Overview

30 September 2019 10:23

Course Homepage
https://bb.imperial.ac.uk/webapps/blackboard/content/listContent.jsp?course_id=_17446_1&content_id=_1582918_1

Maths C
Vector Calculus
https://bb.imperial.ac.uk/webapps/blackboard/content/listContent.jsp?course_id=_17446_1&content_id=_1582939_1
Numerical Analysis
https://bb.imperial.ac.uk/webapps/blackboard/content/listContent.jsp?course_id=_17446_1&content_id=_1582936_1

Book(s)
Advanced_Modern_Engineering_Mathematics_GlynJames
ADVANCED ENGINEERING MATHEMATICS BY ERWIN ERESZIG1

Youtube links
Complex analysis:
https://www.youtube.com/playlist?list=PLHjOMouVJ7UXj9RqccYzktlz9DQdE6l66

Aims:
Introduce the students to concepts of Vector Calculus and Numerical Analysis required for the second and later years of their Electrical and Electronic Engineering course.

Learning Outcomes:
At the end of the course, the students will be able to

1. Understand the notation used to describe scalar and vector fields;
2. Visualise scalar and vector fields;
3. Carry out mathematical operations on scalar and vector fields;
4. Differentiate and integrate scalar and vector fields;
5. Solve partial differential equations involving scalar and vector fields;
6. Understand the notation and methods used in numerical analysis;
7. Understand the derivation of Euler’s, Heun’s, midpoint, Runge-Kutta, finite differences methods;
8. Understand the error analysis underpinning some of these methods;
9. Apply these methods to the solution of ordinary and partial differential equations, including error analysis;
10. Implement and visualise the solutions using Matlab.

Syllabus:
1. Scalar Fields, Partial Differential Equations, Vector Fields, Integrals, Differential Operators.
2. ODE methods: Euler’s, Heun’s, midpoint, Runge-Kutta. Finite differences method for ODE and PDE. Error Analysis. Implementation with Matlab.

From <http://intranet.ee.ic.ac.uk/electricalengineering/eecourses_t4/course_content.asp?c=EE2-08C&s=I2#start>

Vector calculus revision notes

12 October 2019 09:39

∇ = 'gradient operator' $= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial x}\underline{j} + \frac{\partial}{\partial z}\underline{k}$

$$\nabla\phi(x,y,z) = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}$$

$\nabla\phi$ is a vector giving the direction of most rapid change in ϕ.

$$\operatorname{div} B = \nabla\cdot B = \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}$$

div B is a scalar - it expresses how vector B changes spacially through the 6 faces of a 3D box

$$\nabla\cdot B \neq B\cdot\nabla$$

$$\operatorname{curl} B = \nabla\times B = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix} = \underline{i}[B_{3_y} - B_{2_z}] - \underline{j}[B_{3_x} - B_{1_z}] + \underline{k}[B_{2_x} - B_{1_y}]$$

The curl expresses how much swirl is in a vector field

Identities

$$\operatorname{curl}(\nabla\psi) = \nabla\times\nabla\psi = 0$$

$$\operatorname{div}(\operatorname{curl}\underline{B}) = \nabla\cdot(\nabla\times\underline{B}) = 0$$

for two scalars: ψ, ϕ

$$\nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi$$

for vector $\underline{b}$ and scalar ψ

$$\operatorname{div}(\psi\underline{B}) = \psi \operatorname{div}\underline{B} + (\nabla\psi)\cdot\underline{B}$$

$$\operatorname{curl}(\psi\underline{B}) = \psi \operatorname{curl}\underline{B} + (\nabla\psi)\times\underline{B}$$

The max value of $D_{\underline{u}} f(x)$ is given by $\|\nabla f(x)\|$ and occurs in the direction given by $\nabla f(x)$

and occurs in the direction given ...

Proof: $D_{\vec{u}} f = \nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos\theta$ $\rightarrow = 1$ as is unit vector which has magnitude 1

$= \|\nabla f\| \cos\theta$ $\hookrightarrow$ θ is the angle between ∇ and $\vec{u}$

$\theta = 0 \rightarrow \cos\theta = 1$

$\hookrightarrow \max[\|\nabla f\| \cos\theta]$

$= \|\nabla f\|$

At this point, the angle between ∇ and $\vec{u} = 0$

i.e. the vector $\vec{u}$ is pointing in the same direction as the gradient ∇

$$\text{scalar field} \xrightarrow{\text{grad}} \text{vector field} \xrightarrow{\text{div}} \text{scalar field}$$
$$\text{scalar field} \xrightarrow{\text{grad}} \text{vector field} \xrightarrow{\text{curl}} \text{vector field}$$
$$\text{vector field} \xrightarrow{\text{div}} \text{scalar field} \xrightarrow{\text{grad}} \text{vector field}$$
$$\text{vector field} \xrightarrow{\text{curl}} \text{vector field} \xrightarrow{\text{div}} \text{scalar field}$$
$$\text{vector field} \xrightarrow{\text{curl}} \text{vector field} \xrightarrow{\text{curl}} \text{vector field}.$$

Fact

(a) *For any scalar field f we have* $\text{curl}(\text{grad}(f)) = \nabla \times (\nabla(f)) = 0.$

(b) *For any vector field* $\mathbf{u}$ *we have* $\text{div}(\text{curl}(\mathbf{u})) = \nabla.(\nabla \times \mathbf{u}) = 0.$

Max Rate of Change

$D_{\underline{u}} f = \nabla f \cdot \underline{u} = |\nabla f||u| \cos\theta$

$= |\nabla f| \cos\theta$

The max rate of change is when $\cos\theta = 1$

- The unit vector is parallel to the gradient
- The max value is $|\nabla f|$ ← magnitude of the gradient
- The gradient points in the direction of max rate of change
- Magnitude of gradient is max rate of change

Line / path integration notes

12 October 2019 10:02

Line Integrals of Scalar Fields (ψ)

$$\int_C \psi(x,y,z)\, ds$$ ← General Formula

$$ds = \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$ { where $\dot{x}$, $\dot{y}$ denotes dx/dt, dy/dt

Example $\int_C x^2 y\, ds$ where C is the arc of $x^2+y^2=1$ in the first quadrant

1. Parametrise:

$x = \cos t$ $\quad y = \sin t$ $\quad ds = \sqrt{\cos^2 t + \sin^2 t}\; dt = 1 \cdot dt = dt$

