M361K Homework 2

Joshua Dong

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0.1 2.1.1

$a, b \in \mathbb{R}$. Prove:
a) $a + b = 0 \rightarrow b = -a$
$a + b = 0$
$a + b + (-a) = 0 + (-a)$
$b + a + (-a) = -a$ by A1
$b + (a - a) = -a$ by A2
$b = -a$
$\therefore a + b = 0 \rightarrow b = -a$.
b) $-(-a) = a$
$-(-a) = -1 \cdot (-1 \cdot a) =$
by M2 $(-1 \cdot -1) \cdot a =$
$1 \cdot a = a$
$\therefore -(-a) = a$.
c) $(-1)a = -a$
$(-1)a = -(1a) = -a$ by M2 and M3, respectively
$\therefore (-1)a = -a$.
d) $(-1)(-1) = 1$
$(-1)(-1) = -(-1) = 1$ by b)
$\therefore (-1)(-1) = 1$.

0.2 2.1.2

$a, b \in \mathbb{R}$. Prove:
a) $-(a + b) = (-a) + (-b)$
$-(a + b) = (-a) + (-b)$ by the distributive law of multiplication of reals
$\therefore -a + -b = (-a) + (-b)$.
b) $(-a) \cdot (-b) = a \cdot b(-a) \cdot (-b) = (-1 \cdot a) \cdot (-1 \cdot b) =$
by M1 and M2 $(-1 \cdot -1) \cdot (a \cdot b) =$
$a \cdot b$
$\therefore (-a) \cdot (-b) = a \cdot b$.
c) $\frac{1}{-a} = -(\frac{1}{a})$
$\frac{1}{-a} =$
$\frac{1}{-1 \cdot a} =$

by M1 $-1 \cdot \frac{1}{a} =$
$-(\frac{1}{a})$.
$\therefore \frac{1}{-a} = -(\frac{1}{a})$.
d) $b \neq 0 \to -(\frac{a}{b}) = \frac{-a}{b}$
Since b^{-1} exists,
$-(\frac{a}{b}) = -1 \cdot \frac{a}{b} =$
by M1 $\frac{-1 \cdot a}{b} =$
$\frac{a}{b}$
$\therefore b \neq 0 \to -(\frac{a}{b}) = \frac{-a}{b}$.

0.3 2.1.4

$a \in \mathbb{R}$ and $a \cdot a = a$. Prove $a \in \{0, 1\}$.
Let $n \in \mathbb{R}$ where $a = n + 1$.
$a \cdot a = a$
$(1 + n) \cdot (1 + n) = (1 + n)$
$n^2 + 2n + 1 = n + 1$
$n^2 + n = 0$
$n(n + 1) = 0$.
The above is only true if $n \in \{-1, 0\}$
$\therefore\ a \in \{-1 + 1, 0 + 1\} = \{0, 1\}$

0.4 2.1.5

Show $a \neq 0$ and $b \neq 0 \to \frac{1}{ab} = (\frac{1}{a})(\frac{1}{b})$.
$a \neq 0$ and $b \neq 0$
$\therefore \exists a^{-1}, \exists b^{-1}$.
Let $n \in \mathbb{R}$ where $\frac{1}{ab} = n$.
Note that $n \cdot ab = \frac{1}{ab} \cdot ab = (ab)^{-1} \cdot ab = 1$.
$\frac{1}{ab} = n$
$ab \cdot \frac{1}{ab} = ab \cdot n$
$a^{-1}b^{-1}ab \cdot \frac{1}{ab} = a^{-1}b^{-1}ab \cdot n = a^{-1}b^{-1} \cdot 1$.
$\therefore \frac{1}{ab} = a^{-1}b^{-1}$
$\therefore a \neq 0$ and $b \neq 0 \to \frac{1}{ab} = (\frac{1}{a})(\frac{1}{b})$.

