Duality Between Sugihara Monoids and Girdle Algebras

Jen Robertson

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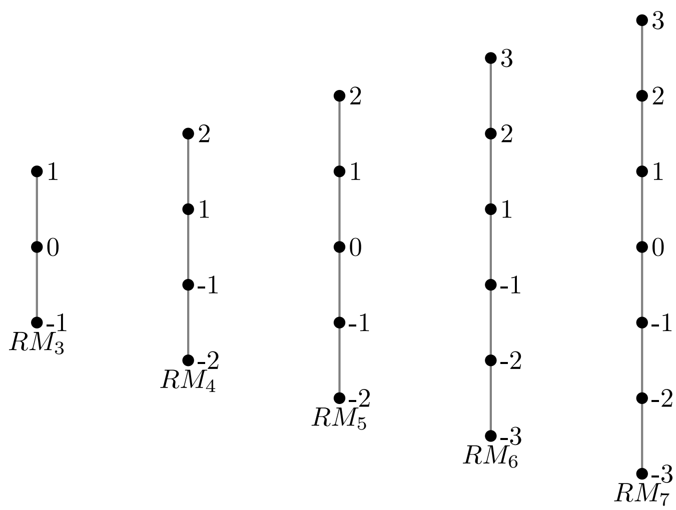
1 Introduction

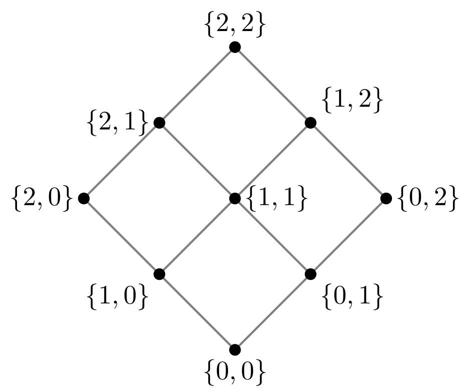
The following images are the visualization of different sized Sugihara monoids.



In the software, the numbering of the RM_n monoid goes from 0 to $n-1$. Each of these monoids have a meet, join, negate, implication, the constant element, and a * operation. Once the software computes the algebra, it will then compute the product algebra and its subalgebras. For the purpose of this analysis, we will split the Sugihara monoids up into even an odd sizes. The reason for doing so is that for all of the even sized monoids, the implication operation builds from that of the previous odd size, and the same can be said for that of the even sizes.

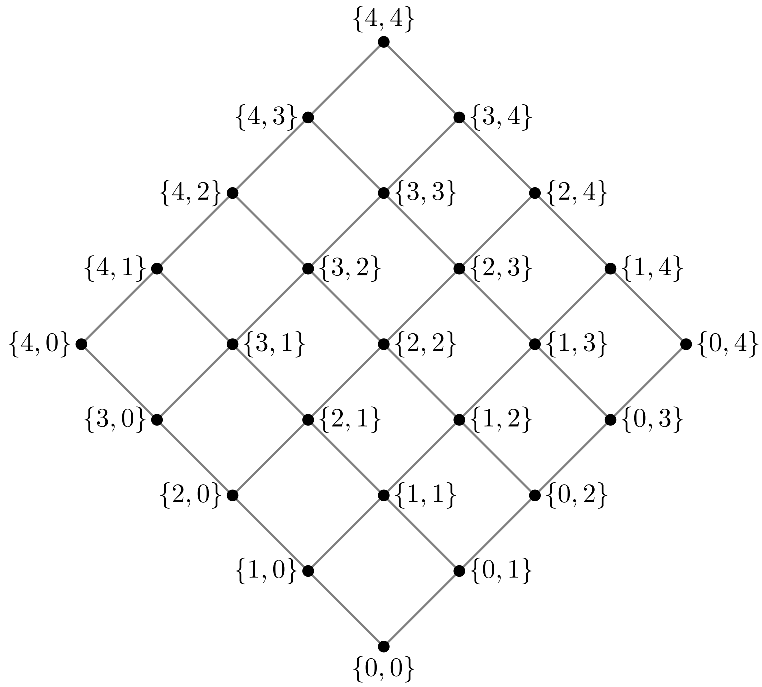
2 Odd Sugihara Monoids

For RM_3, we renumber the values $\{-1, 0, 1\}$ to be $\{0, 1, 2\}$ respectively. The product algebra for this looks as follows:



This product algebra has 5 subalgebras which are as follows:
$A_1 = RM_3 \times RM_3$
$A_2 = \{1\} \times RM_3$
$A_3 = RM_3 \times \{1\}$
$A_4 = \triangle_{RM_3}$
$A_5 = \{1\} \times \{1\}$
The subalgebras to note are those that cannot be written as a products of RM_3. In this product algebra we can note that the identity element, 1, is important.

For RM_5, we renumber the values $\{-2, -1, 0, 1, 2\}$ to be $\{0, 1, 2, 3, 4\}$ respectively. The product algebra for this looks as follows:



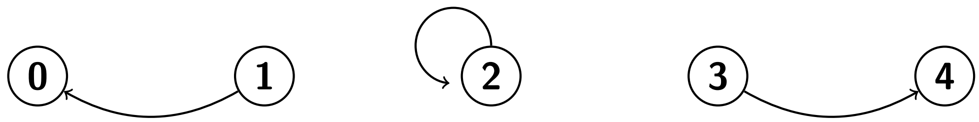
This product algebra has 26 subalgebras which are as follows:

$A_1 = RM_3 \times RM_3$
$A_2 = \{0, 2, 4\} \times \{0, 2, 4\}$
$A_3 = RM_5 \times RM_5$
$A_4 = \{2\} \times \{2\}$
$A_5 = \triangle_{RM_3}$
$A_6 = \triangle_{\{0,2,4\}}$
$A_7 = \triangle_{RM_5}$
$A_8 = \{2\} \times RM_3$
$A_9 = RM_3 \times \{2\}$
$A_{10} = \{2\} \times \{0, 2, 4\}$
$A_{11} = \{0, 2, 4\} \times \{2\}$
$A_{12} = \{2\} \times RM_5$
$A_{13} = RM_5 \times \{2\}$
$A_{14} = RM_3 \times \{0, 2, 4\}$
$A_{15} = \{0, 2, 4\} \times RM_3$
$A_{16} = RM_3 \times RM_5$
$A_{17} = RM_5 \times RM_3$
$A_{18} = RM_5 \times \{0, 2, 4\}$
$A_{19} = \{0, 2, 4\} \times RM_5$
$A_{20} = \{(1, 0), (2, 2), (3, 4)\}$
$A_{21} = \{(0, 1), (2, 2), (4, 3)\}$
$A_{22} = \{(0, 0), (2, 1), (2, 2), (2, 3), (4, 4)\}$
$A_{23} = \{(0, 0), (1, 2), (2, 2), (3, 2), (4, 4)\}$
$A_{24} = \{(0, 1), (1, 2), (2, 2), (3, 2), (4, 3)\}$
$A_{25} = \{(1, 0), (2, 1), (2, 2), (2, 3), (3, 4)\}$

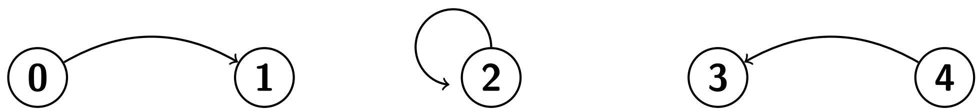
$$A_{26} = \{(0, 0), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4)\}$$

From the data gathered we can observe that our identity element, 2, is extremely important in many subalgebras as well as the set $\{0, 2, 4\}$. Along with this information, if we observe $A_{20} - A_{26}$ we notice that these subalgebras can not be written as a product of elements or algebras. As such, these are the algebras that contain the most pertinent information about the algebra. If we first observe A_{20} and A_{21} we notice that these are reversals of the coordinate pairs. If we were to observe them as a graph where each coordinate pair is a start position and an end position we would get the following:

A_{20}

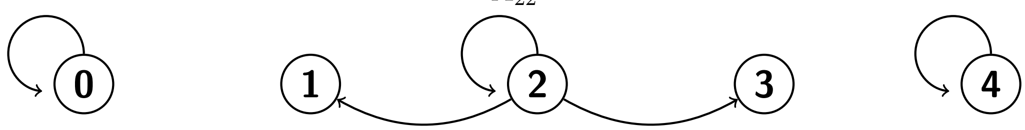


A_{21}

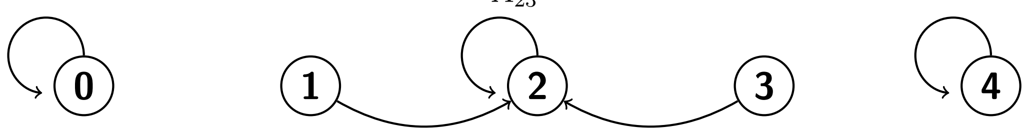


This same pattern can be seen with the pairs of subalgebras A_{22}, A_{23} and A_{24}, A_{25} we can see these relations modeled below:

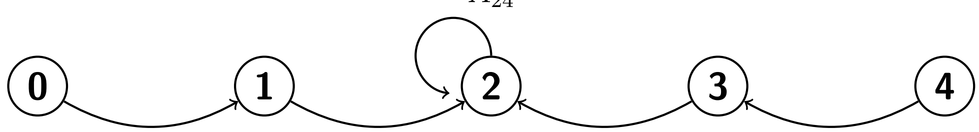
A_{22}



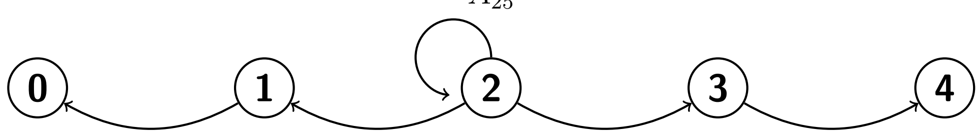
A_{23}



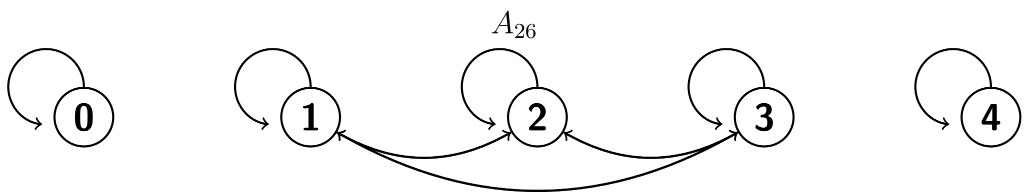
A_{24}



A_{25}

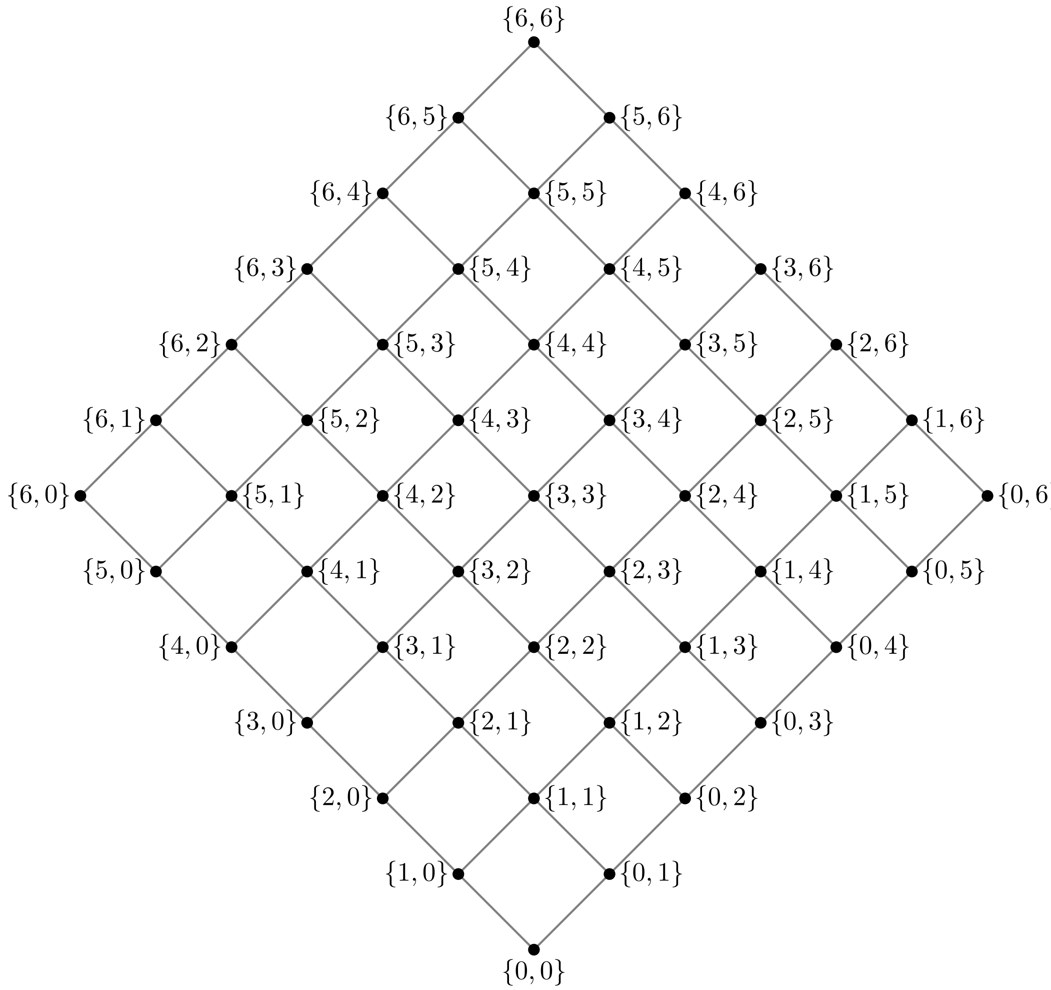


The last subalgebra to observe that cannot be written as a product of elements or algebras is A_{26} This one does not have a pair with its reversal or coordinate points because it would be the same subalgebra. Below is the graph of the subalgebra:



With all of the graphs listed out we can really see just how important the identity element, 2, is here.

For RM_7, we renumber the values $\{-3, -2, -1, 0, 1, 2, 3\}$ to be $\{0, 1, 2, 3, 4, 5, 6\}$ respectively. The product algebra for this looks as follows:



Here it is important to note that for calculating the subalgebras of the product algebra of RM_7 that if we include as constants the top and bottom values of the algebra we get 38 subalgebras, whereas if we calculate the subalgebras without these constants we get 110 subalgebras. Given, that in the other sized algebras we did not include the top and bottom values as constants we will do the same here. It is important to note that the set $\{2, 3, 4\}$ is equivalent to RM_3 as it is the elements $\{-1, 0, 1\}$ before the software converts the values. Similarly, $\{1, 2, 3, 4, 5\}$ is equivalent to RM_5

$A_1 = RM_3 \times RM_3$
$A_2 = RM_3 \times \{1, 3, 5\}$
$A_3 = RM_3 \times \{0, 3, 6\}$
$A_4 = \{1, 3, 5\} \times \{0, 3, 6\}$
$A_5 = \{1, 3, 5\} \times \{1, 3, 5\}$
$A_6 = \{1, 3, 5\} \times RM_3$
$A_7 = \{0, 3, 6\} \times \{0, 3, 6\}$
$A_8 = \{0, 3, 6\} \times \{1, 3, 5\}$