$\dot{x} = -\sin t$ $\quad \dot{y} = \cos t$ $\quad 0 \le t \le \pi/2$

2. Substitute into Formula & Solve

$$\int_C x^2 y\, ds = \int_0^{\pi/2} \cos^2 t \sin t\, dt = \left[-\frac{\cos^3 t}{3}\right]_0^{\pi/2} = 1/3$$

Line Integrals of Vector Fields

$$\int_C \underline{F}(x,y,z)\cdot d\underline{r} = \int_C (F_1\, dx + F_2\, dy + F_3\, dz)$$ (General formula)

Example: find $\int_C \underline{F} \cdot d\underline{r}$ where $\underline{F} = x^2 y\, \underline{i} + (x-z)\, \underline{j} + xyz\, \underline{k}$ and C is $y = x^2$ in plane $z=2$ from $(0,0,2) \to (1,1,2)$

1. Substitute into Formula

$$\int_C (x^2 y\, dx + (x-z)\, dy + xyz\, dz)$$

2. Find ways to remove variables / simplify

If we are in plane $z=2$ then $dz = 0$

If $y = x^2$ $\quad dy = 2x\, dx$ $\quad$ so we try to get integral in terms of x and dx only

$$\int_C (x^2(x^2)dx + (x-2)2x\, dx + xyz \cdot 0) = \int_C (x^4\, dx + (2x^2 - 4x)\, dx)$$

3. Find limits & solve

$(\underline{0}, 0, 2) \to (\underline{1}, 1, 2)$ we choose the x values as we are dx

$$\int_0^1 (x^4 + 2x^2 - 4x)\, dx = \left[x^5/5 + x^3 - 2x\right]_0^1 = -17/15$$

Independence Of Path

1. $\int_C \underline{F} \cdot d\underline{r}$ is independant of path if

$$\boxed{\text{curl } \underline{F} = 0}$$

2. If C is a closed path

$$\boxed{\oint_C \underline{F} \cdot d\underline{r} = 0}$$

Double integrals notes

15 October 2019 20:11

Change Of Variables

$$\iint_R f(x,y)\,dA = \iint f(x(u,v)), f(y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\,dr$$

$$J_{u,v}(x,y) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Example: $\iint_R (x-3y)\,dA$ where R is the region with vertices $(0,0)$, $(2,1)$, $(1,2)$ using $x = 2u+v$ $\quad y = u+2v$

1. Find the jacobian:

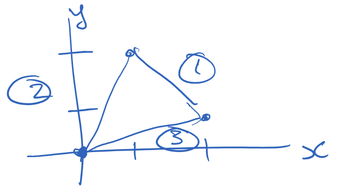
$$J_{u,v}^{x,y} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4-1 = 3$$

2. Sub v and u into the $\int$ expression $\rightarrow \iint (-u-5v)\,|3|\,du\,dv$

$$x - 3y = 2u + v - 3u - 6v = -u - 5v$$

3. Find limits

Verticies: $(0,0)$, $(2,1)$, $(1,2)$ $\longrightarrow$



Find eq of each line

1 $\Rightarrow$ $y = 3 - x$

2 $\Rightarrow$ $y = 2x$

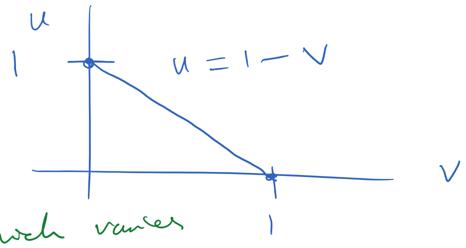
3 $\Rightarrow$ $y = \frac{1}{2}x$

Sub u, v into these equations $(y = u+2v \quad x = 2u+v)$

1 $\Rightarrow$ $u + 2v = 3 - 2u - v$ $\rightarrow$ $3u = 3 - 3v$ $\rightarrow$ $\boxed{u = 1 - v}$

2 $\Rightarrow$ $u + 2v = 4u + 2v$ $\rightarrow$ $3u = 0$ $\rightarrow$ $\boxed{u = 0}$

3 $\Rightarrow$ $u + 2v = u + \frac{v}{2}$ $\rightarrow$ $\frac{3v}{2} = 0$ $\rightarrow$ $\boxed{v = 0}$



4. Substitute Into Integral

$$3\int_0^1\int_0^{1-v} (-u-5v)\,du\,dv \leftarrow$$

we integrate wrt u first, which varies between 0 and 1−v then wrt v, which varies between 0 and 1.

5. Integrate Outer, then Inner Expressions

$$3\int_0^1 \left[-\frac{u^2}{2} - 5vu\right]_0^{1-v} dv = 3\int_0^1 \left[-\frac{(1-v)^2}{2} - 5v(1-v)\right] dv$$

$$3\int_0 \left[-\frac{u^2}{2} - 5vu\right]_0 dv = 3\int_0 \left[-\frac{(1-v)^2}{2} - 5v(1-v)\right] dv$$

$$= 3\int_0^1 \left(-\frac{(1-2v+v^2)}{2} - 5v + 5v^2\right) dv = 3\int_0^1 \left(\frac{9}{2}v^2 - 4v - \frac{1}{2}\right) dv$$

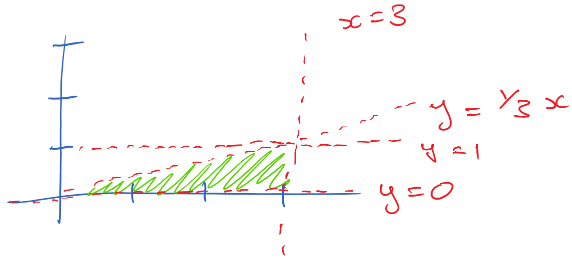
$$= 3\left[\frac{3}{2}v^3 - 2v^2 - \frac{1}{2}v\right]_0^1 = 3\left[\frac{3}{2} - 2 - \frac{1}{2}\right] = 3[-1] = -3$$

Changing Order of Integration

Example: $\int_0^1\int_{3y}^3 e^{x^2}\, dx\, dy$

1. Set the inner limits = first variable of integration, and outer to the second

$$\int_{0=y}^{1=y}\int_{3y=x}^{3=x}$$



2. Graph these lines/curves →

3. Find the new limits from the graph

   In this case lower limit $y=0$, and the upper limit = line $y = \frac{1}{3}x$

   Then for the outer integral the lower limit $x=0$ and upper limit $x=3$

   So the integral becomes: $\int_0^3\int_0^{\frac{1}{3}x} e^{x^2}\, dy\, dx$

4. Integrate :)

$$\int_0^3 \left[e^{x^2} y\right]_0^{\frac{1}{3}x} dx = \int_0^3 \frac{1}{3}x e^{x^2}\, dx$$

→ $u = x^2$

$du = 2x\, dx$

$x=0 \quad u=0$

$x=3 \quad u=9$

$$\frac{1}{3}\int_0^3 \frac{1}{2} e^u\, du = \frac{1}{6}\left[e^u\right]_0^9 = \boxed{\frac{1}{6}e^9}$$

Line integrals

16 October 2019 11:55

Line Integrals

Line Integrals are the same as normal integrals, but we integrate over a curve in a space/plane.