0.5 2.1.6

Show $\nexists s \in \mathbb{R}$ such that $s^2 = 6$.
Suppose, on the contrary, that p and q are integers such that $(\frac{p}{q})^2 = 6$. We may assume that p and q are positive and $p \perp q$. Since $p^2 = 6q^2$, we see that $2 \mid p^2$ and $3 \mid p^2$, thus $2 \mid p$ and $3 \mid p$ and $6 \mid p$. $p \perp q \to 6 \nmid q$.
Since $6 \mid p$, then $p = 6m$ for some $m \in \mathbb{N}$, and hence $36m^2 = 6q^2$, so that $6m^2 = q^2$. $\therefore 6 \mid q^2$, and it follows that $2 \mid q^2$ and $3 \mid q^2$, thus $2 \mid q$ and $3 \mid q$ and $6 \mid q$. Since $(\frac{p}{q})^2 = 6 \to 6 \mid q$ and $6 \mid p$, but $q \perp p$, the hypothesis must be false.

0.6 2.2.1

$a, b \in \mathbb{R}$. $b \neq 0$. Show:
a) $|a| = \sqrt{(a^2)}$
$|a|^2 = a^2$ by 2.2.2.b.
$\sqrt{|a|^2} = \sqrt{a^2}$.
$|a| = \sqrt{a^2}$, since $|a| > 0$.
$\therefore |a| = \sqrt{(a^2)}$.

b) $|\frac{a}{b}| = \frac{|a|}{|b|}$
5 cases:
1) $a < 0, b < 0$
2) $a < 0, b > 0$
3) $a > 0, b > 0$
4) $a > 0, b < 0$
5) $a = 0, b \neq 0$

1) $a < 0, b < 0$
$|a| = -a, |b| = -b$.
$\frac{|a|}{|b|} = \frac{-a}{-b} = \frac{a}{b}$.
$b < 0$. $1 > 0$. $\therefore b^{-1} < 0$.
$\frac{a}{b} = ab^{-1} > 0$, since the product of two negatives is positive.
$\therefore |\frac{a}{b}| = \frac{a}{b}$.
$\frac{|a|}{|b|} = \frac{a}{b} = |\frac{a}{b}|$.
$\therefore a < 0, b < 0 \rightarrow |\frac{a}{b}| = \frac{|a|}{|b|}$.

2) $a < 0, b > 0$
$|a| = -a, |b| = b$.
$\frac{|a|}{|b|} = \frac{-a}{b} = -\frac{a}{b}$.
$b > 0$. $1 > 0$. $\therefore b^{-1} > 0$.
$\frac{a}{b} = ab^{-1} < 0$, since the product of negative and positive is negative.
$\therefore |\frac{a}{b}| = -\frac{a}{b}$.
$\frac{|a|}{|b|} = -\frac{a}{b} = |\frac{a}{b}|$.
$\therefore a < 0, b > 0 \rightarrow |\frac{a}{b}| = \frac{|a|}{|b|}$.

3) $a > 0, b > 0$
$|a| = a, |b| = b$.
$\frac{|a|}{|b|} = \frac{a}{b} = \frac{a}{b}$.
$b > 0$. $1 > 0$. $\therefore b^{-1} > 0$.
$\frac{a}{b} = ab^{-1} > 0$, since the product of two positives is positive.
$\therefore |\frac{a}{b}| = \frac{a}{b}$.
$\frac{|a|}{|b|} = \frac{a}{b} = |\frac{a}{b}|$.
$\therefore a > 0, b > 0 \rightarrow |\frac{a}{b}| = \frac{|a|}{|b|}$.

4) $a > 0, b < 0$
$|a| = a, |b| = -b$.
$\frac{|a|}{|b|} = \frac{-a}{b} = -\frac{a}{b}$.
$b < 0$. $1 > 0$. $\therefore b^{-1} < 0$.
$\frac{a}{b} = ab^{-1} < 0$, since the product of negative and positive is negative.
$\therefore |\frac{a}{b}| = -\frac{a}{b}$.
$\frac{|a|}{|b|} = -\frac{a}{b} = |\frac{a}{b}|$.
$\therefore a > 0, b < 0 \to |\frac{a}{b}| = \frac{|a|}{|b|}$.

5) $a = 0, b \neq 0$.
$\frac{|a|}{|b|} = 0 = |0| = |\frac{a}{b}|$.
$\therefore a = 0, b \neq 0 \to |\frac{a}{b}| = \frac{|a|}{|b|}$.

$\therefore |\frac{a}{b}| = \frac{|a|}{|b|} \forall a, b \in \mathbb{R}, b \neq 0$.

