1st Summative Problem Sheet

Deadline: Friday, October 13, 5pm

1. Consider the unbiased random walk $(X_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ starting at $X_0 = 0$ (that is, for any $x \in \mathbb{Z}$ if the walk is at x at step n it moves to $x+1$ with probability $1/2$ or to $x-1$ with probability $1/2$ independently of everything else). Prove that it is a Markov chain.

2. Consider the Markov chain $(X_n)_{n\in\mathbb{N}_0}$ that is defined as follows. A box has two compartments A and B and it contains $N \geq 1$ balls. At each step we select one of the balls uniformly at random and place it to the other compartment. The random variable X_n is the number of balls in compartment A after n steps in the above process.

   (a) Show that the Markov chain is irreducible.

   (b) Show that the Markov chain is not aperiodic.

   (c) Show that the binomial distribution $\mathrm{Bin}(N, 1/2)$ is a stationary distribution.

3. Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ having the following transition matrix (row/column i correspond to state i):

$$P = \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 & 0 \\ 0 & 1/2 & 1/6 & 1/6 & 1/6 \\ 0 & 1/4 & 1/8 & 1/8 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Draw a directed graph with weights on its edges that illustrates this chain. Identify in your diagram the communication classes.