- This means we must represent a curve by a parametric representation:

$$\underline{r}(t) = [x(t), y(t), z(t)] = x\underline{i} + y\underline{j} + z\underline{k} \qquad a \leq t \leq b$$

- This is the 'path of integration', from $\underline{r}(a)$ to $\underline{r}(b)$
- If A and B coincide then the path is closed.

The line integral is defined by:

dot product

$$\int_C \underline{F}(\underline{r}) \cdot d\underline{r} = \int_a^b \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt$$

scalar due to dot product

$$= \int_a^b \left(F_1 x' + F_2 y' + F_3 z'\right) dt$$

A closed path is denoted by: $\oint_C$

The line integral may be split up

$$\int_C \underline{F} \cdot d\underline{r} = \int_{C_1} \underline{F} \cdot d\underline{r} + \int_{C_2} \underline{F} \cdot d\underline{r}$$

Example: Line Integral on a Plane

$\underline{F}(\underline{r}) = -y\underline{i} - xy\underline{j}$ $\qquad C =$ circular arc $r = 1$ in 1st quadrant

1. Parametrise into t variable, we are using a circle

so $C \Rightarrow \quad r(t) = \cos t\, \underline{i} + \sin t\, \underline{j} \qquad 0 \leq t \leq \pi/2$

$x(t) = \cos t \qquad y(t) = \sin t$

2. Substitute

$$F(r(t)) = -\sin t\, \underline{i} - \sin t \cos t\, \underline{j}$$

$$r'(t) = -\sin t\, \underline{i} + \cos t\, \underline{j}$$

$$\int_0^{\pi/2} (-\sin t, -\sin t \cos t) \cdot (-\sin t, \cos t)\ dt$$

$$\int_0^{\pi/2} \sin^2 t - \sin t \cos^2 t\ dt$$

3. Solve $\quad u = \cos t \quad t = \pi/2, u = 0 \quad t = 0, u = 1$

$$\int_0^{\pi/2} \frac{1}{2}(1 - \cos 2t)\ dt + \int_1^0 u^2\ du$$

$$\frac{1}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi/2} + \left[u^3/3\right]_1^0$$

$$\frac{1}{2}\left[\pi/2\right] + \left[-1/3\right] = \underline{\underline{\pi/4 - 1/3}}$$

Example: Line Integral in Space

$\underline{F}(\underline{r}) = z\,\underline{i} + x\,\underline{j} + y\,\underline{k} \quad C = \text{helix} \quad r(t) = \cos t\,\underline{i} + \sin t\,\underline{j} + 3t\,\underline{k}$

1. Parametrize

from $r(t)$, we know $x(t) = \cos t \quad y(t) = \sin t \quad z(t) = 3t$

$$r'(t) = (-\sin t, \cos t, 3)$$

2. Substitute

$$\int_0^{2\pi} (3t, \cos t, \sin t) \cdot (-\sin t, \cos t, 3)\ dt$$

3. Solve

$$\int_0^{2\pi} -3t\sin t + \cos^2 t + 3\sin t \; dt$$

$u = t \quad v = -\cos t$

$u' = 1 \quad v' = \sin t$

$$-3\left\{\left[-t\cos t\right]_0^{2\pi} + \int_0^{2\pi} \cos t \, dt\right\} + \int_0^{2\pi} \frac{1}{2}\cos 2t + 1 \; dt + 3\left[-\cos t\right]_0^{2\pi}$$

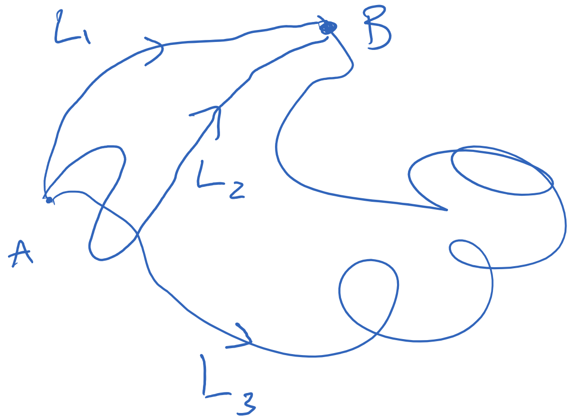
$$6\pi - 3\left[\sin t\right]_0^{2\pi} + \frac{1}{2}\left[\frac{-1}{2}\sin 2t + t\right]_0^{2\pi} + 3\left[-\cos t\right]_0^{2\pi}$$

$$6\pi - 0 + \frac{1}{2}(2\pi) + 0 \qquad = 6\pi + \pi = 7\pi$$

Path Independance

A line integral is path independant if for every start & endpoints A, B, the line integral evaluates to the same value for all paths

i.e No matter what path you take, if you start at A and end at B the line integral value is the same



all = the same value

For a domain D, path independence holds if:

1. $\underline{F} = \text{grad } f$
2. Integration around closed curves $C = 0$
3. curl $\underline{F} = \underline{0}$

(1) $\underline{F} = \text{grad } f \Rightarrow F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}$

$$\int_A^B (F_1\, dx + F_2\, dy + F_3\, dz) = f(B) - f(A)$$

Example: Show $\int_C (2x\, dx + 2y\, dy + 4z\, dz)$ is path independant & find its value from $(0,0,0)$ to $(2,2,2)$

