GRAPH THEORY

Module I

Lecture #1

What is a graph ?

→ A linear graph (or graph) $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \ldots\}$ called vertices and another set $E = \{e_1, e_2, \ldots\}$ whose elements are called edges.

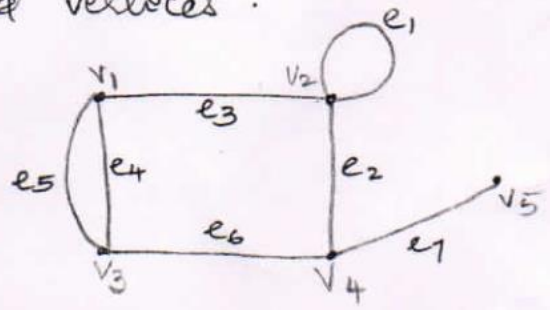
→ edge e_k is identified with an unordered pair (v_i, v_j) of vertices.

→ The vertices v_i, v_j associated with edge e_k are called end vertices of e_k.

→ Most common representation of graph is by means of a diagram

→ vertices are represented as points

→ each edge as a line segment joining its end vertices.



→ An edge is associated with a vertex pair.

→ An edge having the same vertex as both its end vertices is called a <u>self loop</u>

eg: Edge e_1 in figure

→ <u>Parallel edges</u> :- More than one edge associated with a given pair of vertices.

eg: Edges e_4 & e_5

→ <u>Simple graph</u> :- A graph that has neither self loops nor parallel edges.

→ Another names for graph

- linear complex, 1-complex, or a one-dimensional complex.

→ A vertex is also referred to as

- node, junction, point, 0-cell or 0-simplex.

→ Other terms for edge

- branch, line, element, 1-cell, arc, 1-simplex.

Applications of graphs

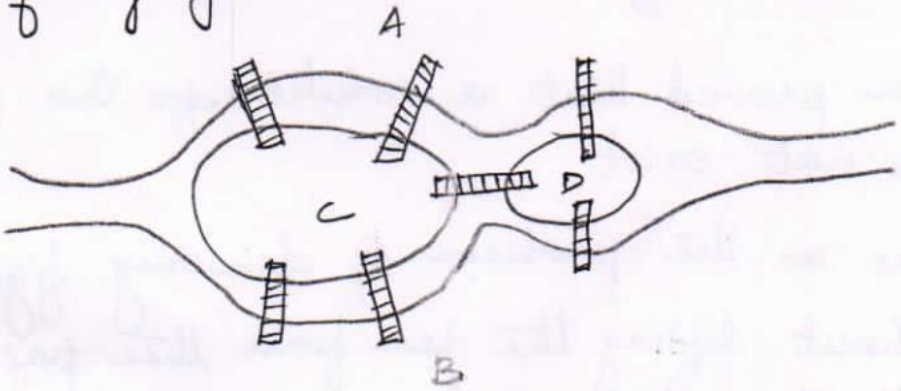
→ Wide range of applications in
  → engineering
  → physical, social & biological sciences
  → linguistics etc.

→ Graph can be used to represent physical situation involving discrete objects and a relationship among them.

(i) Konigsberg Bridge Problem

→ Best known example of graph theory.

→ It was a long-standing problem until solved by Leonhard Euler in 1736, by means of a graph.



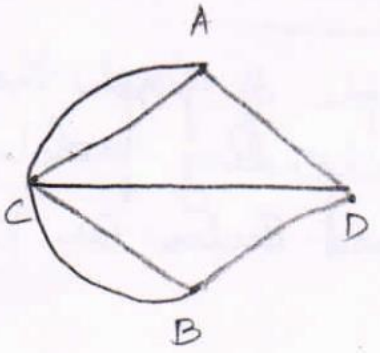
→ Two island C & D → formed by Pregel River in Konigsberg (in West Soviet Russia).

→ C & D were connected to each other and to the banks A and B with seven bridges.

→ Problem was to start at any of the 4 land areas of the city A, B, C or D → walk over each bridges exactly once and return to the starting point (without swimming across the river).

→ Euler represented this situation by means of a graph.

→ Vertices represent the land areas and the edges represented the bridges.



→ Euler proved that a solution for this problem does not exist.

