M351K Homework 2

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1: Disease

Let $D :=$ the event that a person has the disease.
Let $\overline{D} :=$ the event that a person does not have the disease.
Let $T :=$ the event that a person tests positive.
Let $\overline{T} :=$ the event that a person tests negative.
Given $P(D) = 0.0005, P(T|D) = 0.99, P(\overline{T}|\overline{D}) = 0.995$.
Then $P(T|\overline{D}) = 1 - P(\overline{T}|\overline{D}) = 1 - 0.995 = 0.005$.
$P(T) = P(T|D)P(D) + P(T|\overline{D})P(\overline{D}) = 0.99(0.0005) + 0.005(1 - 0.0005) = 0.0054925$.
Then by Bayes, $P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{(0.99)(0.0005)}{0.0054925} = 0.09012289485662267 \approx 9\%$.
The probability that a person has the disease given they tested positive is about 9However, the more important percentage is $P(T|D) = 99\%$.

2: Independence

a)

Let $A = B = \mathbb{U}$, where $\mathbb{U}$ is the universe.
Then $P(A|B) = P(A) = 1$.
Then A and B are pairwise independent.

b)

A and B are independent. A and C are independent.
Then $P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C)) = P(A \cap B) + P(A \cap C) - P(((A \cap B) \cap (A \cap C))) = P(A)P(B) + P(A)P(C) - P(A \cap B \cap C) = P(A)(P(B) + P(C) - \frac{P(A \cap (B \cap C))}{P(A)})$.
If A and $B \cap C$ are independent, then $P(A \cap (B \cup C)) = P(A)(P(B) + P(C) - P(B \cap C)) = P(A)P(B \cup C)$, and A is independent of $B \cup C$.
Else, A is not independent of $B \cup C$.
Therefore, supposing that A is independent of and A is independent of C does not imply that A is independent of $B \cup C$ since we never asserted that A was independent of $B \cap C$.

c)

If A, B, and C are independent then $P(A \cap B \cap C) = P(A)P(B)P(C)$ and A, B, and C are pairwise independent.
Then $P(A \cap (B \cup C))$
$= P((A \cap B) \cup (A \cap C))$
$= P(A \cap B) + P(A \cap C)) - P((A \cap B) \cap (A \cap C))$
$= P(A)P(B) + P(A)P(C) - P(A \cap B \cap C)$
$= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C)$
$= P(A)P(B) + P(A)P(C) - P(A)P(B \cap C)$
$= P(A)(P(B) + P(C) - P(B \cap C))$
$= P(A)P(B \cup C)$.
Then A and $B \cup C$ are independent.

3: Chairs

Let S be the sample space.
Then $S = \{(a, b) \in \mathbb{N} \mid a, b \leq n, a \neq b\}$.
Let A be the event that $|a - b| \leq 3$.
Let $n \geq 7$.
Then $P(A) = P(A \mid 3 < a \leq n - 3)P(3 < a \leq n - 3)$
$+ P(A \mid a = 1)P(a = 1)$
$+ P(A \mid a = 2)P(a = 2)$
$+ P(A \mid a = 3)P(a = 3)$
$+ P(A \mid a = n - 2)P(a = n - 2)$
$+ P(A \mid a = n - 1)P(a = n - 1)$
$+ P(A \mid a = n)P(a = n)$
$= P(A \mid 3 < a \leq n - 3)P(3 < a \leq n - 3)$
$+ 2P(A \mid a = 1)P(a = 1)$
$+ 2P(A \mid a = 2)P(a = 2)$
$+ 2P(A \mid a = 3)P(a = 3)$
$= \frac{6}{n-1} \cdot \frac{n-6}{n} + \frac{5}{n-1} \cdot \frac{2}{n} + \frac{4}{n-1} \cdot \frac{2}{n} + \frac{3}{n-1} \cdot \frac{2}{n}$
$= \frac{6}{n-1} \cdot \frac{n-6}{n} + \frac{2}{n} \cdot (\frac{5}{n-1} + \frac{4}{n-1} + \frac{3}{n-1})$
$= \frac{6(n-2)}{(n-1)n}$.
If $n = 6$ then $P(A) = \frac{1}{3} \cdot (\frac{5}{5} + \frac{4}{5} + \frac{3}{5}) = \frac{4}{5}$.
If $n = 5$ then $P(A) = \frac{1}{5} + \frac{2}{5} \cdot (\frac{4}{4} + \frac{3}{4}) = \frac{9}{10}$.
If $n = 4$ then $P(A) = 1$.
If $n = 3$ then $P(A) = 1$.
If $n = 2$ then $P(A) = 1$.

4: Set Algebra

$P(A \cup B \mid C)$
$= \frac{P((A \cup B) \cap C)}{P(C)}$
$= \frac{P((A + B - (A \cap B)) \cap C)}{P(C)}$
$= \frac{P(A \cap C) + P(B \cap C) - P((A \cap B) \cap C)}{P(C)}$
$= \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P((A \cap B) \cap C)}{P(C)}$
$= P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$.

5: Candy

We use partitions. There are 151 choices for each partition, since we may partition the front or past the end. Then there are 151^4 unique possible partitions, since partition positions may be repeated. Each unique partition corresponds to a unique color distribution, so there are a maximum of 519885601 unique jars the factory can produce without repeats.

6: Coins

$P(\text{toss } 5 = \text{Head}) = P(\text{Head})$
$= P(\text{Head} \mid \text{coin a})P(\text{coin a}) + P(\text{Head} \mid \text{coin b})P(\text{coin b})$
$= 0.7 \cdot \frac{1}{2} + 0.3 \cdot \frac{1}{2}$
$= \frac{1}{2}$.
$P(\text{toss } 6 = \text{Head} \mid \text{first 5 tosses are Heads}) = P(\text{Head}) = \frac{1}{2}$.

7: Coins

a)

Fix one man in one position. This removes repeats due to rotation. Then there are $7!$ ways to place the rest of the men.
For no two women to sit side-by-side, the women must be placed in a seat between two men.
There are 8 positions between the men, and there are $\binom{8}{5}$ ways to select seats for the women. Order matters, so there are $5! \cdot \binom{8}{5}$ ways to seat the women.
Then there are a total of $7! \cdot 5! \cdot \binom{8}{5} = 33868800$ unique ways to seat the 13 people independent of rotation.

b)

There are two couples and 9 people who are not couples.
Fix one of the couples to account for rotations. There are 4 ways to do this.
There are then 10 spots for the next couple to be in, and 2 ways to order the couple. Thus there are 20 ways to place the second couple.
In the remaining 9 slots, there are $9!$ ways to place the remaining people.
Then there are a total of $4 \cdot 20 \cdot 9! = 29030400$ ways to seat the 13 people independent of rotation.