$F = (2x, 2y, 4z) = \text{grad } f$

$f = x^2 + y^2 + 2z^2 \qquad \frac{\partial f}{\partial x} = F_1 = 2x \quad \frac{\partial f}{\partial y} = 2y = F_2 \quad \frac{\partial f}{\partial z} = F_3 = 4z$

so independant so

$$\int_C \underline{F}\, d\underline{r} = f(B) - f(A) = (4+4+8) - (0+0+0) = 16$$

Double integrals

16 October 2019 14:18

The double integral of $f(x,y)$ over the region R is:

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA$$

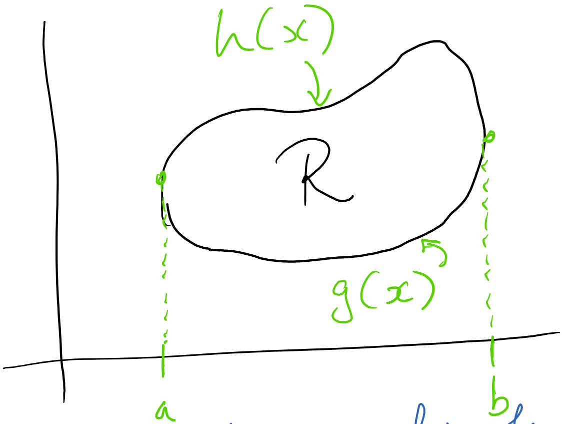
Evaluation by two Successive Integrations

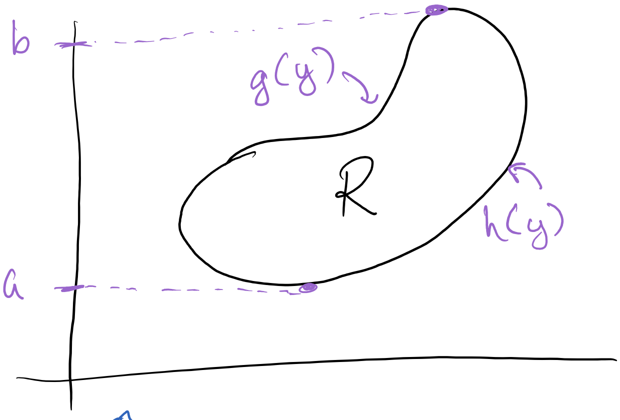
$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\,dy \right] dx$$

$h(x)$ and $g(x)$ represent the boundary curve of R.

- the result of the inner integral will be a function of x (due to limits).

$$= \int_a^b \left[\int_{g(y)}^{h(y)} f(x,y)\,dx \right] dy$$





finding limits for $dy\,dx$ generally

for $dx\,dy$

Right = top
Left = bottom

Top = top
Bottom = Bottom (limits)

Change of variables (Jacobian)

$$\iint f(x,y)\,dx\,dy = \iint f(x(u,v),\ y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

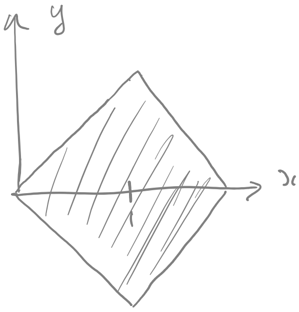
$$\iint_R f(x,y)\,dx\,dy = \iint_R f(x(u,v),\, y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

this can also be written

$$J_{u,v}(x,y) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

the two lines are 'absolute value' lines

Example: evaluate $\iint_R (x^2+y^2)\,dx\,dy$



1. (If not given) find transformation

$x+y=u \qquad x-y=v$

$x=\frac{1}{2}(u+v) \qquad y=\frac{1}{2}(u-v)$

2. Find the Jacobian $\Rightarrow \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$

this will just be $+\frac{1}{2}$ in integral as we take $|-\frac{1}{2}| = \frac{1}{2}$

3. Sub into formula

$$\frac{1}{2}\int_0^2\!\!\int_0^2 \left[\tfrac{1}{2}(u+v)\right]^2 + \left[\tfrac{1}{2}(u-v)\right]^2 du\,dv$$

$$= \frac{1}{8}\int_0^2\!\!\int_0^2 u^2+v^2+\cancel{2uv}+u^2+v^2-\cancel{2uv}\quad du\,dv$$

$$= \frac{1}{4}\int_0^2\!\!\int_0^2 u^2+v^2\quad du\,dv = \frac{8}{3}$$

Often we use polar coordinates $x=r\cos\theta \quad y=r\sin\theta$

where $J_{r,\theta}\,x,y = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$

$= r(\cos^2\theta + \sin^2\theta)$

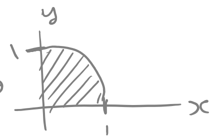
$= r \cdot 1 = \underline{\underline{r}}$

$$\boxed{\iint f(x,y)\,dx\,dy = \iint f(r\cos\theta,\, r\sin\theta)\, r\,dr\,d\theta}$$

$$\boxed{\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Example: Polar Coordinates

$f(x,y) = 1 \rightarrow$ mass density in $\rightarrow$



find the total mass

$$M = \iint_R dx\,dy = \int_0^{\pi/2}\int_0^1 r\,dr\,d\theta = \int_0^{\pi/2}\left[r^2/2\right]_0^1 d\theta = \int_0^{\pi/2}\frac{1}{2}\,d\theta$$

$$= \left[\tfrac{1}{2}\theta\right]_0^{\pi/2} = \pi/4$$

using formula above.

Integral Theorems

16 October 2019 19:16

Greens Theorem

Greens Theorem transforms double integrals onto line integrals & vice versa

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \oint_c (F_1\, dx + F_2\, dy)$$

Example: $F_1 = y^2 - 7y$ $F_2 = 2xy + 2x$ for circle $x^2 + y^2 = 1$

$$\iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\; dx\, dy = \iint_R (2y+2) - (2y-7)\; dx\, dy$$

$$= \iint_R 9 \; dx\, dy = 9 \iint dx\, dy = 9\pi \leftarrow \text{LHS}$$

RHS: $\underline{r}(t) = (\cos t, \sin t)$ $\underline{r}'(t) = (-\sin t, \cos t)$

$F_1 = \sin^2 t - 7\sin t$ $F_2 = 2\cos t \sin t + 2\cos t$

$$\oint_c (F_1 x' + F_2 y')\, dt = \int_0^{2\pi} 7\sin^2 t - \sin^3 t + 2\cos^2 t \sin t + 2\cos^2 t \quad dt$$

change of variables

$$= 7\int_0^{2\pi} \sin^2 t - \int_0^{2\pi} \sin^3 t + 2\int_0^{2\pi} \cos^2 t \sin t + 2\int_0^{2\pi} \cos^2 t$$

