle_t = t$

$\tilde{c}$) X_t is a continuous process

$\tilde{d}$) $X_0 = 0$

First to check $\tilde{a}$)

In question 1a we have shown that the integral $\int_0^t B_s ds$ is an Ito integral.

From theorem 1.4 property (iii)

$$c I(t) = \int_0^t c M(u) dW(u)$$

In this case $c = \sqrt{3}/t$

$\Rightarrow \quad \sqrt{3}/t \, I(t) = \int_0^t \sqrt{3}/t \,(t-s)\, dB_s$

$\Rightarrow \quad I_2(t) = \int_0^t \sqrt{3}/t \,(t-s)\, dB_s$

where $M(s) = \sqrt{3}/t \,(t-s)$ is a deterministic integrand

and dB_s is a Brownian motion

So $I_2(t)$ is an Itô integral and from theorem 1.4 property (iv), ~~this~~ the itô integral is a martingale.

$\Rightarrow$ X_t is a martingale and $\tilde{a})$ is satisfied

$\tilde{b})$ $X_t = I_2(t)$ from above. therefore X_t is an itô integral

Therefore using property (vi) from theorem 1.4 the quadratic variation of X_t is:

$$\langle X \rangle_t = \langle X, X \rangle_t = \int_0^t M^2(u)\, du$$

$$= \int_0^t \left(\frac{\sqrt{3}}{t}(t-u)\right)^2 du$$

$$= \int_0^t \frac{3}{t^2}(t-u)^2\, du$$

$$= \int_0^t \frac{3}{t^2}(t^2 - 2tu + u^2)\, du$$

$$= \frac{3}{t^2}\left[t^2 u - \cancel{2}\,tu^2 + \frac{u^3}{3} \right]_0^t$$

$$= \frac{3}{t^2}\left(\left[t^3 - t^3 + \frac{t^3}{3} \right] - 0\right)$$

$$= \frac{3}{t^2}\left(\frac{t^3}{3} \right)$$

$$= t$$

so $\forall t \geq 0$ $\langle X \rangle_t = t$ and $\tilde{b})$ is satissfied

$\tilde{c})$ From above, we have shown that X_t is an ~~integral~~ Ito' integral.

Thus, from theorem 1.4 property (i), ~~X_t~~ X_t is continuous, since the paths of an Ito' integral $I(t)$ are continuous.

so $\tilde{c})$ is satissfied

$\tilde{d})$ when $t=0$ $X_t = 0$ from the definition of X_t.

Thus, $\tilde{d})$ is satissfied

~~All four~~ X_t satissfies all four properties of the Martingale characterisation of a Brownian motion.

Therefore X_t is a Brownian motion.

X +10

Q2. Let B_t be a standard Brownian motion.

Suppose X_t follows the Brownian bridge process with SDE

$$\begin{cases} dX_t = \frac{y - X_t}{1-t}\, dt + dB_t \\ X_1 = y \end{cases}$$

where the end value of X_t at $t=1$ is y

a) Show that under the condition $X_0 = x$ and for $0 \leq t < 1$,

$$X_t = yt + (1-t)\left(x + \int_0^t \frac{1}{1-s}\, dB_s\right)$$

$$dX_t = \frac{y - X_t}{1-t}\, dt + dB_t \quad X_1 = y,\ X_0 = x,\ 0 \leq t < 1$$

$$= \frac{y}{1-t}\, dt - \frac{X_t\, dt}{1-t} + dB_t$$

$$dX_t + \frac{X_t}{1-t}\, dt = \frac{y}{1-t}\, dt + dB_t$$

divide through by $1-t$

$$\frac{dX_t}{1-t} + \frac{X_t}{(1-t)^2}\, dt = \frac{y}{(1-t)^2}\, dt + \frac{dB_t}{(1-t)}$$

The first two terms on the left hand side are a result of a chain rule, specifically.

