Axiom 1.1.1 : Properties of Equivalence
The following is assumed to be true for all $\alpha, \beta,$ and, γ in the real numbers.

$$\forall \alpha, \beta, \gamma \in \mathbb{R}$$

An element α is equal to itself, this property is called reflexivity. If an element α is equal to an element β then β is also equal to α, this property is called symmetry. If and element α is equivalent to an element β, and β is also equivalent γ then α is equivalent to γ, this property is called transitivity.

$$\text{(i) } \alpha = \alpha, \text{(relexivity)}$$
$$\text{(ii) } \alpha = \beta \Rightarrow \beta = \alpha, \text{(symmetry)}$$
$$\text{(iii) } \alpha = \beta \wedge \beta = \gamma \Rightarrow \alpha = \gamma, \text{(transitivity)}$$

Axiom 1.1.2 : Well Defined Operation
An operation is said to be well defined if given the same inputs, the operation will always give the same outputs. Or symbolically,

$$\forall \alpha, \beta \in \mathbb{R}, f \text{ is an operation}, \alpha = \beta \Rightarrow f[\alpha] = f[\beta]$$

at the moment we will take for granted the fact that the normal operations of numbers is well defined, that is addition, subtraction, multiplication, and division of two numbers.

Example 1.1.3 : Isolate x
We can apply the remark about well defined operations to manipulate our equations in the following manner.
Suppose

$$x + 2 = 3$$

then

$$x + 2 - 2 = 3 - 2 \tag{1}$$
$$x = 1 \tag{2}$$

Remark 1.1.4 : Interpretation of Well Defined Operation in Terms of Equation Manipulation
For those that are not convinced by the manipulation above, or how the definition of a well defined operation is related to it. A way to interperet a well defined operation is as an operation which has the property of if you apply the operation to a single number, you should always get the same result.

In step (1) we know that we have two equivalent objects $x + 2$ and 3, the well defined operation that we apply to the two equivalent objects is "minus 2". Since minus is well defined, we conclude that $x + 2 - 2$ is equivalent to $3 - 2$,

we collect the terms and conclude that x is equivalent to 1.

Example 1.1.5 : Isolate x

$$2x = 6$$
$$\frac{1}{2} \cdot 2x = \frac{1}{2} \cdot 6$$
$$x = 3$$

Example 1.1.6 : Isolate x

$$\frac{x}{3} = 4$$
$$\frac{x}{3} \cdot 3 = 4 \cdot 3$$
$$x = 12$$

Example 1.1.7 : Isolate y

$$\frac{y + 2}{3} = 4x$$
$$3 \cdot \frac{y + 2}{3} = 3 \cdot 4x$$
$$y + 2 = 12x$$
$$y = 12x - 2$$

Example 1.1.8 : Isolate y

$$\frac{3 - 2y}{5} - y = x$$
$$\frac{3 - 2y}{5} = x + y$$
$$3 - 2y = 5x + 5y$$
$$3 - 5x = 7y$$
$$y = \frac{3 - 5x}{7}$$

Axiom 1.1.9 : Distribution
For α, β, γ the reals, left and right multiplication is distributed over addition.

$$\forall \alpha, \beta, \gamma \in \mathbb{R}$$
$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$$

Distribution is a rule that lets us rewrite an expression into an equivalent form, giving us an equation.

Remark 1.1.10 : Geometric Demonstration of Distribution If we accept that the the area of a rectangle is its length multiplied by its width, and that its area is also the sum of the parts its made of, then it follows that multiplication is distributed over addition.

insert illustration here

$$\text{Area} = \alpha(\beta + \gamma), \text{ length by width}$$
$$\text{Area} = \alpha\beta + \alpha\gamma, \text{ sum of area of parts}$$

Example 1.1.11 : Expand $3\alpha(2\beta + \gamma)$

$$3\alpha(2\beta + \gamma) = 6\alpha\beta + 3\alpha\gamma$$

Example 1.1.12 : Multiply By Distribution $12 \cdot 231$

$$12 \cdot 231 = 12(200 + 30 + 1) \qquad = 2400 + 360 + 12 = 2772$$

It turns out that long multiplication is really just distribution in disguise.

Remark 1.1.12 : Number Line (optional)
A number line is a visual tool that is able to visually represent a variable that takes on some value. It consists of a single axis, with regularly spaced markings of numbers that increase towards the right with a point on the marking on which its the variable's value.

Example 1.1.13 : $x = 4$ Plotted On A Number Line

insert illustration here

Remark 1.1.14 : Cartesian Plane (optional)
A Cartesian plane is an extended version of a number line that is able to visually represent two variables which take on some value. It consists of two perpendicular axes, both with markings like the number line. There is a point on the intersection of the two markings that take the values of the two variables intersect.

Example 1.1.15 : $x = 3 \land y = 4$ Plotted On A Cartesian Plane
Often abbreviated as $(x, y) = (3, 4)$

insert illustration here

Remark 1.1.16 : Linear Equation
A linear equation is an equation of the form $\alpha x + \beta y + \gamma = 0,$ where α, β, γ are in the reals.