$u = \cos t$

$du = -\sin t\, dt \longleftrightarrow$ use same sub

$$= 7\int \frac{1-\cos 2t}{2} + \int 1-u^2\, du - 2\int u^2 + \cancel{2}\int \frac{\cos 2t + 1}{\cancel{2}}$$

$$= \frac{7}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^{2\pi} + \left[\cos t - \frac{1}{3}\cos^3 t\right]_0^{2\pi} - 2\left[\frac{1}{3}\cos^3 t\right]_0^{2\pi} + \left[\frac{1}{2}\sin 2t + t\right]_0^{2\pi}$$

$$= 7\pi + 0 - 0 + 2\pi = 9\pi$$

The area A enclosed by circle C

$$A = \iint_A 1\, dx\, dy = \oint_C x\, dy = -\oint y\, dx = \frac{1}{2}\oint_C (-y\,dx + x\,dy)$$

$$x = x(u,v) \qquad y = y(u,v)$$

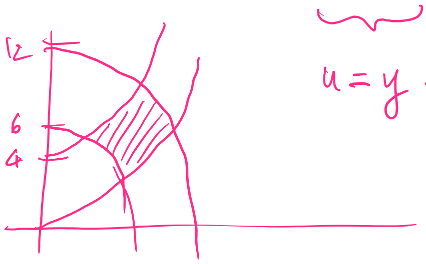
$$A' = \iint_{A'} du\, dv = \oint_{C'} u\, dv = \oint_{C'} u\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)$$

..... etc

$$\iint_A \underbrace{\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)}_{\text{Jacobian}} dx\,dy$$

Example: $\iint xy\, dx\,dy$ over region bounded by

$y = x^2+4$, $y = x^2$, $y = 6 - x^2$, $y = 12 - x^2$ $\quad y \geq 0 \quad x \geq 0$



$$u = y + x^2 \qquad v = y - x^2$$

$$J(x,y)_{(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \left[\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}\right]^{-1}$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial x} = -2x \qquad \frac{\partial v}{\partial y} = 1$$

$$\Rightarrow [2x + 2x]^{-1} = 1/4x \qquad y = (u+v)/2$$

$$\Rightarrow \iint_{A'} xy \frac{1}{4x}\, du\, dv = \frac{1}{4}\iint_{A'} y\, du\, dv = \frac{1}{8}\iint (u+v)\, du\, dv$$

$u \Rightarrow 6 \to 12$ $\qquad$ $v \Rightarrow 4 \to 4$

$$\frac{1}{8}\int^{4} dv \int^{12} (u+v)\, du = \frac{1}{8}\int^{4}\left[\frac{u^2}{2} + vu\right]_6^{12} dv$$

$$\frac{1}{8}\int_0^4 dv\int_6^{12}(u+v)\,du = \frac{1}{8}\int_0^4\left[\frac{u^2}{2}+vu\right]_6^{12} dv$$

$$= \frac{1}{8}\int_0^4\left[(72+12v)-(18+6v)\right] dv$$

$$= \frac{1}{8}\int_0^4 54+6v\ dv = \frac{1}{8}\left[54v+3v^2\right]_0^4$$

$$= \frac{1}{8}\left[216+48\right] = \frac{264}{8} = 8\overline{)264}^{\,033} = 33$$

Surface Integrals

We may integrate over two types of Surfaces:

a) $\iint_S f(x,y,z)\,dS$ b) $\iint_S \underline{F}(\underline{r})\cdot\hat{\underline{n}}\,dS = \iint_S \underline{F}(\underline{r})\cdot d\underline{S}$

Scalar field Vector fields

$$\iint_S \underline{F}(\underline{r})\cdot d\underline{S} = \iint_A \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right) dx\,dy$$

$$\iint_S f(x,y,z)\,dS = \iint_A f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$$

Example

Evaluate $\iint_S (x+y+z)\,dS$ $S \Rightarrow x^2+y^2+z^2=1$ in 1^{st} quad

$\sqrt{\quad}$ $\frac{\partial z}{\quad} = \frac{-x}{\quad}$ $\frac{\partial z}{\quad} = \frac{-y}{\quad}$

$$z = \sqrt{1 - y^2 - x^2} \qquad \frac{\partial z}{\partial x} = \frac{-x}{\sqrt{1 - y^2 - x^2}} \qquad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{1-y^2-x^2}}$$

$$\iint x + y + \sqrt{1 - y^2 - x^2} \qquad \underline{etc}$$

Exercises

24 October 2019 16:41

Glyn James

find total diff $u(x,y) = x^y$

$\frac{\partial u}{\partial x} = yx^{y-1} \qquad \frac{\partial u}{\partial y} = x^y \ln x$

$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = yx^{y-1} dx + x^y \ln x \, dy$

- find grad f for $f(r) = 3x^2 + 2y^2 + z^2$ at $(1,2,3)$
  Calculate the directional derivative at $(1,2,3)$ in
  the direction of $\underline{u} = \frac{1}{3}(2,2,1)$

$\frac{\partial f}{\partial x} = 6x \quad \frac{\partial y}{\partial x} = 4y \quad \frac{\partial z}{\partial x} = 2z$

grad $f = 6x\underline{i} + 4y\underline{j} + 2z\underline{k}$

grad $f(1,2,3) = 6\underline{i} + 8\underline{j} + 6\underline{k}$

directional derivative $= (6,8,6)\cdot\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

$= 4 + \frac{16}{3} + \frac{\cancel{6}\,2}{\cancel{3}} = 34/3$

- find max rate of change of f at $(1,2,3)$ and
  direction.