$$d\left(\frac{1}{1-t}X_t\right) = \frac{1}{1-t}dX_t + \frac{X_t}{(1-t)^2}dt$$

$$\Rightarrow \quad d\left(\frac{1}{1-t}X_t\right) = \frac{y}{(1-t)^2}dt + \frac{dB_t}{(1-t)}$$

integrate:

$$\int_0^t d\left(\frac{1}{1-u}X_u\right) = y\int_0^t \frac{1}{(1-u)^2}du + \int_0^t \frac{1}{1-u}dB_u$$

$$\frac{1}{1-t}X_t - X_0 = y\left(\frac{1}{1-u}\Big|_0^t\right) + \int_0^t \frac{1}{1-u}dB_u$$

$$\frac{1}{1-t}X_t - x = y\left(\frac{1}{1-t} - 1\right) + \int_0^t \frac{1}{1-u}dB_u$$

$$\frac{1}{1-t}X_t = x + y\left(\frac{t}{1-t}\right) + \int_0^t \frac{1}{1-u}dB_u$$

$$X_t = yt + (1-t)\left(x + \int_0^t \frac{1}{1-s}dB_s\right)$$

✓+20

b) Using the above expression, find the mean and variance of X_t, given $X_0 = x$.

First expand the equation of X_t.

$$X_t = yt + x - tx + (1-t)\left(\int_0^t \frac{1}{1-s}\, dB_s\right)$$

$$\Rightarrow E[X_t] = E\left[yt + x - tx + (1-t)\left(\int_0^t \frac{1}{1-s}\, dB_s\right)\right]$$

$$= E[x] + E[t(y-x)] + E\left[(1-t)\left(\int_0^t \frac{1}{1-s}\, dB_s\right)\right]$$

The last term inside the expectation is an Ito' Integral

$$I(t) = \int_0^t \frac{1-t}{1-s}\, dB_s$$

where $\frac{1-t}{1-s} = M(s)$ is a deterministic integrand and dB_s is a Brownian motion.

From theorem 1.21 in the lecture notes this ito' integral has a mean of 0.

$$\Rightarrow E[X_t] = E[x] + E[(y-x)t] + E[I(t)]$$
$$= E[x] + E[(y-x)t] + 0$$

and since x and $(y-x)t$ are constants, expectation can be removed.

$$\Rightarrow E[X_t] = x + (y-x)t$$

$$Var[X_t] = Var\left[x + (y-x)t + (1-t)\int_0^t \frac{1}{1-s} dB_s \right]$$

Since $Var[a+b+cX] = c^2 Var[X]$, then

$$Var[X_t] = (1-t)^2 \, Var\left[\int_0^t \frac{1}{1-s} dB_s \right]$$

The integral within the variance is also an Ito' integral with a deterministic integrand, $M(s) = 1/_{1-s}$ and dB_s being a Brownian motion.

From theorem 1.21 in the lecture notes, an Ito' integral with a deterministic integrand has a variance:

$$Var[I(t)] = E[I^2(t)]$$
$$= \int_0^t M^2(s)\, ds$$

$$\Rightarrow Var[X_t] = (1-t)^2 \, Var[I(t)]$$
$$= (1-t)^2 \int_0^t M^2(s)\, ds$$
$$= (1-t)^2 \int_0^t \frac{1}{(1-s)^2} ds$$
$$= (1-t)^2 \left. \frac{1}{1-s} \right|_{s=0}^{t}$$
$$= (1-t)^2 \left(\frac{1}{1-t} - 1 \right)$$
$$= t(1-t)$$

✓+20

c) Show X_t follows a normal distribution

$$X_t = x + (y-x)t + (1-t)\int_0^t \frac{1}{1-s}\, dB_s$$

As we have shown in part b, the last term is an Itó integral with a deterministic integrand.

From theorem 1.21, ~~An~~ an Itó integral with a deterministic integrand ~~to~~ follows a normal distribution.

The addition of terms ~~s~~, x and $(y-x)t$ to the normally distributed Itó integral still make X_t follow a normal distribution.

The only randomness of X_t is from the Itó integral which follows a normal distribution.

The prescence of a deterministic term $(y-x)t$ does not ~~alter~~ affect the distribution of X_t.

This is because the definition of a stochastic process X_t following a normal distribution is for each t, X_t follows a normal distribution.

This is true for $X_t = x + (y-x)t + (1-t)\int_0^t \frac{1}{1-s}\, dB_s$

and therefore X_t follows a normal distribution $\square$

✓ +18