M351K Homework 5

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1: pdf

$\int_{-\infty}^{\infty} h(x)dx$
$= \int_{-\infty}^{\infty}(\alpha f(x) + (1-\alpha)g(x))dx$
$= \alpha \int_{-\infty}^{\infty} f(x)dx + (1-\alpha)\int_{-\infty}^{\infty} g(x)dx$
$= \alpha + (1-\alpha)$
$= 1.$
$\alpha, (1-\alpha), f(x), g(x) \geq 0 \ \ \forall x \in \mathbb{R}.$
Then $\alpha f(x) + (1-\alpha)g(x) \geq 0 \ \ \forall x \in \mathbb{R}.$
Then $h(x) \geq 0 \ \ \forall x \in \mathbb{R}.$
Then $h(x)$ is a valid probability density function.

2: C

a)

$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} Ce^{-\alpha x}dx.$
Then knowing properties of exponential distributions, we conclude that $C = \alpha$ is the only possibility.

b) plot

Knowing properties of exponential distributions, we recall
$F(x) = \begin{cases} 1 - e^{-\alpha x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}.$

c)

$y = F_Y(y) = g(F_Z(z)) = g(1 - e^{-z}).$
Then $g(y) = \begin{cases} -\ln(1-y) & \text{if } y \in [0,1] \\ 0 & \text{if } x < 0 \\ \infty & \text{if } x > 1 \end{cases}.$

3: pdf

a)

$F(0) = 0$.

b)

$F(1) = \int_0^1 (\frac{3}{2}y - \frac{3}{4}y^2)dx = (\frac{3}{4}y^2 - \frac{3}{12}y^3)\Big|_0^1 = \frac{9}{12} - \frac{3}{12} = \frac{1}{2}$.

c)

$F(2) = 1$.

d)

$F(4) = 1$.

maximization

To find $\operatorname{argmax} F(a + 0.2) - F(a)$, we can use derivatives.
$f(a + 0.2) - f(a)$
$= (\frac{3}{2}(a + \frac{1}{5}) - \frac{3}{4}(a + \frac{1}{5})^2) - (\frac{3}{2}a - \frac{3}{4}a^2)$
$= \frac{3}{2}a + \frac{3}{10} - \frac{3}{4}a^2 - \frac{6}{20}a - \frac{1}{25} - \frac{3}{2}a + \frac{3}{4}a^2$
$= \frac{3}{10} - \frac{6}{20}a$
$= \frac{3}{10}(1 - a)$.
Convinently, $1 \in [0, 2]$.
Since the pdf is non-negative and there is only one critical point, we know that $\operatorname{argmax} F(a + 0.2) - F(a) = 1$

4: integration

integration

$$1 = \int_1^\infty \int_1^\infty ce^{-2(x+y)} dxdy$$
$$= c\int_1^\infty \int_1^\infty e^{-2x}e^{-2y} dxdy$$
$$= c\int_1^\infty e^{-2y}dy(-\frac{1}{2}e^{-2x}\Big|_1^\infty)$$
$$= c(0-(-\frac{1}{2}e^{-2}))\int_1^\infty e^{-2y}dy$$
$$= \frac{1}{2}ce^{-2}(-\frac{1}{2}e^{-2x}\Big|_1^\infty)$$
$$= \frac{1}{4}ce^{-4}.$$

Then $c = 4e^4$.

more integration

$f_X(x) = \int_1^\infty f_{XY}(x,y)dy$
$= \int_1^\infty ce^{-2x}e^{-2y}dy$
$= ce^{-2x}(-\frac{1}{2}e^{-2y}\Big|_1^\infty)$
$= 2e^2e^{-2x}$.
By a symmetry argument, we see that $f_Y(y) = 2e^2e^{-2y}$.
We also notice that $f_{XY}(x,y) = f_Y(y)f_X(x)$.
Thus X and Y are independent random variables.

5: integration

$F_X(x)$
$$= \int_{-\infty}^x \frac{\frac{\alpha}{\pi}}{x^2+\alpha^2}$$
$$= \frac{1}{\alpha\pi}\int_{-\infty}^x \frac{1}{(\frac{x}{\alpha})^2+1}$$
$$= \frac{1}{\pi}(\tan^{-1}(\frac{x}{\alpha})\Big|_{-\infty}^x)$$
$$= \frac{1}{\pi}\tan^{-1}(\frac{x}{\alpha})-(-\frac{1}{2})$$
$$= \frac{1}{\pi}\tan^{-1}(\frac{x}{\alpha})+\frac{1}{2}.$$

6: integration

a) plot pdf

$1 = \int_{-\infty}^{\infty} f(x)dx = c\int_0^1 (x - x^3)dx.$
$= c(\frac{1}{2}x^2 - \frac{1}{4}x^4)\Big|_0^1$
$= \frac{c}{4}.$
Then $c = 4$.

b) plot cdf

$$\int_{-\infty}^{x} f_X(x)dx = \begin{cases} 2x^2 - x^4 & \text{if } x \in [0,1] \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x > 1 \end{cases}.$$

c) pdf

$P(0 < X < 0.5) = F(0.5) - F(0) = F(0.5) = \frac{7}{16}.$
$P(X = 1) = 0.$
$P(0.25 < X < 0.5) = F(0.5) - F(0.25) = \frac{7}{16} - \frac{31}{256} = \frac{81}{256}.$