Remark 1.1.17 : Slope Intercept Form of a Linear Equation
From the original form of the linear equation, we can derive the following alternate form of a linear equation.

$$\alpha x + \beta y + \gamma = 0$$
$$\beta y = -\alpha x - \gamma$$
$$y = -\frac{\alpha}{\beta}x - \frac{\gamma}{\beta}$$

Suppose we subsitute $\sigma = -\frac{\alpha}{\beta}, \iota = -\frac{\gamma}{\beta}$, then we can rewrite out equation as

$$y = \sigma x + \iota$$

Remark 1.1.18 : Linear Equation Constructor (optional)
Consider a contructor for linear equations

$$\texttt{LinearEquation} : \mathbb{R}^3 \mapsto \texttt{Equation}$$
$$(\alpha, \beta, \gamma) \rightarrow \alpha x + \beta y + \gamma = 0$$

which is able to construct linear equations of standard form, the following alternate constructor is able to construct linear equation of slope intercept form

$$\texttt{LinearEquation} : \mathbb{R}^2 \mapsto \texttt{Equation}$$
$$(\sigma, \iota) \rightarrow y = \sigma x + \iota$$

Remark 1.1.19 : Substitution

In an equation of more than one variables, we can suppose without proof that one or more variables takes on a concrete value in order to find the implied values of other variables. Consider the following equation

$$\alpha = \frac{\beta + \gamma + 5}{2}$$

from this equation, we can only find the value of variables in terms of other variables. Suppose that $\gamma = 2\beta$, then

$$\alpha = \frac{3\beta}{5}$$

Suppose that $\beta = 5$, then

$$\alpha = 3$$

So it turns out that in our original equation, if we suppose that $\gamma = 2\beta \wedge \beta = 5$, then $\alpha = 3$

This technique will become useful when we try to understand why equations of a certain form are called linear equations.

Remark 1.1.20 : Linear Equation Represented by Cartesian Plane

On a cartesian plane, a point requires the value of two variables. A linear equation is an equation of two variables, and thing to note about equations of two variables is that if you know the value of one variable, you can find out the value of the other.

Suppose

$$\texttt{LinearEquation[sigma= 1, iota= 1]} \rightarrow y = 1x + 1$$

is our linear equation. We can suppose that x takes on certain values, and use the subsitution method to find values of y.

Suppose that x takes on the values $-2, -1, 0, 1, 2$ then y takes on the values $-1, 0, 1, 2, 3$ respectively, since the linear equation $y = x + 1$ tells us that y is precisely 1 more than any given value of x. So we have generated the pairs of numbers $(x, y) = (-2, -1) \vee (-1, 0) \vee (0, 1) \vee (1, 2) \vee (2, 3)$. Plotting these pairs on the cartesian plane we end up with this figure

<div align="center">insert figure</div>

All of the points that we generated using this subsitution lie on the same line, and it turns out that every single point generated from this equation does also lie on this line. So we choose represent our equation on the cartesian plane as the line on which every single point generated by the equation lies on.

<div align="center">insert figure</div>

Remark 1.1.21 : Slope

Slope is a measure of how steep the a linear equation is, defined as increase in the value of y for when x is increased by 1.

Positive slope is manifested as liens increasing in hight towards the right, with high positive slope being steeper. Zero slope is seen as a flat line, while negative slope has the line decrease in height towards the right.

insert figure here (positive slope, high slope, flat, and neg slope)

We know that we can find the slope of a linear equation by increasing x by 1 and finding out how much y has increased by, which is what we are going to do. Suppose

$$\texttt{LinearEquation[sigma= 4, iota= 3]} \rightarrow y = 4x + 3$$

is our linear equation. Suppose x is any number α, then

$$y = 4x + 3$$
$$= 4\alpha + 3$$

now suppose x is one more that before, $x = \alpha + 1$, let us call our new y, y^* so we can differentiate between the old y. Then

$$y^* = 4x + 3$$
$$= 4(\alpha + 1) + 3$$
$$= 4\alpha + 3 + 4$$
$$= y + 4$$

since y^* is 4 greater than y, and x is 1 greater in the computation of y^* than y, by the definition of slope, we can say that the slope of our linear equation is 4.

Theorem 1.1.22 : Slope of a Linear Equation
The slope of a linear equation

$$\texttt{LinearEquation[sigma=}\ \sigma\texttt{, iota=}\ \iota\texttt{]} \ \rightarrow y = \sigma x + \iota$$

is σ.