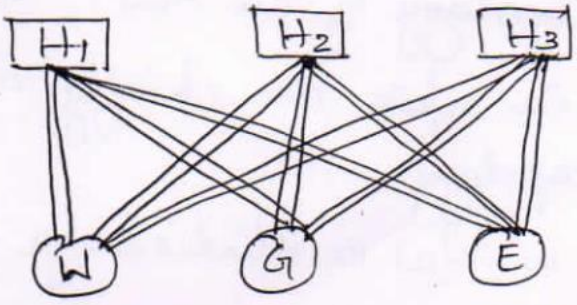
→ Same as the problem of drawing figures without lifting the pen from the paper and without retracing a line.

(ii) Utilities Problem

→ There are three houses H_1, H_2 & H_3.

→ Each to be connected to each of the 3 utilities - water (W), gas (G) and electricity (E) by means of conduits

→ ~~This~~



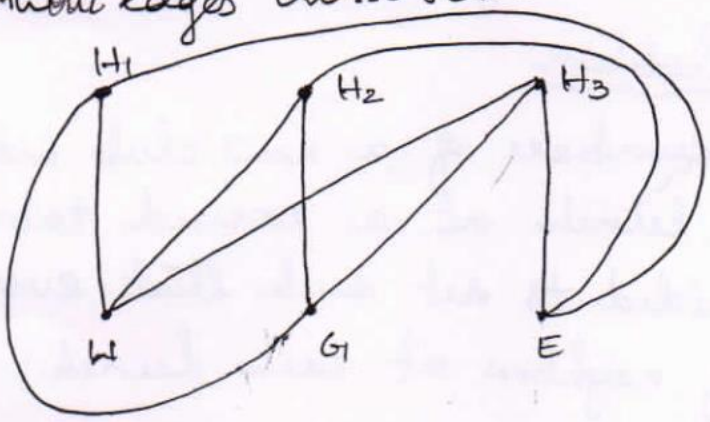
→ Make connections without any crossovers of the conduits.

→ This problem can be represented using a graph.

→ Conduits are edges

→ Houses & utility supply centers are vertices.

→ The graph cannot be drawn in a plane without edges crossover.



(iii) Electrical Network Problems

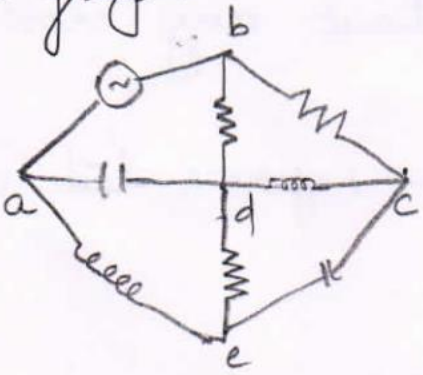
→ Properties (such as transfer fn and input impedence) of an electrical n/w are fns of 2 factors

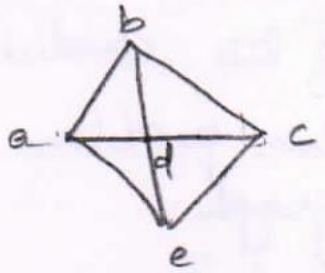
(a) Nature & value of the elements forming the n/w. → such as resistors, inductors, transistors etc.

(b) The way these elements are connected together → ie the topology of the n/w.

→ The variation in n/ws are chiefly due to the variations in topology.

→ The topology of a n/w is studied by means of a graph.





→ The junctions are represented by vertices

→ Branches are represented by edges.

(iv) Seating Problem

→ Nine members of a new club meet each day for lunch at a round table.

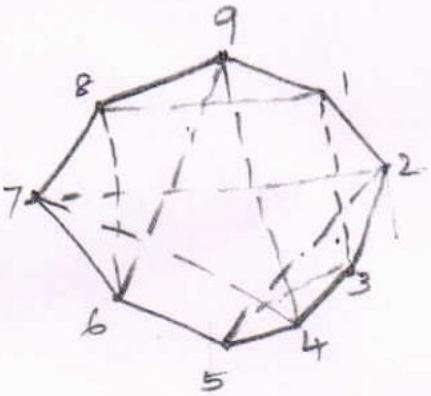
→ They decided to sit such that every member has diff. neighbors at each lunch.

→ How many days can this arrangement last?

→ This can be represented by a graph with 9 vertices

→ Each vertex represents a member.