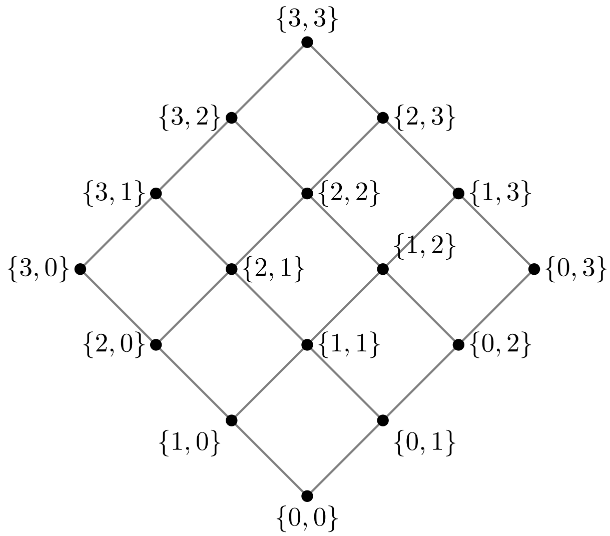
$A_9 = \{0,3,6\} \times RM_3$
$A_{10} = \triangle_{RM_3}$
$A_{11} = RM_3 \times \{3\}$
$A_{12} = \{1,3,5\} \times \{3\}$
$A_{13} = \{3\} \times \{0,3,6\}$
$A_{14} = \triangle_{\{1,3,5\}}$
$A_{15} = \{(1,0),(3,3),(5,6)\}$
$A_{16} = \{3\} \times \{1,3,5\}$
$A_{17} = \{0,3,6\} \times \{3\}$
$A_{18} = \{3\} \times RM_3$
$A_{19} = \{(0,1),(3,3),(6,5)\}$
$A_{20} = \triangle_{\{0,3,6\}}$
$A_{21} = \{3\} \times \{3\}$
$A_{22} = RM_3 \times RM_5$
$A_{23} = RM_3 \times \{0,2,3,4,6\}$
$A_{24} = RM_5 \times \{0,2,3,4,6\}$
$A_{25} = RM_5 \times RM_5$
$A_{26} = RM_5 \times RM_3$
$A_{27} = \{0,2,3,4,6\} \times \{0,2,3,4,6\}$
$A_{28} = \{0,2,3,4,6\} \times RM_5$
$A_{29} = \{0,2,3,4,6\} \times RM_3$
$A_{30} = \triangle_{\{0,1,3,5,6\}}$
$A_{31} = \triangle_{RM_7}$
$A_{32} = \{3\} \times \{0,2,3,4,6\}$
$A_{33} = \{3\} \times RM_7$
$A_{34} = RM_3 \times \{0,1,3,5,6\}$
$A_{35} = RM_5 \times \{0,1,3,5,6\}$
$A_{36} = RM_5 \times \{1,3,5\}$
$A_{37} = \{0,2,3,4,6\} \times \{0,1,3,5,6\}$
$A_{38} = \{0,2,3,4,6\} \times \{1,3,5\}$
$A_{39} = RM_3 \times RM_7$
$A_{40} = RM_5 \times RM_7$
$A_{41} = \{0,2,3,4,6\} \times RM_7$
$A_{42} = RM_5 \times \{0,3,6\}$
$A_{43} = \{0,2,3,4,6\} \times \{0,3,6\}$
$A_{44} = \{1,3,5\} \times \{0,1,3,5,6\}$
$A_{45} = \{1,3,5\} \times \{0,2,3,4,6\}$
$A_{46} = \{0,1,3,5,6\} \times \{0,3,6\}$
$A_{47} = \{0,1,3,5,6\} \times \{0,1,3,5,6\}$
$A_{48} = \{0,1,3,5,6\} \times \{0,2,3,4,6\}$
$A_{49} = RM_7 \times \{0,2,3,4,6\}$
$A_{50} = RM_7 \times RM_7$
$A_{51} = RM_7 \times \{0,1,3,5,6\}$
$A_{52} = RM_7 \times \{0,3,6\}$
$A_{53} = \{1,3,5\} \times RM_5$
$A_{54} = \{0,1,3,5,6\} \times \{1,3,5\}$
$A_{55} = \{0,1,3,5,6\} \times RM_5$
$A_{56} = RM_7 \times RM_5$
$A_{57} = RM_7 \times \{1,3,5\}$
$A_{58} = \{1,3,5\} \times RM_7$
$A_{59} = \{0,1,3,5,6\} \times RM_7$
$A_{60} = \{0,1,3,5,6\} \times RM_3$
$A_{61} = RM_7 \times RM_3$
$A_{62} = \{0,3,6\} \times \{0,1,3,5,6\}$
$A_{63} = \{0,3,6\} \times \{0,2,3,4,6\}$
$A_{64} = \{0,3,6\} \times RM_5$
$A_{65} = \{0,3,6\} \times RM_7$
$A_{66} = \triangle_{RM_5}$
$A_{67} = \triangle_{\{0,2,3,4,6\}}$
$A_{68} = RM_5 \times \{3\}$
$A_{69} = \{0,2,3,4,6\} \times \{3\}$
$A_{70} = \{0,1,3,5,6\} \times \{3\}$
$A_{71} = RM_7 \times \{3\}$
$A_{72} = \{3\} \times \{0,1,3,5,6\}$
$A_{73} = \{3\} \times \{0,2,3,4,6\}$