0.7 2.2.5

If $a < x < b$ and $a < y < b$, show that $|x-y| < b-a$. Interpret this geometrically.
$a < x < b, a < y < b$.
$-a > -y > -b$.
$a - a > a - y > a - b$.
$a - b < a - y$.

$-a > -y > -b$.
$b - a > b - y > b - b$.
$b - y < b - a$.

$a < x < b$
$a - y < x - y < b - y$
$a - b < a - y < x - y < b - y < b - a$.
$a - b < x - y < b - a$.

$x - y > 0 \to |x - y| = x - y \to |x - y| < b - a$.
$x - y < 0 \to |x - y| = -(x - y)$.
$-(x - y) < -(a - b) = b - a$.
$\therefore x - y < 0 \to |x - y| < b - a$.
$\therefore |x - y| < b - a$.
Geometrically, this means that any two points selected within a range will not have a distance greater than the span of the range.

0.8 2.2.16

Let $\varepsilon > 0$ and $\delta > 0$, and $a, b \in \mathbb{R}$. Show that $V_\varepsilon(a) \cap V_\delta(a)$ and $V_\varepsilon(a) \cup V_\delta(a)$ are γ-neighborhoods of a for appropriate values of γ.
$V_\varepsilon(a) \cap V_\delta(a) = \{x \in \mathbb{R} \mid |x - a| < \varepsilon\} \cap \{x \in \mathbb{R} \mid |x - a| < \delta\} =$
$\{x \in \mathbb{R} \mid |x - a| < min(\varepsilon, \delta)\} = V_{min(\varepsilon,\delta)}(a)$.
$min(\varepsilon, \delta) \in \{\varepsilon, \delta\} \subseteq \mathbb{R}$.
$\therefore \exists \gamma \in \mathbb{R}$ where $V_\gamma(a) \subseteq V_\varepsilon(a) \cap V_\delta(a)$.

$V_\varepsilon(a) \cup V_\delta(a) = \{x \in \mathbb{R} \mid |x - a| < \varepsilon\} \cup \{x \in \mathbb{R} \mid |x - a| < \delta\} =$
$\{x \in \mathbb{R} \mid |x - a| < max(\varepsilon, \delta)\} = V_{max(\varepsilon,\delta)}(a)$.
$max(\varepsilon, \delta) \in \{\varepsilon, \delta\} \subseteq \mathbb{R}$.
$\therefore \exists \gamma \in \mathbb{R}$ where $V_\gamma(a) \subseteq V_\varepsilon(a) \cup V_\delta(a)$.

$\therefore V_\varepsilon(a) \cap V_\delta(a)$ and $V_\varepsilon(a) \cup V_\delta(a)$ are γ-neighborhoods of a for appropriate values of γ.

0.9 2.2.17

Show that $a, b \in \mathbb{R}, a \neq b \to \exists$ ε-neighborhoods U of a and V of b such that $U \cap V = \emptyset$.
Let $\alpha \in \mathbb{R}$ where $V_\alpha(a) = U$ and let $\beta \in \mathbb{R}$ where $V_\beta(b) = V$.
Without loss of generality, let $a < b$.
If we can show that $max(V_\alpha(a)) < min(V_\beta(b))$, then U and V are disjoint, as any intersection between them would contain the points $max(V_\alpha(a))$ and $min(V_\beta(b))$.
Suppose $\alpha = \beta = \frac{b-a}{3}$.
Then $\max(V_\alpha(a)) = a + \alpha = a + \frac{b-a}{3}$ and $\min(V_\beta(b)) = b - \beta = b - \frac{b-a}{3}$.

$max(V_\alpha(a)) - min(V_\beta(b)) = a + \frac{b-a}{3} - (b - \frac{b-a}{3}) = a - b + \frac{2(b-a)}{3} = (a-b)(1 - \frac{2}{3}) = (a-b)(\frac{1}{3})$.
$(a-b)(\frac{1}{3}) < 0$ since the product of positive and negative is negative.
$\therefore max(V_\alpha(a)) < min(V_\beta(b))$
$\therefore U$ and V are disjoint.
$\therefore U \cap V = \emptyset$ when $\alpha = \beta = \frac{b-a}{3}$.
$\therefore a, b \in \mathbb{R}, a \neq b \to \exists$ ε-neighborhoods U of a and V of b such that $U \cap V = \emptyset$.