→ An edge joining two vertices represents the relationship of sitting next to each other.



→ Solid lines - 1 2 3 4 5 6 7 8 9

→ Dashed lines - 1 3 5 2 7 4 9 6 8 1

→ Two more arrangements are possible.

a) 1 5 7 3 9 2 8 4 6 1

b) 1 7 9 5 8 3 6 2 4 1

→ In general, → for n people the number of possible arrangements is

$$\frac{n-1}{2}, \text{ if } n \text{ is odd}$$

and

$$\frac{n-2}{2}, \text{ if } n \text{ is even.}$$

Questions

1) What is a graph?
2) What are the applications of graphs?

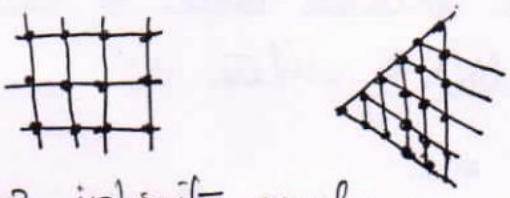
Lecture #2

Finite and Infinite Graphs

→ A graph with a finite no: of vertices as well as finite no: of edges is called a finite graph.

→ Otherwise it is an infinite graph.

eg:

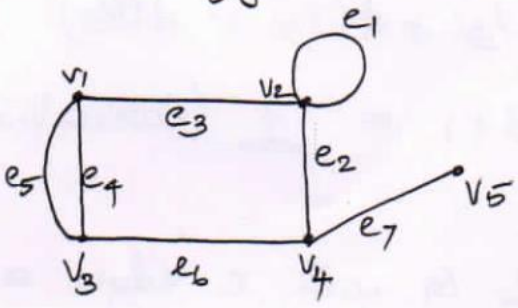


Portions of 2 infinite graphs.

Incidence and Degree

→ When a vertex v_i is an end vertex of some edge e_j → v_i & e_j are incident with each other.

→ Consider the figure



→ edges e_2, e_6 & e_7 are incident ~~on a~~ with vertex v_4.

Adjacent

→ Two nonparallel edges are said to be adjacent if they are incident on a common vertex.

eg: e_2 & e_7 are adjacent

→ Illrly two vertices are said to be <u>adjacent</u> if they are end vertices of the same edge.

eg: V_4 & V_5 are adjacent
V_1 & V_4 are not.

<u>Degree</u>

→ The no: of edges incident on a vertex v_i, with self loops counted twice is called the degree, $d(v_i)$, of vertex v_i

for eg: $d(v_1) = 3$
$d(v_2) = 4$
$d(v_3) = 3$
$d(v_4) = 3$
$d(v_5) = 1$.

→ Degree of a vertex is also known as <u>valency</u>.

$d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5)$

$= 3+4+3+3+1 = \underline{\underline{14}}$ (twice the no: of edge)

→ In general.

→ Consider a graph G with e edges and n vertices $v_1, v_2, \ldots v_n$.

→ Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the no: of edges in G. That is

$$\sum_{i=1}^{n} d(v_i) = 2e.$$

Theorem

"The no: of vertices of odd degree in a graph is always even".

Proof

Consider the vertices with odd and even degrees seperately, the sum of the degrees of all vertices $\sum_{i=1}^{n} d(v_i)$ can be expressed as the sum of two sums, each taken over vertices of even and odd degrees, resptly.

$$\sum_{i=1}^{n} d(v_i) = \sum_{even} d(v_j) + \sum_{odd} d(v_k) \quad \text{——} \quad (1)$$

Since the left hand side of eqn (1) is even, and the first expression on the RHS is even, the 2nd expression must also be even:

$$\text{ie } \sum_{odd} d(v_k) = \text{an even number} \quad \text{——} \quad (2)$$

Bcoz in eqn (2) each $d(v_k)$ is odd, the total no: of terms in the sum must be even to make the sum an even number.

Hence the theorem.

Regular graph

→ Graph in which all vertices are of equal degree is called regular graph.

