$\boxed{8\cancel{0}4}$ $-10 = 74$

Stochastic Process Problem Sheet 3

Question 1a. ~~14~~ 15

Let $B = (B_t)_{t \in \mathbb{R}_+}$ be a stochastic process on $\mathbb{R}_+$, valued in $\mathbb{R}$.

Definition of a standard Brownian motion is

B is said to be a Brownian motion iff:

a) B has independent increments:

$\forall\ t < u$ in $\mathbb{R}_+$, $B_u - B_t$ independent of $(B_s)_{s \leq t}$

(i.e. : $\forall n \in \mathbb{N}^*$, $\forall\ 0.t_0 < t_1 < t_2 < \dots t_n$ in $\mathbb{R}_+$,

5 the RVs $B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}$ are ✓✓ independent).

b) $\forall\ s < t$ in $\mathbb{R}_+$, $B_t - B_s \sim N(0, t-s)$ ✓

c) With probability 1, the trajectory $t \to B_t$ is continuous ✓

d) $B_0 = 0$ ✓

Question 1b

5

A stochastic process B is a Brownian motion iff:

~~4~~

$\bar{a}$) B is a Gaussian process ✓

$\bar{b}$) $\mu_B = 0$ ✓ and $\forall s,t \geq 0$, $C_B(s,t) = \min(s,t)$ ✓

$\bar{c}$) With probability 1, ✓ the trajectories of B are continuous.

Question 1c

Let B be a stochastic process. B is Brownian motion iff:

5

$\tilde{a}$) B is a martingale ✓

$\tilde{b}$) $\forall t \geq 0$, $\langle B \rangle_t = \langle B, B \rangle_t = t$ ✓✓

$\tilde{c}$) B is a continuous process ✓

$\tilde{d}$) $B_0 = 0$ ✓

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18 ~~15~~ -5 (formatting)

Question 2

Let $\alpha > 0$ and B^1, B^2 be two independent Brownian motions.

$W = \alpha B^1 + \sqrt{1-\alpha^2}\, B^2$ is a Brownian motion

if ~~it~~ W satisfies the four ~~as~~ properties in the definition stated in question 1a.

(4) a) show that W has independent increments:

i.e. $\forall\ 0 \le t_1 \le t_2 \le \dots \le t_n$

$W(t_1), W(t_2) - W(t_1), \quad \dots, \quad W(t_n) - W(t_{n-1})$

are independent random variables.

Substitute ~~each~~ the formula into each term.

1^{st} term: $\quad \alpha B^1(t_1) + \sqrt{1-\alpha^2}\, B^2(t_1)$

2^{nd} term: $\quad \alpha B^1(t_2) + \sqrt{1-\alpha^2}\, B^2(t_2) - \alpha B^1(t_1) - \sqrt{1-\alpha^2}\, B^2(t_1)$

$\vdots$

n^{th} term: $\quad \alpha B^1(t_n) + \sqrt{1-\alpha^2}\, B^2(t_n) - \alpha B^1(t_{n-1}) - \sqrt{1-\alpha^2}\, B^2(t_{n-1})$

rearrange each term as follows:

1^{st} term: $\quad \alpha B^1(t_1) + \sqrt{1-\alpha^2}\, B^2(t_1)$

2^{nd} term: $\alpha(B^1(t_2) - B^1(t_1)) + \sqrt{1-\alpha^2}\,(B^2(t_2) - B^2(t_1))$

$\vdots$

n^{th} term : $\alpha(B^1(t_n)-B^1(t_{n-1}))+\sqrt{1-\alpha^2}(B^2(t_n)-B^2(t_{n-1}))$

~~Each~~ increments : $B^1(t_1), B^1(t_2)-B^1(t_1), \ldots, B^1(t_n)-B^1(t_{n-1})$
$B^2(t_1), B^2(t_2)-B^2(t_1), \ldots, B^2(t_n)-B^2(t_{n-1})$

are independent random variables because of part a of the definition of a brownian motion

Each increments are multiplied by constants, either by α or $\sqrt{1-\alpha^2}$. This is simply the rescaling property.

Therefore, these rescaled increments are also independent random variables. Also because B^1 and B^2 are independent of each other.

From the 1^{st} term to the n^{th} term, the increments are independent random variables, as required

$\square$

Property of 1(b).