Max rate of change occurs on the direction parallel to grad f at $(1,2,3) \Rightarrow (6,8,6)$

$\underline{u} = \frac{1}{\sqrt{36+64+36}}(6,8,6) = \frac{1}{2\sqrt{34}}(6,8,6)$

$= \frac{1}{\sqrt{34}}(3,4,3)$

max rate of $f(r)$ is $|\text{grad } f|$

$|\text{grad } f| = \sqrt{36+64+36} = 2\sqrt{34}$

- $2z = x^2 + y^2$ find $\underline{u}$ at $(1,3,5)$ obtain eq of normal and the tangent plane to the surface at $(1,3,5)$

$f = x^2 + y^2 - 2z \quad \text{grad } f = (2x, 2y, -2)$

at $(1,3,5) \rightarrow \; = (2, 6, -2)$

$$\underline{u} = \frac{1}{\sqrt{4+36+4}}(2, 6, -2) = \frac{1}{\sqrt{11}}(1, 3, -1)$$

eq of line in direct of normal ie:

$$\frac{x-1}{1} = \frac{y-3}{3} = \frac{z-5}{-1}$$

eq of tangent plane

$$(1)(x-1) + 3(y-3) + (-1)(z-5) = 0$$

$$x + 3y - z = 5$$

Notes for test revision

28 October 2019 13:26

Question type	Solution
Plot the vector fields	Follow until pattern is spotted 1. Find values along the axes 2. Find along y=±x 3. Find along y=±2x or 1/2 x
Find ∇(ϕ) Find grad(ϕ)	$\nabla(\varphi) = \boldsymbol{i}\frac{\partial \phi}{\partial x} + \boldsymbol{j}\frac{\partial \phi}{\partial y} + \boldsymbol{k}\frac{\partial \phi}{\partial z}$
Find div(ϕ)	$div(\varphi) = \nabla \cdot \phi = \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y} + \frac{\partial \phi_3}{\partial z}$
Find curl(ϕ)	$curl(\phi) = \nabla \times \phi = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_1 & \phi_2 & \phi_3 \end{vmatrix}$
Find directional derivative	$\nabla(\phi(p_1, p_2, p_3))$ (e.g. find del(phi) and substitute the point given)
Max rate of change at point P	Find $\lvert\nabla(\phi(p_1, p_2, p_3))\rvert$
Direction of max rate of change at point P	1. Find ∇(ϕ) 2. Normalise the vector of ∇(ϕ) at point P to unit vector
Unit normal to the surface ϕ(x, y, z) at point P	1. Vector normal to surface = ∇(ϕ) 2. Normalise for unit normal
Equation of the normal to the surface ϕ(x, y, z) at point P	1. Find vector normal = ∇(ϕ) call this v. 2. Equation of the normal line is found from P and step 1: 3. $\frac{x - P_1}{v_1} = \frac{y - P_2}{v_2} = \frac{z - P_3}{v_3}$
Equation of the tangent plane to the surface at point P	1. Find vector normal 2. Equation of tangent plane: 3. $(v_1)(x - P_1) + (v_2)(y - P_2) + (v_3)(z - P_3) = 0$
Change of coordinates	

Test notes

09 November 2019 12:09

- Divergence is always a scalar
- $\text{Div} = \nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$
- $\text{Curl} = \nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

- Conservative vector field: we can express the vector field in terms of the scalar field ϕ

- Path definition: $\int_a^b \underline{F}(r(t)) \cdot \underline{r}'(t)\, dt = \int_A^B \underline{F}(r(t)) \cdot d\underline{r}$
- Closed path: $\oint_C \underline{F}(r(t)) \cdot d\underline{r}$

Parametrization: Circles / Cylinders / etc $\Rightarrow$ Polar
Straight lines / curves $\Rightarrow t \rightarrow$ find from $x_1 \to x_2$, $y_1 \to y_2$ ....
$0 \leq t \leq 1$

- If vector field is conservative:
  - $E_x = \frac{\partial \phi}{\partial x} \quad E_y = \frac{\partial \phi}{\partial y} \quad E_z = \frac{\partial \phi}{\partial z}$
  - $E_x\, dx + E_y\, dy + E_z\, dz = d\phi$
  - $\int_C E_x\, dx + E_y\, dy + E_z\, dz = \int_C d\phi$
  - $\int_C d\phi = \phi(B) - \phi(A)$

- Double Integrals: $\int_a^b \int_{g_2(x)}^{g_1(x)} f(x,y) \underbrace{dy\, dx}_{dA}$ — order doesn't matter until limits set
  - Change of Variables:
  - $\text{Jacobian} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$\Rightarrow \iint f(x,y)\, dx\, dy = \iint f(x(u,v), y(u,v))\, J\frac{(x,y)}{(u,v)}\, du\, dv$

$\Rightarrow$ Polar - Cartesian Relationship:

$$\iint f(x,y)\, dx\, dy \rightarrow \iint f(r\cos\theta, r\sin\theta)\, \boxed{r}\, dr\, d\theta$$

- Green's Theorem: $\oint_C \underline{F} \cdot d\underline{r} = \iint_R (\nabla \times F) \cdot \underline{k}\, dA$ — unit vector $= 1$

Also written as

$$= \oint_C F_1\, dx + F_2\, dy = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dx\, dy$$

- **Surface Integrals & Fluxes:**

$$\iint_S dS = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy$$

'Projection of the surface on the x, y plane'

$$\iint_S f(x, y, z)\, dS = \iint_D f(\underline{r}(u, v)) \left| \underline{r}_u \times \underline{r}_v \right| du\, dv$$

$r(u, v) = x(u, v)\,\underline{i} + y(u, v)\,\underline{j} + z(u, v)\,\underline{k}$

and

$$\iint_S dS = \iint_D \left| \underline{r}_u \times \underline{r}_v \right| du\, dv$$

normal vector associated with element of surface area

D is the domain which features the projection onto u, v plane.
(or x, y ... etc)

- **Fluxes:** $\Phi = \iint_S \underline{E} \cdot \hat{n}\, dS =$

$f(x, y, z) = z - g(x, y)$

∇f is normal to surface given by $f(x, y, z) = 0$, so we have normal vector to surface

$$\boxed{\hat{n} = \frac{\nabla f}{|\nabla f|}}$$

— Parametrized surfaces:

$$\hat{n} = \frac{\underline{r}_u \times \underline{r}_v}{|\underline{r}_u \times \underline{r}_v|} \quad \text{and} \quad \iint_S F \cdot dS = \iint_D \underline{E} \cdot (\underline{r}_u \times \underline{r}_v)\, dA$$