Isolated Vertex

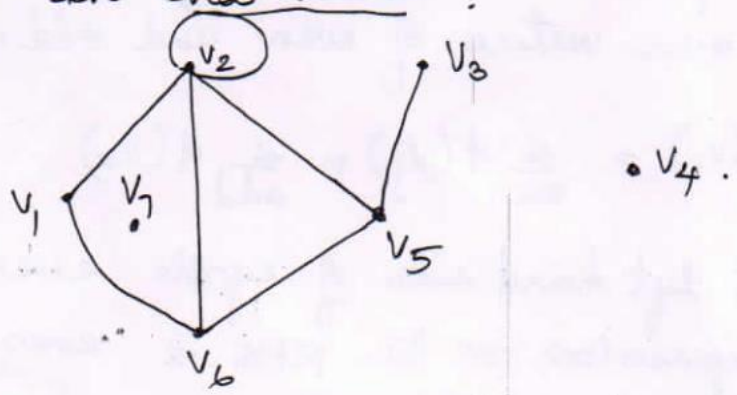
→ A vertex having no incident edge is called an isolated vertex.

→ i.e Vertices with zero degree

Pendant vertex

→ A vertex of degree one is called pendant vertex or an end vertex.

→ eg:



→ V_4 & V_7 are isolated vertices

→ V_3 is a pendant vertex.

→ Two adjacent edges are said to be in a series if their common vertex is of degree two.

eg: two edges incident on V_1 are in series

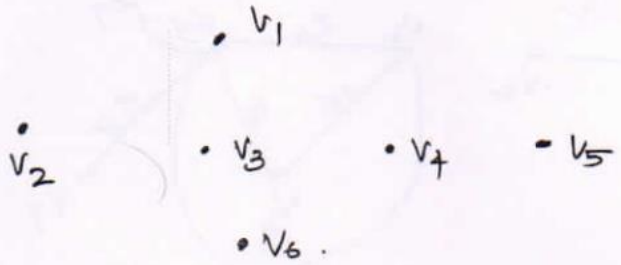
Null graph

→ Graph without any edges.

→ i.e In the definition of a graph $G = (V, E)$, the edge set E is empty.

→ In other words every vertex in a null graph is ~~any~~ an isolated vertex.

eg: A null graph of six vertices

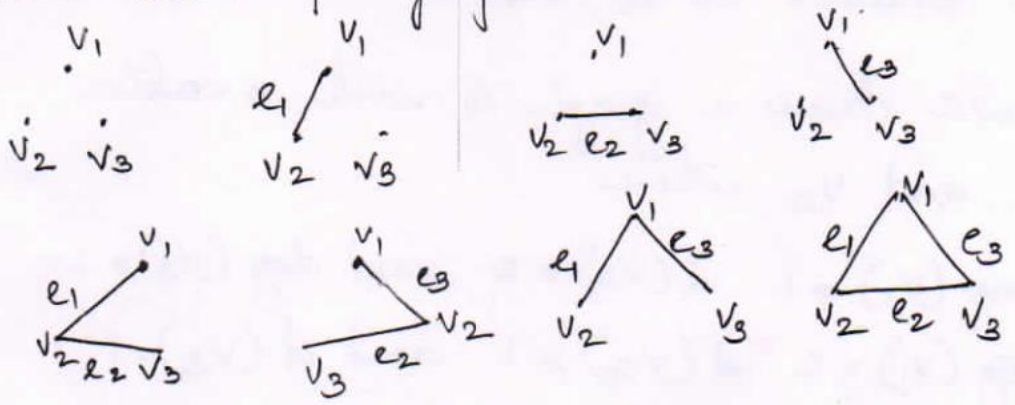


→ Although the edge set E may be empty, the vertex set V must not be empty.

→ So by definition, a graph must have at least one vertex.