(4) b) ~~Show that $\forall s \le t$ in $\mathbb{R}_+$ $W_t - W_s \sim N(0, t-s)$~~

~~$E[W_t - W_s] =$~~

Show that $\forall t \geq 0$ and $h \geq 0$ the increments are normally distributed with zero mean and variance h

X • Gaussian?

/ • $E[W(t+h)-W(t)] = E[W(t+h)] - E[W(t)]$

since independence due to gaussian random vector

$$= E[\alpha B^1(t+h) + \sqrt{1-\alpha^2} B^2(t+h)]$$
$$- E[\alpha B^1(t) + \sqrt{1-\alpha^2} B^2(t)]$$

$$= \alpha E[B^1(t+h)] + \sqrt{1-\alpha^2} E[B^2(t+h)]$$
$$- \alpha E[B^1(t)] + \sqrt{1-\alpha^2} E[B^2(t)]$$

since $\forall t \geq 0 \quad B^1_t \sim N(0,t)$ and $B^2_t \sim N(0,t)$

$$\Rightarrow \quad E[B^1_t] = E[B^2_t] = 0$$

$$\Rightarrow E[W(t+h) - W(t)] = \alpha(0) + \sqrt{1-\alpha^2}(0) - \alpha(0)$$
$$- \sqrt{1-\alpha^2}(0)$$
$$= 0$$

$$\cancel{Var[W(t+h) - W(t)] = Cov[W(t+h) - W(t), W(t+h) - W(t)]}$$
$$\cancel{= C_W(t+h, t+h)}$$

$$Var[W(t+h) - W(t)] = Var[\alpha(B^1(t+h) - B^2(t))$$
?
$$+ \sqrt{1-\alpha^2}(B^1(t+h) - B^2(t))]$$
$$= \alpha^2 Var[B^1(t+h) - B^2(t)] + (1-\alpha^2) Var[B^1(t+h) - B^2(t)]$$

$$= Var[B^1(t+h) - B^2(t)]$$

$$= C_{B^1}(t+h, t+h) - 2C_{B^1B^2}(t+h, t) + C_{B^2}(t,t)$$

$$= t+h \quad -2t \quad +t \qquad \text{since } C_B(s,t) = \min(s,t)$$
$$= h$$

as required $\square$

(5) c) almost surely (i.e, with probability 1)

the function $t \to B(t)$ is continuous.

if $\alpha \in \mathbb{R}^+$ and $\sqrt{1-\alpha^2} \in \mathbb{R}$, then the multiplication of each to continuous function of Brownian motion

then αB^1 and $\sqrt{1-\alpha^2} B^2$ is continuous

The addition of two continuous functions is also a continuous function.

These follow from the properties of composition of continuous functions.

Thus W_t is continuous, as required $\square$.

(5) d) $W(0)=0$

if B^1 and B^2 are two independent Brownian motions then

$$B^1(0)=0 \quad \text{and} \quad B^2(0)=0$$

$$\Rightarrow W(0) = \alpha B^1(0) + \sqrt{1-\alpha^2} B^2(0)$$
$$= 0 + 0$$
$$= 0$$

as required. $\square$

The four properties are satisfied, therefore

W is ~~a~~ a Brownian motion

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51 Question 3a -5 (formatting)

Let $X_0, a \in \mathbb{R}$ $b, T \in \mathbb{R}_+^*$ and let W be a Brownian motion.

$\forall t \in \mathbb{R}_+$ $X_t = X_0 + at + bW_t$

$X_0 = 100$, $a = -15$, $b = 3$, $T = 4$

a) $$\begin{aligned} E[X_T] &= E[X_4] \\ &= E[X_0 + 4a + bW_4] \\ &= E[X_0] + E[4a] + E[bW_4] \\ &= X_0 + 4a + bE[W_4] \\ &= 100 + 4\times -15 + 3\times 0 \end{aligned}$$

Since $W_t \sim N(0,t)$ $\forall t \geq 0$

$$\begin{aligned} &= 100 - 60 \\ &= 40 \end{aligned}$$

✓7

Question 3b

$$\begin{aligned} Var[X_T] &= Var[X_4] \\ &= Var[X_0 + 4a + bW_4] \\ &= Var[bW_4] \end{aligned}$$

Since $Var[X] = Var[X+c]$ where c is a constant.

