M328K Homework 7

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0.1 4.3.10

Find $x \in \mathbb{Z}$ such that
$x \equiv 9 \pmod{10}$,
$x \equiv 9 \pmod{11}$, and
$x \equiv 0 \pmod{13}$.

Using the first congruence,
Let $q \in \mathbb{Z}$ where $x = 10q + 9$.
Substituting into the second congruence,
$10q + 9 \equiv 9 \pmod{11}$.
It follows that $10q \equiv 0 \pmod{11}$,
$q \equiv 0 \pmod{11}$.
Let $r \in \mathbb{Z}$ where $q = 11r + 0$.
Now we have $x = 10(11r) + 9 = 110r + 9$.
Substituting into the third congruence,
$110r + 9 \equiv 0 \pmod{13}$.
It follows that $110r \equiv 4 \pmod{13}$,
$6r \equiv 4 \pmod{13}$,
$r \equiv (\bar{6})(4) \equiv 44 \equiv 5 \pmod{13}$.
Let $s \in \mathbb{Z}$ where $r = 13s + 5$.
Now we have $x = 110(13s + 5) + 9 = 1430s + 559$.
559 is a vaild value for x.

0.2 4.3.20.b

Find $x \in \mathbb{Z}$ such that
$x \equiv 2 \pmod{14}$,
$x \equiv 16 \pmod{21}$, and
$x \equiv 10 \pmod{30}$.

We derive an equivalent, minimal set of congruences using the Chinese remainder theorem.
$x \equiv 2 \pmod 2$
$x \equiv 2 \pmod 7$
$x \equiv 16 \pmod 3$
$x \equiv 16 \pmod 7$
$x \equiv 10 \pmod 2$
$x \equiv 10 \pmod 3$
$x \equiv 10 \pmod 5$
These reduce to:
$x \equiv 0 \pmod 2$,
$x \equiv 1 \pmod 3$,
$x \equiv 0 \pmod 5$,
$x \equiv 2 \pmod 7$.

Using the first congruence,
Let $q \in \mathbb{Z}$ where $x = 2q$.
Substituting into the second congruence,
$2q \equiv 1 \pmod 3$.
It follows that $q \equiv 2 \pmod 3$.
Let $r \in \mathbb{Z}$ where $q = 3r + 2$.
Now we have $x = 2(3r + 2) = 6r + 4$.
Substituting into the third congruence,
$6r + 4 \equiv 0 \pmod 5$.
It follows that $r \equiv 1 \pmod 5$.
Let $s \in \mathbb{Z}$ where $r = 5s + 1$.
Now we have $x = 6(5s + 1) + 4 = 30s + 10$.
Substituting into the fourth congruence,
$30s + 10 \equiv 2 \pmod 7$.
It follows that $s \equiv (\bar{2})(6) \equiv 3 \pmod 7$.
Let $t \in \mathbb{Z}$ where $s = 7t + 3$.
Now we have $x = 30(7t + 3) + 10 = 210t + 100$.
Thus any $x \in S$ where $S = \{x \mid 210t + 100 \ \ \forall t \in \mathbb{Z}\}$ will be a solution to the congruence.

0.3 4.3.20.e

Find $x \in \mathbb{Z}$ such that
$x \equiv 7 \pmod{9}$,
$x \equiv 2 \pmod{10}$,
$x \equiv 3 \pmod{12}$, and
$x \equiv 6 \pmod{15}$.

We derive an equivalent, minimal set of congruences using the Chinese remainder theorem.
$x \equiv 7 \pmod{9}$
$x \equiv 2 \pmod{2}$
$x \equiv 2 \pmod{5}$
$x \equiv 3 \pmod{3}$
$x \equiv 3 \pmod{4}$
$x \equiv 6 \pmod{3}$
$x \equiv 6 \pmod{5}$
These reduce to:
$x \equiv 0 \pmod{2}$,
$x \equiv 0 \pmod{3}$,
$x \equiv 3 \pmod{4}$,
$x \equiv 1 \pmod{5}$,
$x \equiv 2 \pmod{5}$,
$x \equiv 7 \pmod{9}$.