YouTube notes

11 November 2019 18:55

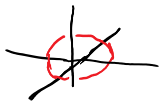
Curve	Parametric Equations
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ellipse)	$x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$
$x^2 + y^2 = r^2$ (circle)	$x = r\cos t$, $y = r\sin t$, $0 \le t \le 2\pi$
$y = f(x)$	$x = t \quad y = f(t)$
$x = f(y)$	$y = t \quad x = f(t)$
$(x_0, y_0, z_0) \rightarrow (x_1, y_1, z_1)$	$x = (1-t_0)x_0 + tx_1$, $y = (1-t_0)y_0 + ty_1$, $z = (1-t_0)z_0 + tz_1$, $0 \le t \le 1$

Cylindrical Coordinates: (r, θ, z)

$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

$r^2 = x^2 + y^2 \quad z = z \quad \tan\theta = y/x$

r is distance from origin to projection on $x-y$ plane



eg. if doesn't travel up z plane (is flat)

Spherical Coordinates: (ρ, θ, ϕ)

$(x = r\cos\theta, \; y = r\sin\theta, \; r = \rho\sin\phi)$

$\Rightarrow \; x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$

$\Rightarrow \; \rho^2 = x^2 + y^2 + z^2 \quad \tan\theta = y/x \quad \cos\phi = z/\rho$

Gradient Vector & Stuff

∇f is orthogonal to vector/surface f at point (x_0, y_0, z_0)

'normal'

- find tangent plane & normal line to $f(x, y, z) = a$

1. $\nabla F(x, y, z)$
2. $\nabla F(x_0, y_0, z_0) = (a, b, c)$
3. $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ (tangent plane)
4. $\underline{r}(t) = (x_0, y_0, z_0) + t(a, b, c)$ (normal line)

Double Integrals

Polar coordinates: $dA = r\,dr\,d\theta$

Cylindrical coords: $dV = r\,dz\,dr\,d\theta$

Spherical coords: $dV = \rho^2 \sin\varphi\,d\rho\,d\theta\,d\varphi$

Change of Variables

Something

02 January 2020 13:08

1. Show satisfies laplace:

$$u_{xx} + u_{yy} = 0$$

2. find conjugate v:
   - Step 1
   - $u_x = v_y \quad u_y = -v_x$
   - Integrate v_y & v_x
   - Solve $A(x)$ & $B(y)$
   - final v

3. find $f(z)$
   - Step 1 & 2
   - $f(z) = u + iv$ (sub)

Year 1 Revision Notes

09 October 2019 21:07

Planes

$ax + by + cz = d$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Equivalent

To find eq of a plane:

1. find a point and a normal vector

This may be found

1. (a, b, c) from plane equation will be normal
2. finding 2 vectors from points / or given and taking cross product

Partial Derivatives

$$f_x = \frac{\partial}{\partial x} \qquad f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

Order is opposite

Remember:

$\boxed{f = \text{first}}$

eg first letter is done first, so $f_{xy} \to \partial/\partial x \to \frac{\partial}{\partial y}$

Total Differential

$$df = \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy$$

← doesn't need to be df $\rightsquigarrow$ dz, dw, ...

$z = e^{x^2+y^2} \tan(2x)$

$\partial/\partial x = 2xe^{x^2+y^2} \tan 2x + e^{x^2+y^2} 2 \tan 2x$

$\partial/\partial y = 2ye^{x^2+y^2} \tan 2x$

$dz = (2xe^{x^2+y^2} \tan 2x + 2e^{x^2+y^2} \tan 2x)\, dx + (2ye^{x^2+y^2} \tan 2x)\, dy$

Chain Rule

$dy/dt = dx/dt \; dy/dx$

$z = f(x, y) \qquad x = g(t) \qquad y = h(t)$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

$z = xe^{xy} \qquad x = t^2 \qquad y = t^{-1}$

$dx/dt = 2t \qquad dy/dt = -t^{-2}$

$\partial f/\partial x = e^{xy} + yxe^{xy} \qquad \partial f/\partial y = x^2 e^{xy}$

$$\frac{dz}{dt} = (e^{xy} + yxe^{xy})(2t) + (x^2 e^{xy})(-t^{-2})$$

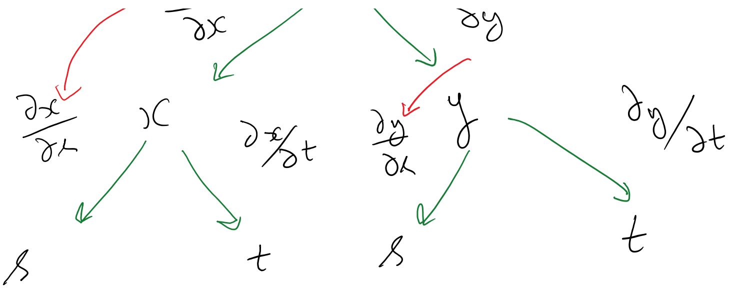
Tree Diagram:

z

$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y}$

$z = f(x, y)$

Tree Diagram



$z = f(x, y)$

$x = g(s, t)$

$y = h(s, t)$

So for $\dfrac{dz}{ds} \rightarrow \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}} \rightarrow \begin{aligned} F_x &= \frac{\partial}{\partial x}(f(x,y)) \\ F_y &= \partial/\partial y(f(x,y)) \end{aligned}$$

find dy/dx for $x\cos(3y) + x^3y^5 - 3x + e^{xy} = 0$

$F_x = \cos(3y) + 3x^2y^5 - 3 + ye^{xy}$

$F_y = -3x\sin 3y + 5x^3y^4 + xe^{xy}$

$dy/dx = -F_x/F_y$

Lecture 1 - Vector Calculus: Gradient, Divergence and Curl

07 October 2019 19:00

$$f(x,y,z) = 2x^2y^2 + yz - x$$

The Nabla ∇ is the differential operator

Which can be written like this:

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

So the differential of function f is the derivative of the function with respect to each of its variables

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Which in this example would be this

$$= \langle 4xy^2 - 1, 4x^2y + z, y \rangle$$

Divergence

The divergence of vector F is equal to the vector F dot product with the differential operator, which is the same as the sum of the differential of the i expression with respect to x, j with respect to y, and k with respect to z.

$$\operatorname{div} \underline{F} = \nabla \cdot \underline{F} = P_x + Q_y + R_z$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$

So for this example

$$\text{if} \quad \underline{F} = \langle x^2 + y^2, y^2 + z^2, x^2 + z^2 \rangle$$