$$Var[X_4] = b^2 Var[W_4]$$
$$= b^2 t$$
$$= b^2 4$$
$$= 3^2 4$$
$$= 36$$

✓ 7

Question 3c

$P(X_T \in [L, U])$ with $L = 30$ $U = 35$

$$P(X_4 \in [30, 35]) = P(30 \leq X_4 \leq 35)$$
$$= P(30 \leq X_0 + 4a + bW_4 \leq 35)$$
$$= P(-10 \leq bW_4 \leq -5)$$
$$= P(-10/3 \leq W_4 \leq -5/3)$$
$$= P(W_4 \leq -5/3) - P(W_4 \leq -10/3)$$

~~since $W_4 \sim N(0,4)$~~

if $W_4 \sim N(0,4)$, then we can transform that using the standard normal $Z \sim N(0,1)$

$W_4 = \sigma Z + \mu$ where $W_4 \sim N(\mu, \sigma^2)$

$$= P(\sigma Z + \mu \leq -5/3) - P(\sigma Z + \mu \leq -10/3)$$

$$= P\left(Z \leq \frac{-5/3 - \mu}{\sigma}\right) - P\left(Z \leq \frac{-10/3 - \mu}{\sigma}\right)$$

where $\mu = 0$ and $\sigma = 2$

$$\Rightarrow = P(Z \leq -5/6) - P(Z \leq -10/6)$$

$= 0.2033 - 0.0475$ ✓7

$= 0.1558$

$= 0.16$ (using the standard normal table)

Question 3d

$X_t = X_0 + at + bW_t$ is comprised of a random variable W_t that is gaussian with

$E[X_t] = X_0 + at$

$Var[X_t] = b^2 t$

$X_t \sim N(X_0 + at, b^2 t)$ ✓10

$$f(x) = \frac{1}{\sqrt{2\pi b^2 t}} \cdot \exp\left(-\frac{(x-(X_0+at))^2}{2b^2 t}\right)$$

where the pdf of the normal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Question 3e

~~X is Gaussi~~ Let $X = (X_{t_1}, X_{t_2}, \ldots, X_{t_n})$

X is a Gaussian random vector if and only if

$\forall \lambda_{t_i} = (\lambda_{t_1}, \lambda_{t_2}, \dots \lambda_{t_n}) \in \mathbb{R}^n$

$\langle \lambda | X \rangle = \sum \lambda_i X_i$ is a Gaussian random variable

$$= \sum \underbrace{\lambda_i X_0}_{\text{constant}} + \underbrace{\lambda_i a t_i}_{\text{constant}} - \underbrace{b \lambda_i W_{t_i}}_{\text{Gaussian RV}}$$

two constant terms added to a GRV is also a Gaussian RV.

Therefore X_t is gaussian random vector, thus a gaussian process. ✓10

Question 3 5

For X to be a martingale it must satisfy three conditions.

i) X_t is adapted to the filtration $\mathcal{F}_s$.

X_t is a function of a random process W_t, thus $\mathcal{F}_s$ measurable

ii) X_t is integrable $E[|X_t|] < \infty \quad \forall t$

iii) Let $s \leq t$ in $\mathbb{R}_+$

$$E[X_t | \mathcal{F}_s] = X_s$$
$$= E[X_s + X_t - X_s | \mathcal{F}_s]$$
$$= E[X_s | \mathcal{F}_s] + \underbrace{E[X_t - X_s | \mathcal{F}_s]}_{=0}$$
$$= X_s$$

inorder for the martingale property to be satisfied

$$E[X_t - X_s | \mathcal{F}_s] = 0$$
$$= E[X_0 + at + bW_t - X_0 - as - bW_s | \mathcal{F}_s]$$
$$= E[at + bW_t - as\, bW_s | \mathcal{F}_s]$$
$$= E[at - as] + E[bW_t - bW_s | \mathcal{F}_s]$$
$$= a(t-s) + b\,E[W_t - W_s | \mathcal{F}_s]$$
$$= a(t-s) + b\,E[W_t - W_s]$$

$E[W_t - W_s] = 0$ from the definition of a Brownian motion.

Thus $a = 0$ inorder for $E[X_t - X_s | \mathcal{F}_s] = 0$

therefore the necessary conditions must be

$a = 0$, $X_0 \in \mathbb{R}$ and $b \in \mathbb{R}_+^*$ ✓10