Problem Sheet 4

1. X_n be the position after n steps of a particle that performs an unbiased random walk on $\mathbb{Z}$ starting at 0.

show that $(X_n^2 - n)_{n\geq 0}$ is a martingale w.r.t $(X_n)_{n\geq 0}$

define $(Y_n)_{n\geq 0} = (X_n^2 - n)_{n\geq 0}$. $(Y_n)_{n\geq 0}$ is a martingale if the following three conditions hold.

i) $E(|Y_n|) < \infty$ — This is great!

$E(|X_n^2 - n|) \leq E(|X_n^2|) + E(|n|)$: triangle inequality

$= E(X_n^2) + E(n)$: X_n^2 and $n \geq 0$ $\therefore$ no need for the modulus

$= E(X_n^2) + n$: expectation of a constant is a constant

$\max(X_n^2) = n^2$

$\therefore \quad \ldots \leq n^2 + n$

$= E((\sum_{i=1}^n \xi_i)^2) + n$: $X_n = \sum_{i=1}^n \xi_i$ where $\xi_i = \pm 1$ w.p. $1/2$

$= E(\underbrace{\sum_{i=1}^n \xi_i^2}_{①} + \underbrace{\sum_{i\neq j} \xi_i \xi_j}_{②}) + n$

① represents all times where both ξ's are equal to 1 or -1
② represents all times where one $\xi = 1$ and the other $\xi = -1$

$= \sum_{i=1}^n E(\xi_i^2) + \sum_{i\neq j} E(\xi_i \xi_j) + n$

$= \sum_{i=1}^n 1 + \sum_{i\neq j} 0 + n$

where $E(\xi_i^2) = (1)^2(1/2) + (-1)^2(1/2) = 1$

and $E(\xi_i \xi_j) = \underbrace{E(\xi_i)E(\xi_j)}_{\text{independence}} = ((1)(1/2) + (-1)(1/2))((-1)(1/2) + (1)(1/2)) = 0$

✓ +10

$= n + n \leq 2n < \infty$ □

ii) $E(Y_n \mid X_{n-1}, \dots, X_0) = Y_{n-1}$

$E(X_n^2 - n \mid X_{n-1}, \dots, X_0) = \underbrace{E(X_n^2 \mid X_{n-1}, \dots, X_0) - E(n \mid X_{n-1}, \dots, X_0)}_{\text{linearity}}$:

$X_n^2 = (X_{n-1} + \xi_n)^2 \Rightarrow \quad = E((X_{n-1} + \xi_n)^2 \mid X_{n-1}, \dots, X_0) - n$: independence and expectation of a constant

now expand the quadratic

$= \underbrace{E(X_{n-1}^2 \mid X_{n-1}, \dots, X_0)}_{\text{measurable}} + 2E(\underset{\text{measurable}}{X_{n-1}} \underset{\text{independent}}{\xi_n} \mid X_{n-1}, \dots, X_0) + \underbrace{E(\xi_n^2 \mid X_{n-1}, \dots, X_0)}_{\text{independent}} - n$

$= X_{n-1}^2 + 2X_{n-1}E(\xi_n) + E(\xi_n^2) - n$

from i) we know that $E(\xi_n) = 0$ and $E(\xi_n^2) = 1$

$\Rightarrow \quad = X_{n-1}^2 + 2X_{n-1}(0) + 1 - n$

$= X_{n-1}^2 - n + 1$

$= X_{n-1}^2 - (n-1)$

$= Y_{n-1}$ $\square$ ✓+14

iii) $Y_n = X_n^2 - n$ is adapted to $(X_n)_{n \in \mathbb{N}_0}$ because Y_n is a function of X_n. $\square$

The three conditions hold, and therefore $(Y_n)_{n \geq 0} = (X_n^2 - n)_{n \geq 0}$ is a martingale w.r.t $(X_n)_{n \geq 0}$ ✓+5

2. X_n represents an unbiased random walk and is the same as the random variable X_n seen in the first question.

Show that $P(|X_n| \geq t) \leq 2e^{-\frac{t^2}{2n}}$

We will use Hoeffding's inequality.

The Hoeffding inequality states that a martingale $(X_n)_{n \in \mathbb{N}_0}$ satisfies the bounded difference condition, if for any $n \geq 1$

$$P(|X_n - X_{n-1}| \leq k_n) = 1 \quad \text{for some } k_n > 0$$

$$P(|X_n - X_0| \geq x) \leq 2e^{-\frac{1}{2} \cdot \frac{x^2}{\sum_{i=1}^{n} k_i^2}}$$

If X_n represents an unbiased random walk: $X_n = \begin{cases} X_{n-1}+1 & \text{w.p. } \frac{1}{2} \\ X_{n-1}-1 & \text{w.p. } \frac{1}{2} \end{cases}$ ~~where~~ $\cancel{X_0 = 0}$ starting at 0 then:

i) we know that this X_n is a martingale (proven in lecture notes)

✓+5

ii) $|X_n - X_{n-1}| = 1 = k_n \Rightarrow P(|X_n - X_{n-1}| = 1) = 1 \quad \forall n \geq 1$

$\Rightarrow$ the bounded difference condition is satisfied

iii) $X_0 = 0$ ✓+5

Using Hoeffding's inequality, setting $x = t$ and substituting the above properties:

$$P(|X_n - 0| \geq t) \leq 2 \cdot e^{-\frac{1}{2} \frac{t^2}{\sum_{i=1}^{n} 1^2}}$$

$$\Rightarrow P(|X_n| \geq t) \leq 2 \cdot e^{-\frac{t^2}{2n}}$$ ✓+5

Since $\sum_{i=1}^{n} 1^2 = \sum_{i=1}^{n} 1 = n$ $\square$

3. Z_n : size of n^{th} generation ; $Z_0=1$; $E(Z)<1$

show that $\lim_{n\to\infty} P(Z_n>0)=0$

$$P(Z_n>0) = P(Z_n \geq 1)$$
$$\leq E(Z_n)$$: markov inequality ; $P(X\geq \epsilon) \leq \frac{E(X)}{\epsilon}$

$(Z_n)_{n\geq 0}$ is <u>not</u> a martingale, however, we ~~can~~ rescale.

