M361K Homework 2

Joshua Dong

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0.1 2.3.1

Let $S_1 = \{x \in \mathbb{R} : x \geq 0\}$. Show S_1 has lower bounds, but no upper bounds. Show that $\inf S_1 = 0$.
Assume by that an upper bound, $v \in \mathbb{R}$, exists.
$v \geq 0$ (since $\exists x \in S_1$ where $x \geq 0$, and an upper bound must be greater than any given element of the set).
$v + 1 \geq 1 \geq 0$.
$v + 1 \geq 0 \rightarrow v + 1 \in S_1$. Therefore v is not an upper bound, contradiction.
$\therefore \nexists v \in \mathbb{R}$ where v is a lower bound of S_3.

0 is a lower bound of S_1, by definition of S_1.
Let $t \in \mathbb{R}$ where $t > 0$.
$t > 0$ and $2 > 0$.
$\therefore \frac{t}{2} > 0$.
$t > \frac{t}{2} > 0$.
$\therefore$ t is not a lower bound.
Thus $\inf S_1 = 0$.

0.2 2.3.3

Let $S_3 = \{\frac{1}{n} : n \in \mathbb{N}\}$. Show that $\sup S_3 = 1$ and $\inf S_3 \geq 0$ (Archemedian property 2.4).
First we show that $\frac{1}{n} > \frac{1}{n+1}$:
$\frac{1}{n} - \frac{1}{n+1} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)}$.
$1, n, (n+1) > 0$.
$\therefore \frac{1}{n(n+1)} > 0$.
$\therefore \frac{1}{n} > \frac{1}{n+1}$.
This means that the smallest $n \in \mathbb{N}$ will produce the greatest $\frac{1}{n}$.
$\therefore \sup S_3 = \frac{1}{1} = 1$.

Now we show that the infinimum is 0 because $\frac{1}{n}$ can become arbitrarily close to 0.
$1, n > 0 \to \frac{1}{n} > 0$.
$\therefore$ 0 is a lower bound for S_3.
$\therefore \exists w = \inf S, w \geq 0$.
$\forall \varepsilon > 0 \quad \frac{1}{\varepsilon} \in \mathbb{R} \to \exists n \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < n \to \frac{1}{n} < \varepsilon$
$0 \leq w \leq \frac{1}{n} < \varepsilon$
$\forall \varepsilon > 0 \quad 0 \leq w < \varepsilon$
$\therefore w = 0$.
$\therefore \inf S_3 = 0$.

0.3 2.3.4

Let $S_4 = \{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\}$. Find $\inf S_4$ and $\sup S_4$.
Let $S_4 = 1 + S_5 = 1 + \{-\frac{(-1)^n}{n} : n \in \mathbb{N}\}$.
$-\frac{(-1)^n}{n} \leq |-\frac{(-1)^n}{n}| \forall n \in \mathbb{N}$.
$\therefore \sup S_5 \leq \sup \{|-\frac{(-1)^n}{n}| : n \in \mathbb{N}\}$.
$\therefore$ 1 is an upper bound on S_5.
1 is an element of S_5 (when $n = 1$).
$\therefore 1 = \sup S_5$.
$\therefore 2 = \sup S_4$.

By symmetry, $\inf S_5 \geq \inf \{-|-\frac{(-1)^n}{n}| : n \in \mathbb{N}\}$.
Observe that this set is increasing. This means that the first value that is in both sets will be the infinimum of S_5.
$-1 \notin \inf \{-\frac{(-1)^n}{n} : n \in \mathbb{N}\}$.
$-\frac{1}{2} \in \inf \{-\frac{(-1)^n}{n} : n \in \mathbb{N}\}$.
$\therefore -\frac{1}{2} = \inf S_5$.
$\therefore \frac{1}{2} = \inf S_4$.

0.4 2.3.6

Let S be a nonempty subset of $\mathbb{R}$ that is bounded below.
Prove that $\inf S = -\sup \{-s : s \in S\}$.

By definition of infinimum, $\forall s \in S$, $\inf S \leq s$ and $\nexists$ a lower bound m such that $m > \inf S$.
$\therefore -\inf S > -s \quad \forall -s \in S$.
This implies $-\inf S$ is an upper bound for $\{-s : s \in S\}$.

Since $\nexists$ a lower bound m such that $m > \inf S$,
$m \leq \inf S \quad \forall m$ where m is a lower bound.
Using similar logic to the previous argument,
m is a lower bound $\to -m > -s \quad \forall -s \in S \to -m$ is an upper bound.
$\therefore -m \geq -\inf S \quad \forall -m$ where $-m$ is an upper bound.

$\therefore -\inf S = \sup\{-s : s \in S\}$, by definition of supremum.
$\therefore \inf S = -\sup\{-s : s \in S\}$.

0.5 2.3.7

If a set $S \subseteq \mathbb{R}$ contains one of its upper bounds, show that this upper bound is the supremum of S.
We need only prove that any upper bound of S must be greater than or equal to its higest element:
Let v be an upper bound of S. We want to show that the greatest element, u, is less than or equal to v.
Assume the contrary, that is, $u > v$. Since u is an element of S, v is not an upper bound, contradiction.
Therefore if a set $S \subseteq \mathbb{R}$ contains one of its upper bounds, this upper bound is the supremum of S.

0.6 2.3.8

Let $S \subseteq \mathbb{R}$ be nonempty. Show that $u \in \mathbb{R}$ is an upper bound of S iff ($t \in \mathbb{R}$ and $t > u$) $\to t \notin S$.
($t \in \mathbb{R}$ and $t > u$) $\to t \notin S$
$t \notin S \quad \forall t \in \mathbb{R}$ where $t > u$.
$\neg(\exists t \in \mathbb{R}$ where $t > u$ and $t \in S)$.
$t \leq u \quad \forall t \in \mathbb{R}$ where $t \in S$.
$t \leq u \quad \forall t \in S$.
This is the definition of an upper bound.