There are no solutions to this contradictory set of congruences. (The intersection of the system is the null set).

0.4 4.3.32

Show that the system
$x \equiv 1 \pmod{2}$,
$x \equiv 0 \pmod{4}$,
$x \equiv 0 \pmod{3}$,
$x \equiv 2 \pmod{12}$,
$x \equiv 2 \pmod{8}$,
$x \equiv 22 \pmod{24}$
is a covering set of congruences.

We derive an equivalent set of congruences using the Chinese remainder theorem.
$x \equiv 1 \pmod{2}$,
$x \equiv 0 \pmod{4}$,
$x \equiv 0 \pmod{3}$,
$x \equiv 2 \pmod{3}$,
$x \equiv 2 \pmod{4}$,
$x \equiv 2 \pmod{8}$,
$x \equiv 22 \pmod{3}$,
$x \equiv 22 \pmod{8}$.
These reduce to:
$x \equiv 1 \pmod{2}$,
$x \equiv 0 \pmod{3}$,
$x \equiv 1 \pmod{3}$,
$x \equiv 2 \pmod{3}$,
$x \equiv 0 \pmod{4}$,
$x \equiv 2 \pmod{4}$,
$x \equiv 2 \pmod{8}$,
$x \equiv 6 \pmod{8}$.

The union of:
$x \equiv 0 \pmod{3}$,
$x \equiv 1 \pmod{3}$,
$x \equiv 2 \pmod{3}$
is U, the universal set of all integers.
Therefore the system is a covering set of congruences.

0.5 4.3.36

Find all solutions of the congruence $x^2 + 6x - 31 \equiv 0 \pmod{2^3 3^2}$.

$x^2 + 6x - 31 \equiv 0 \pmod{2^3}$.
$x^2 + 6x + 9 \equiv 0 \pmod{8}$
$(x+3)^2 \equiv 0 \pmod{8}$.
$\therefore x \equiv 1 \pmod{8}$ or
$x \equiv 5 \pmod{8}$.
These are equivalent to the congruence $x \equiv 1 \pmod{4}$.

$x^2 + 6x - 31 \equiv 0 \pmod{3^2}$.
$x^2 - 3x - 4 \equiv 0 \pmod{9}$
$(x-4)(x+1) \equiv 0 \pmod{9}$.
$\therefore x \equiv 4 \pmod{9}$ or
$x \equiv 8 \pmod{9}$.

We want the intersection of the modulo 8 and modulo 9 solutions:
$x \equiv 1 \pmod{4}$ and
$x \equiv 4 \pmod{9}$ or $x \equiv 8 \pmod{9}$.
This is equivalent to:
$x \equiv 1 \pmod{4}$ and $x \equiv 4 \pmod{9}$ or
$x \equiv 1 \pmod{4}$ and $x \equiv 8 \pmod{9}$.
Now we solve the systems of congruences.

$x \equiv 1 \pmod{4}$,
$x \equiv 4 \pmod{9}$.
Using the first congruence,
Let $q \in \mathbb{Z}$ where $x = 4q + 1$.
Substituting into the second congruence,
$4q + 1 \equiv 4 \pmod{9}$.
It follows that $q \equiv (\bar{4})(3) \equiv 3 \pmod{9}$.
Let $r \in \mathbb{Z}$ where $q = 9r + 3$.
Now we have $x = 4(9r + 3) + 1 = 36r + 13$.

$x \equiv 1 \pmod{4}$,
$x \equiv 8 \pmod{9}$.
Using the first congruence,
Let $q \in \mathbb{Z}$ where $x = 4q + 1$.
Substituting into the second congruence,
$4q + 1 \equiv 8 \pmod{9}$.
It follows that $q \equiv (\bar{4})(7) \equiv 4 \pmod{9}$.
Let $r \in \mathbb{Z}$ where $q = 9r + 4$.
Now we have $x = 4(9r + 4) + 1 = 36r + 17$.

Therefore, the solution set is $\{x \mid x \equiv 13 \pmod{36} \text{ or } x \equiv 17 \pmod{36}\}$