M328K Homework 8

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0.1 4.4.6

Find all solutions of $x^8 - x^4 + 1001 \equiv 0 \pmod{539}$.

$x^8 - x^4 + 462 \equiv 0 \pmod{7^2 11}$.
$\therefore x^8 - x^4 + 462 \equiv 0 \pmod{7^2}$ and $x^8 - x^4 + 462 \equiv 0 \pmod{11}$
We first find solutions to the first congruence.
$x^8 - x^4 + 21 \equiv 0 \pmod{7^2}$
First, we find solutions to $x^8 - x^4 + 21 \equiv 0 \pmod{7}$
$x^8 - x^4 \equiv 0 \pmod{7}$
$x^4(x^4 - 1) \equiv 0 \pmod{7}$
The soultions modulo 7 are 0, 1, and 6.
To see if solutions lift, we can use Hansel's Lemma.
$f(x) = x^8 - x^4 + 21$.
$f'(x) = 8x^7 - 4x^3 \equiv x^7 - 4x^3 \pmod{7}$.
$f'(0) \equiv 0 \pmod{7}$.
$f(0) \not\equiv 0 \pmod{49}$.
$\therefore x \equiv 0 \pmod{7}$ does not lift modulo 49.
$f'(1) \equiv 1 - 4 \equiv -3 \pmod{7}$.
$f(1) \equiv 1 - 1 + 21 \equiv 21 \pmod{4}9$.
$\therefore x \equiv 1 \pmod{7}$ lifts to 8 modulo 49.
$f'(-1) \equiv (-1) - (-4) \equiv 3 \pmod{7}$.
$f(-1) \equiv 1 - 1 + 21 \equiv 21 \pmod{7}$.
$\therefore x \equiv 1 \pmod{7}$ lifts to 41 modulo 49.

We now find solutions to the second congruence.
$x^8 - x^4 + 0 \equiv 0 \pmod{11}$
$x^4(x^4 - 1) \equiv 0 \pmod{11}$
$\therefore x \equiv 0, 1, 10 \pmod{11}$

The intersection is found by solving the linear system of congruences.
Therefore, the solution set is $\{x \mid x \equiv 8 \pmod{49} \text{ or } x \equiv 41 \pmod{49}\}$

0.2 4.4.10

Find all solutions of $x^5 + x - 6 \equiv 0 \pmod{2^4 3^2}$.

We first find soultions to $x^5 + x - 6 \equiv 0 \pmod{2^4}$ and $x^5 + x - 6 \equiv 0 \pmod{3^2}$
$x^5 + x - 6 \equiv 0 \pmod{2^4}$
$x^5 + x - 6 \equiv 0 \pmod{3^2}$.
Solving the first congruence:
$x^5 + x + 2 \equiv 0 \pmod{4}$
By observation, 0 is not a solution, but 1, 2, and 3 are modulo 4.
For some $t \in \mathbb{Z}$, $(4t+1)^5 + (4t+1) - 6 \equiv 1 + 20t + 4t - 5 \equiv 8t - 4 \equiv 0 \pmod{16}$.
But this is never true, thus 1 does not lift.
By trying all the rest of the possible solutions, we find that x is 3, 6, or 11 modulo 16.

Solving the second congruence:
By observation, 0, 1, -1, 2, and -2 are not solutions.
$3^5 + 3 - 6 = 81(3) + 3 - 6 = 240$. 3 is not a solution.
$(-3)^5 - 3 - 6 = -(240 + 12) = 252$. -3 is a solution. (2+5+2 = 9)
$(4)^5 + 4 - 6 = 256(4) - 2 = 800 + 200 + 22$. 4 is not a solution.
$(5)^5 + 5 - 6 = 625(5) - 1 = 3000 + 124$. 5 is not a solution.

Solving the system:
$x \equiv 6 \pmod{9}$
$x \equiv 3, 6, 11 \pmod{16}$.
x is in the form $6 + 144t$, $51 + 144t$, or $123 + 144t$ for any $t \in \mathbb{Z}$.

0.3 5.1.2

Determine the highest power of 2 that divides 1423408.
$1423408 = 711704(2) = 355852(2^2) = 177926(2^3) = 88963(2^4)$. The highest power of 2 is 16.