Proof:

Suppose

$$\texttt{LinearEquation[sigma=}\ \sigma\texttt{, iota=}\ \iota\texttt{]} \ \rightarrow y = \sigma x + \iota$$

is our general linear equation
Suppose $x = \alpha$, then

$$y = \sigma\alpha + \iota$$

Now suppose x is increased by one and y^* is our new y, $x = \alpha + 1$, then

$$y^* = \sigma(\alpha + 1) + \iota$$
$$= \sigma\alpha + \iota + \sigma$$
$$= y + \sigma$$

So we have shown that given a general linear equation in slope intercept form

$$y = \sigma x + \iota$$

the slope is simply σ. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 1.1.23 : Intercepts
The intercepts of a equation are the locations in which the graph intersecs with the y and x axis, these are known as the y and x intercepts respectively.

Theorem 1.1.24 : x, and y Intercepts of a Linear Equation
The x and y intercepts of a linear equation

$$\texttt{LinearEquation[sigma=} \sigma \texttt{, iota=} \iota \texttt{]} \rightarrow y = \sigma x + \iota$$

are

$$\texttt{x-intercept:} \left(-\frac{\iota}{\sigma}, 0\right)$$
$$\texttt{y-intercept:} \left(\frac{0}{\iota}\right)$$

Proof:

Suppose

$$\texttt{LinearEquation[sigma=} \sigma \texttt{, iota=} \iota \texttt{]} \rightarrow y = \sigma x + \iota$$

To find the y intercept, we need to find the location in which the graph representing the equation intersects the y axis, and that is when x is 0. So suppose that $x = 0$, then

$$y = \sigma \cdot 0 + \iota = \iota$$

Then the y-intercept of our equation occurs at $(0, \iota)$
Similarly in order to find the x intercept, we need only to find the position of x when y is 0. Supposing that $y = 0$, then

$$0 = \sigma x + \iota$$
$$\sigma x = -\iota$$
$$x = -\frac{\iota}{\sigma}$$

Example 1.1.25 : Slope of Equation From Two Points
Suppose we have two points $p = (p_x, p_y) = (1, 3), q = (q_x, q_y) = (3, 7)$, where each point is represented as a list of two numbers.

Suppose $\delta x = q_x - p_x$ and $\delta y = q_y - p_y$, δx and, δy are the change (difference) in x, and y respectively, then

$$\delta y = 7 - 3 = 4$$
$$\delta x = 3 - 1 = 2$$

Then it follows that y increases by 4 whenever x increases by 2, interpreted as $\frac{4}{2}$, or that y increases by 2 whenever x increases by 1 which is our definition of slope, interpreted as $\sigma = \frac{2}{1} = 2$.

However we notice that we can simply get the slope from our first fraction $\frac{4}{2} = 2$ without the need to reduce its denominator, since by definition reduction of a fraction does not change its value.

In general suppose we have two points $p = (x_0, y_0), q = (x_1, y_2)$, then

$$\delta y = y_0 - y_1$$
$$\delta x = x_0 - x_1$$

to find the slope, we take the ratio of δy to δx

$$\sigma = \frac{\delta y}{\delta x}$$
$$= \frac{y_0 - y_1}{x_0 - x_1}$$

Theorem 1.1.26 : Intercept From Two Points

Theorem 1.1.27 : Equivalent Definition of Slope (optional) Slope can be equivalently defined as the ratio of the difference between the y co-ordinates and the difference between the x co-ordinates of two points on a line.

Proof:

Suppose

$$p = (x_0, y_0)$$
$$q = (x_1, y_1)$$

are two points that lie on a linear equation, then it follows that

$$\delta y = y_0 - y_1 \delta x = x_0 - x_1$$

According to our first definition of slope

$$\sigma = \frac{\frac{\delta y}{\delta x}}{\frac{\delta x}{\delta x}}$$
$$= \frac{\frac{\delta y}{\delta x}}{1}$$
$$= \frac{\delta y}{\delta x}$$

Which is simply the ratio of the difference between the y and x co-ordinates of two points on a line. □

Remark 1.1.28 : Equation From Slope and Intercept Suppose we are given a desired slope of σ, and y-intercept of ι, and asked to construct a linear equation that meets the criteria. We can simply use our constructor

$$\texttt{LinearEquation[sigma=} \sigma \texttt{, iota=} \iota \texttt{]} \;\to y = \sigma x + \iota$$

to be sure that our equation actually has the correct slope and y-intercept, we can refer back to the theorems that we proved earlier.

So we find that σ is the slope of our equation, and ι is the intercept of the equation.

Remark 1.1.28 : Equation From 2 Points (Derivation) Suppose we have two points $p = (x_0, y_0), q = (x_1, y_1)$, and we are asked to find an equation that passes through those two points.

From the remark just above, we know that if we know that slope, and intercept of our target linear equation, then we can get the equation for free. So in order to find our equation in this case, we will work on trying to extract information for slope and intercept from our points p, q.

Suppose that

$$p = (x_0, y_0)$$
$$q = (x_1, y_1)$$

are two points that our linear equation passes through

$y + s = (x + 1)s + i$

Remark 1.1.29 : Standard Form of Linear Equation hello

$$\texttt{LinearEquation[sigma=} \sigma \texttt{, iota=} \iota \texttt{]} \;\to y = \sigma x + \iota$$