We compute each i, j, k component derivative,

$$\frac{\partial}{\partial x}(x^2 + y^2) = 2x \qquad \frac{\partial}{\partial y}(y^2 + z^2) = 2y$$

$$\frac{\partial}{\partial z}(x^2 + z^2) = 2z$$

and then sum to get the divergence.

$$\nabla \cdot \underline{F} = 2x + 2y + 2z$$

Curl

The curl of a vector F is defined as the cross product of the vector F and the derivative function

$$\operatorname{curl} \underline{F} = \nabla \times \underline{F} =$$

We use a matrix to find the calculations needed, where P, Q and R are just the expressions in the i, j and k components of the vector

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \boxed{\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle}$$

Using the vector F from before and taking the derivatives from the step above gives the following

$$\operatorname{curl} \underline{F} = \langle 0 - 2z, 0 - 2x, 0 - 2y \rangle$$

And then simplify to complete the curl

$$= \langle -2z, -2x, -2y \rangle$$

following

And then simplify to complete the curl.

$$= \langle -2z, -2x, -2y \rangle$$

The directional derivative:

If f is a differentiable function of x any y, then f has a directional derivative in the direction of any unit vector, it is defined by this function (can be extended for more variables but using 2 in this example).

$$D_{\underline{u}} f = \nabla f \cdot \underline{u}$$

Example: find the directional derivative of f(x, y) at (2,1) given by the angle theta

$$f(x,y,z) = x^3 - xy^2 - z \quad \text{at } (1,1,0)$$

$$\underline{v} = 2i - 3j + 6k$$

1. Find the unit vector
   a. If given a non-normal vector, normalise it by dividing by the length

$$\underline{u} = \frac{1}{\sqrt{2^2 + 3^2 + 6^2}}(2,-3,6)$$

$$= \frac{1}{7}(2,-3,6)$$

2. Find the differential of f(x, y, z)

$$\nabla f(x,y,z) = (f_x, f_y, f_z)$$

$$= (3x^2 - y^2, -2xy, -1)$$

$$\nabla f(1,1,0) = (2,-2,1)$$

3. Substitute the differential and normal vector into the equation

$$D_{\underline{u}} f(1,1,0) = \nabla f(1,1,0) \cdot \underline{u}$$

$$= (2,-2,1) \cdot \frac{1}{7}(2,-3,6)$$

4. Simplify and solve

$$= \frac{1}{7}(4 + 6 - 6) = 4/7$$

$$\frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k} = \underline{T}$$

The total differential is

$$\frac{\partial \phi}{dt} = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt} = 0$$

$$\nabla \phi \cdot T = 0$$

$$Z^2 = 4(x^2 + y^2) \quad \text{at } (1,0,2)$$

$$\phi = 4(x^2 + y^2) - z^2 = 0$$

$$\nabla \phi(x,y,z) = \frac{\partial \phi}{\partial x}\underline{i} + \frac{\partial \phi}{\partial x}\underline{j} + \frac{\partial \phi}{\partial x}\underline{k}$$

$$= 4(2x)\underline{i} + 4(2y)\underline{j} + (-2z)\underline{k}$$

$$= 8x\,\underline{i} + 8y\,\underline{j} - 2z\,\underline{k}$$

$$\nabla\varphi(1,0,2) = 8\underline{i} - 4\underline{k}$$

$$\underline{u} = \frac{1}{\sqrt{8^2+4^2}}(8\underline{i} - 4\underline{k}) = \frac{1}{\sqrt{5}}(2\underline{i} - \underline{k})$$

(a) $D_{\vec{u}}f(2,0)$ where $f(x,y) = x\mathbf{e}^{xy} + y$ and $\vec{u}$ is the unit vector in the direction of $\theta = \frac{2\pi}{3}$.

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$$\underrightarrow{u} = (\cos(2\pi/3), \sin(2\pi/3)) = (-1/2, \sqrt{3}/2)$$

$$\nabla = (\partial f/\partial x, \partial f/\partial y) = ((e^{xy} + xye^{xy}), (x^2e^{xy}+1))$$

$$D_{\underrightarrow{u}} f(x,y) = (-1/2)(e^{xy} + xye^{xy}) + (\sqrt{3}/2)(x^2e^{xy}+1)$$

$$D_{\underrightarrow{u}} f(2,0) = (-1/2)(1+0) + (\sqrt{3}/2)(4+1)$$

$$= -1/2 + 5\sqrt{3}/2$$

$$= \frac{5\sqrt{3}-1}{2}$$

(b) $D_{\vec{u}}f(x,y,z)$ where $f(x,y,z) = x^2z + y^3z^2 - xyz$ in the direction of $\vec{v} = \langle -1, 0, 3\rangle$.

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$$\underrightarrow{u} = \frac{1}{\sqrt{1^2+0^2+3^2}}(-1,0,3) = \left(\frac{-1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}}\right)$$

$$\nabla = ((2xz - yz), (3y^2z^2 - xz), (x^2 - xy))$$

$$D_{\underrightarrow{u}} f(x,y,z) = \left(\frac{-1}{\sqrt{10}}\right)(2xz - yz) + 0 + \left(\frac{3}{\sqrt{10}}\right)(x^2 - xy + 2y^3z)$$

$$D_{\underrightarrow{u}} f(x,y,z) = \frac{1}{\sqrt{10}}(3x^2 - 3xy - 2xz + yz + 6y^3z)$$

The max value of $D_{\underrightarrow{u}} f(x)$ is given by $\|\nabla f(x)\|$ and occurs in the direction given by $\nabla f(x)$

Proof: $D_{\underline{u}} f = \nabla f \cdot \underline{u} = \|\nabla f\|\,\|\underline{u}\|\cos\theta$ $= 1$ as is unit vector which has $\|\underline{u}\| = 1$

Proof: $D_{\underline{u}} f = \nabla f \cdot \underline{u} = \|\nabla f\| \, \|\underline{u}\| \cos\theta$

$$= \|\nabla f\| \cos\theta$$

which has magnitude 1

↳ θ is the angle between ∇ and $\underline{u}$

$\theta = 0 \rightarrow \cos\theta = 1$

↳ $\max\left[\|\nabla f\| \cos\theta\right]$

$$= \|\nabla f\|$$

At this point, the angle between ∇ and $\underline{u} = 0$

i.e. the vector $\underline{u}$ is pointing in **the same direction** as the gradient ∇