$A_{74} = \{(0,0),(3,1),(3,3),(3,5),(6,6)\}$
$A_{75} = \{(1,0),(3,1),(3,2),(3,3),(3,4),(3,5),(5,6)\}$
$A_{76} = \{(0,1),(3,2),(3,3),(3,4),(6,5)\}$
$A_{77} = \{(0,0),(3,2),(3,3),(3,4),(6,6)\}$
$A_{78} = \{(1,1),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,2),(4,3),(4,4),(5,5)\}$
$A_{79} = \{(1,0),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,2),(4,3),(4,4),(5,6)\}$
$A_{80} = \{(0,1),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,2),(4,3),(4,4),(6,5)\}$
$A_{81} = \{(0,0),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,2),(4,3),(4,4),(6,6)\}$
$A_{82} = \{(1,0),(2,1),(2,3),(2,5),(3,1),(3,3),(3,5),(4,1),(4,3),(4,5),(5,6)\}$
$A_{83} = \{(0,0),(2,1),(2,3),(2,5),(3,1),(3,3),(3,5),(4,1),(4,3),(4,5),(6,6)\}$
$A_{84} = \{(1,0),(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),$
$(4,1),(4,2),(4,3),(4,4),(4,5),(5,6)\}$

$A_{85} = \{(0,0),(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),$
$(4,1),(4,2),(4,3),(4,4),(4,5),(6,6)\}$
$A_{86} = \{(0,0),(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5),(6,6)\}$
$A_{87} = \{(0,0),(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(2,5),$
$(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5),(6,6)\}$
$A_{88} = \{(0,0),(1,1),(1,3),(1,5),(2,1),(2,3),(2,5),(3,1),(3,3),(3,5),(4,1),$
$(4,3),(4,5),(5,1),(5,3),(5,5),(6,6)\}$
$A_{89} = \{(0,1),(1,2),(1,3),(1,4),(3,2),(3,3),(3,4),(5,2),(5,3),(5,4),(6,5)\}$
$A_{90} = \{(0,0),(1,2),(1,3),(1,4),(3,2),(3,3),(3,4),(5,2),(5,3),(5,4),(6,6)\}$
$A_{91} = \{(0,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,1),$
$(4,3),(4,4),(5,2),(5,3),(5,4),(6,5)\}$
$A_{92} = \{(0,0),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,1),$
$(4,3),(4,4),(5,2),(5,3),(5,4),(6,6)\}$
$A_{93} = \{(0,0),(1,1),(1,2),(1,3),(1,4),(1,5),(2,4),(3,1),(3,2),(3,3),(3,4),$
$(3,5),(5,1),(5,2),(5,3),(5,4),(5,5),(6,6)\}$
$A_{94} = \{(1,0),(2,2),(3,3),(4,4),(5,6)\}$
$A_{95} = \{(0,1),(2,2),(3,3),(4,4),(6,5)\}$
$A_{96} = \{(1,1),(2,3),(3,3),(4,3),(5,5)\}$
$A_{97} = \{(1,0),(2,3),(3,3),(4,3),(5,6)\}$
$A_{98} = \{(0,1),(2,3),(3,3),(4,3),(6,5)\}$
$A_{99} = \{(0,0),(2,3),(3,3),(4,3),(6,6)\}$
$A_{100} = \{(0,1),(1,3),(3,3),(5,3),(6,5)\}$
$A_{101} = \{(0,0),(1,3),(3,3),(5,3),(6,6)\}$
$A_{102} = \{(0,1),(1,3),(2,3),(3,3),(4,3),(5,3),(6,5)\}$
$A_{103} = \{(0,0),(1,3),(2,3),(3,3),(4,3),(5,3),(6,6)\}$
$A_{104} = \{(1,1),(3,2),(3,3),(3,4),(5,5)\}$
$A_{105} = \{(0,0),(1,1),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,2),(4,3),(4,4),(5,5),(6,6)\}$
$A_{106} = \{(0,0),(1,1),(2,3),(3,3),(4,3),(5,5),(6,6)\}$
$A_{107} = \{(1,0),(3,1),(3,3),(3,5),(5,6)\}$
$A_{108} = \{(1,0),(3,2),(3,3),(3,4),(5,6)\}$
$A_{109} = \{(0,0),(1,1),(3,2),(3,3),(3,4),(5,5),(6,6)\}$
$A_{110} = \{(0,0),(3,1),(3,2),(3,3),(3,4),(3,5),(6,6)\}$