Let $W_n=\frac{Z_n}{\mu^n}$, where $(W_n)_{n\geq 0}$ is a martingale w.r.t $(Z_n)_{n\geq 0}$ (proven in lecture notes)

Since $(W_n)_{n\geq 0}$ is a martingale, $E(W_n)=E(W_0)$ $\forall\, n\geq 0$

$$\Rightarrow E\left(\frac{Z_n}{\mu^n}\right)=E\left(\frac{Z_0}{\mu^0}\right)=E(Z_0)=1$$

$$E\left(\frac{Z_n}{\mu^n}\right)=1 \Rightarrow E(Z_n)\,{}^1/_{\mu^n}=1$$
$$E(Z_n)=\mu^n \quad (1)$$

Substitute (1) into the inequality above.

$$P(Z_n>0)\leq \mu^n$$

take the limit of both sides

$$\lim_{n\to\infty} P(Z_n>0) \leq \lim_{n\to\infty} \mu^n$$

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if $E(Z_n)<1$ $\forall n$ then the $\lim_{n\to\infty}\mu^n=0$

$$\lim_{n\to\infty} P(Z_n>0)\leq 0 \Rightarrow \lim_{n\to\infty} P(Z_n>0)=0$$

a probability cannot be less than zero $\square$

very nice.

4.) X_n : biased random walk $X_n = \begin{cases} X_{n-1}+1 & \text{w.p. } p \\ X_{n-1}-1 & \text{w.p. } q=1-p \end{cases}$

where $0<p<1$.

Show that ~~the~~ $(Y_n)_{n\geq 0}$, where $Y_n = (q/p)^{X_n}$ is a martingale with respect to $(X_n)_{n\geq 0}$.

~~Let~~ ~~X_n~~ define $X_n = \sum_{i=1}^{n} \xi_i$ where $\xi_i = \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } q=1-p \end{cases}$

the following three conditions must hold for $(Y_n)_{n\geq 0}$ to be a martingale w.r.t $(X_n)_{n\geq 0}$

i) $E(|Y_n|) < \infty$

$$E(|Y_n|) = E(|(q/p)^{X_n}|)$$

$$= E((q/p)^{X_n})$$ if $0<p<1$ then $q/p > 0$ and $(q/p)^{X_n} \geq 1$ $\forall X_n \in \mathbb{Z}$

0?

$$= E((q/p)^{\xi_1+\xi_2+\dots+\xi_n})$$

$$= E((q/p)^{\xi_1}(q/p)^{\xi_2}\cdots(q/p)^{\xi_n})$$

$$= E((q/p)^{\xi_1}) \cdot E((q/p)^{\xi_2}) \cdots E((q/p)^{\xi_n})$$

$$E((q/p)^{\xi_i}) = (q/p)^{(1)}(p) + (q/p)^{(-1)}(q)$$

$$= \frac{q}{p}\cdot p + \frac{p}{q}\cdot q$$

$$= q + p$$

$$= 1$$

$$\Rightarrow E(|Y_n|) = \prod_{i=1}^{n} E((q/p)^{\xi_i}) = \prod_{i=1}^{n} 1 = 1 < \infty \quad \square$$

ii) $E(Y_n | X_{n-1}, \dots, X_0) = Y_{n-1}$

$$E((q/p)^{X_n} | X_{n-1}, \dots, X_0) = E\left((q/p)^{\sum_{i=1}^{n} \xi_i} \middle| X_{n-1}, \dots, X_0 \right)$$

$$= E\left((q/p)^{\xi_n + \sum_{i=1}^{n-1} \xi_i} \middle| X_{n-1}, \dots, X_0 \right)$$

$$= E\left((q/p)^{\xi_n} (q/p)^{\sum_{i=1}^{n-1} \xi_i} \middle| X_{n-1}, \dots, X_0 \right)$$

$$= \underbrace{E((q/p)^{\xi_n} | X_{n-1}, \dots, X_0)}_{\text{independence}} \underbrace{E\left((q/p)^{\sum_{i=1}^{n-1} \xi_i} \middle| X_{n-1}, \dots, X_0 \right)}_{\text{measurable}}$$

$$= E((q/p)^{\xi_n}) \cdot (q/p)^{\sum_{i=1}^{n-1} \xi_i}$$

from part i) we have shown that $E((q/p)^{\xi_i}) = 1$

$$\Rightarrow E(Y_n | X_{n-1}, \dots, X_0) = 1 \cdot (q/p)^{\sum_{i=1}^{n-1} \xi_i}$$

$$= (q/p)^{X_{n-1}}$$

$$= Y_{n-1} \quad \square$$

iii) $Y_n = (q/p)^{X_n}$ is adapted to $(X_n)_{n \in \mathbb{N}_0}$, because Y_n is a function of X_n.

The three conditions hold and therefore $(Y_n)_{n \in \mathbb{N}_0}$ is a martingale w.r.t $(X_n)_{n \in \mathbb{N}_0}$. $\square$

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