3 Even Sugihara Monoids

For even Sugihara monoids we do not have a "middle" element. If we look at RM_4 we have the elements $\{-2,-1,1,2\}$ that we renumber for the software to be $\{0,1,2,3\}$. The product algebra looks as follows:

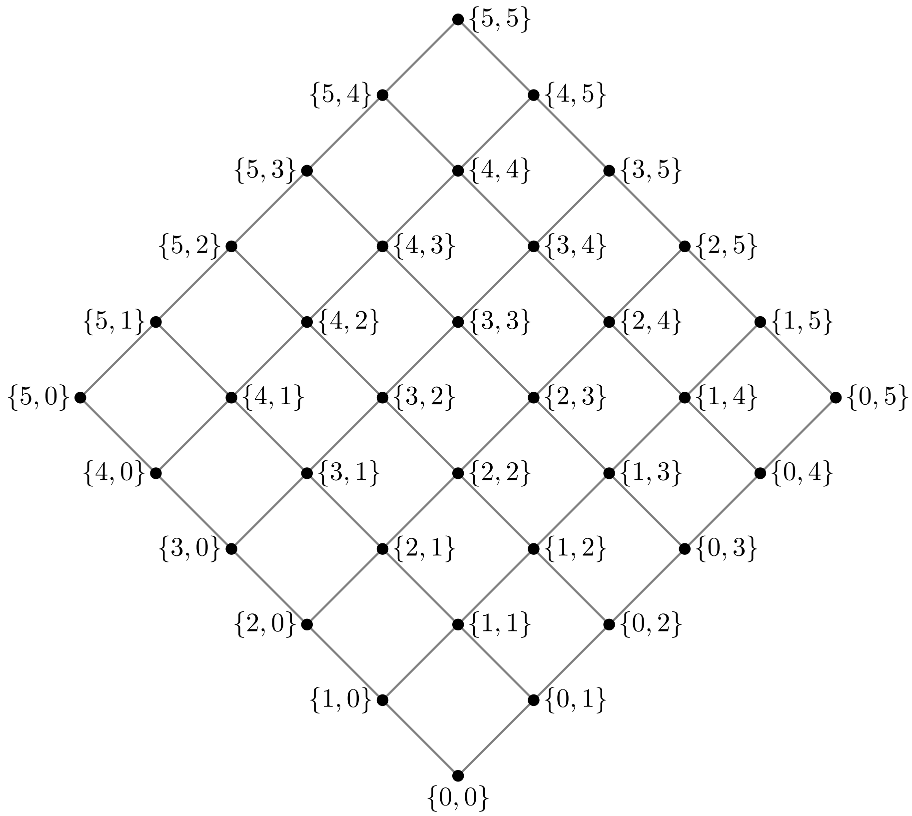


The product algebra generates 7 subalgebras, please note that the set $\{1, 2\}$ is equivalent to RM_2:

$A_1 = RM_4 \times RM_4$
$A_2 = RM_2 \times RM_4$
$A_3 = RM_4 \times RM_2$
$A_4 = RM_2 \times RM_2$
$A_5 = \triangle_{RM_4}$
$A_6 = \triangle_{RM_2}$

$A_7 = \{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

For RM_6 the elements $\{-3, -2, -1, 1, 2, 3\}$ become $\{0, 1, 2, 3, 4, 5\}$ respectively. The product algebra looks as follows:



This product algebra generates 32 subalgebras. Here it is important to know that the set $\{1, 2, 3, 4\}$ is equivalent to RM_4 and the values $\{2, 3\}$ are equivalent to RM_2:

$A_1 = RM_4 \times \{0, 2, 3, 5\}$
$A_2 = RM_4 \times RM_4$
$A_3 = \{0, 2, 3, 5\} \times \{0, 2, 3, 5\}$
$A_4 = \{0, 2, 3, 5\} \times RM_4$
$A_5 = RM_2 \times RM_4$
$A_6 = RM_4 \times RM_2$
$A_7 = RM_2 \times \{0, 2, 3, 5\}$
$A_8 = \{0, 2, 3, 5\} \times RM_2$
$A_9 = RM_2 \times RM_2$
$A_{10} = \triangle_{RM_4}$
$A_{11} = \triangle_{\{0,2,3,5\}}$
$A_{12} = \triangle_{RM_2}$
$A_{13} = RM_4 \times RM_6$
$A_{14} = RM_6 \times \{0, 2, 3, 5\}$
$A_{15} = RM_6 \times RM_6$
$A_{16} = RM_6 \times RM_4$
$A_{17} = \{0, 2, 3, 5\} \times RM_6$
$A_{18} = RM_2 \times RM_6$
$A_{19} = RM_6 \times \{2, 3\}$
$A_{20} = \triangle_{RM_6}$

$A_{21} = \{(1, 0), (2, 2), (3, 3), (4, 5)\}$
$A_{22} = \{(0, 1), (2, 2), (3, 3), (5, 4)\}$
$A_{23} = \{(0, 0), (1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3)(2, 4), (3, 1), (3, 2),$

$(3,3),(3,4),(4,1),(4,2),(4,3),(4,4),(5,5)\}$
$A_{24}=\{(1,0),(2,1),(2,2),(2,3)(2,4),(3,1),(3,2),(3,3),(3,4),(4,5)\}$
$A_{25}=\{(0,0),(2,1),(2,2),(2,3)(2,4),(3,1),(3,2),(3,3),(3,4),(5,5)\}$
$A_{26}=\{(0,1),(1,2),(1,3),(2,2),(2,3),(3,2),(3,3),(4,2),(4,3),(5,4)\}$
$A_{27}=\{(0,0),(1,2),(1,3),(2,2),(2,3),(3,2),(3,3),(4,2),(4,3),(5,5)\}$
$A_{28}=\{(1,0),(2,2),(2,3),(3,2),(3,3),(4,5)\}$
$A_{29}=\{(0,1),(2,2),(2,3),(3,2),(3,3),(5,4)\}$
$A_{30}=\{(1,1),(2,2),(2,3),(3,2),(3,3),(4,4)\}$
$A_{31}=\{(0,0),(2,2),(2,3),(3,2),(3,3),(5,5)\}$
$A_{32}=\{(0,0),(1,1),(2,2),(2,3),(3,2),(3,3),(4,4),(5,5)\}$

4 Conclusions from Sugihara Monoids

5 Girdle Algebras

6 Comparison of Girdle Algebras and Sugihara Monoids