<u>Problems - Questions</u>

1) Draw all simple graphs with 3 vertices



2) For the given graph

(a) Write the vertex set

(b) Write the edge set

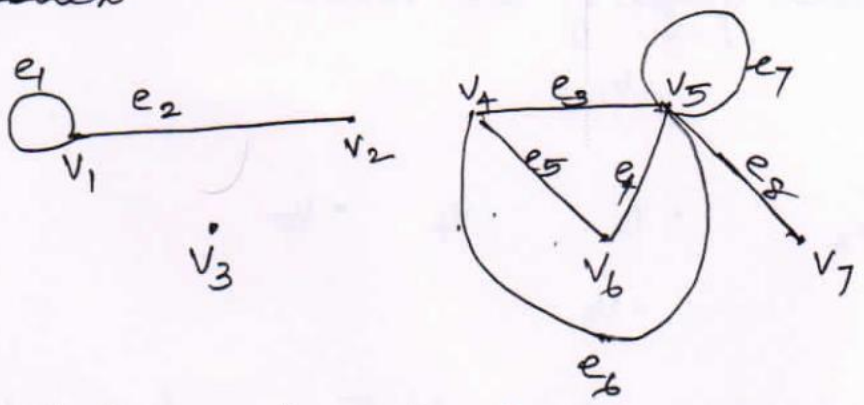
(c) Find all isolated vertices

(d) Find all loops

(e) Find all parallel edges

(f) Find all pendant vertices

(a) Vertex



(a) Vertex set = $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$

(b) Edge set = $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$

(c) Isolated $\rightarrow v_3$

(d) loops $\rightarrow e_1$ & e_7

(e) Parallel edges e_3 & e_6

(f) Pendant $\rightarrow v_2, v_7$

3) Can we draw a graph G with 3 vertices v_1, v_2 and v_3 where

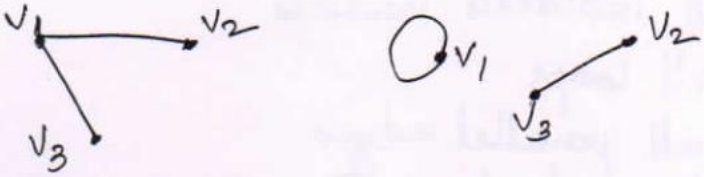
(a) $\deg(v_1) = 1$ $d(v_2) = 2$ and $\deg(v_3) = 2$

(b) $\deg(v_1) = 2$ $d(v_2) = 1$ and $d(v_3) = 1$.

If yes, draw the graph.

(a) No since $d(v_1) + d(v_2) + d(v_3) = 1+2+2 = 5 \rightarrow$ odd.

(b) Yes $2+1+1 = 4$.



4) Determine the no: of edges in a graph with 6 vertices, 2 of degree 4 and 4 of degree 2. Draw 2 such graphs.

Suppose the graph with 6 vertices has e no: of edges.

$$\therefore \sum_{i=1}^{6} d(v_i) = 2e$$

$$\Rightarrow d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) = 2e$$

Given 2 vertices are of degree 4 and 4 vertices are of degree 2.

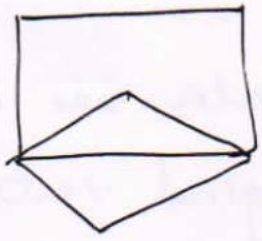
Hence

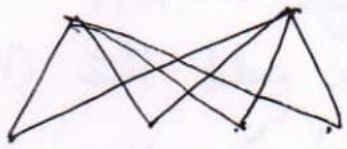
$$\Rightarrow 4+4+2+2+2+2 = 2e$$

$$\Rightarrow 16 = 2e$$

$$\Rightarrow e = \underline{\underline{8}}$$

$\therefore$ No: of edges = 8





5) Show that the max degree of any vertex in a simple graph with n vertices is n-1.

→ Let G be a simple graph with n vertices.

→ Consider any vertex v of G.

→ Since the graph is simple, vertex v can be adjacent to almost remaining n-1 vertices.

→ ∴ Max degree of any vertex in a simple graph with n vertices is n-1.

6) Show that the max no: of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

<u>Proof</u>

By induction.

If $n=1$ $\frac{n(n-1)}{2} = \frac{1(1-1)}{2} = 0$ true

If $n=k$ $\frac{k(k-1)}{2}$ edges are required true

To prove for $n=k+1$

When we add the $(k+1)$ vertex we need to connect it to the k original vertices requiring k additional edges.

Then we have $\frac{k(k-1)}{2} + k$

$$\Rightarrow \frac{k(k-1)}{2} + 2k$$

$$\Rightarrow \frac{k^2 - k + 2k}{2}$$

$$\Rightarrow \frac{k^2 + 2k + 1 - k - 1}{2}$$

$$\Rightarrow \frac{(k+1)^2 - (k+1)}{2}$$

$$\Rightarrow \frac{(k+1)\left[(k+1) - 1\right]}{2}$$

~~Hence~~ true for $n = k+1$

Hence proved

Questions

1) Define Finite Graph, Infinite graph, ~~Sa~~ Simple graph. with examples.

Lecture #3

Paths and Circuits

Isomorphism

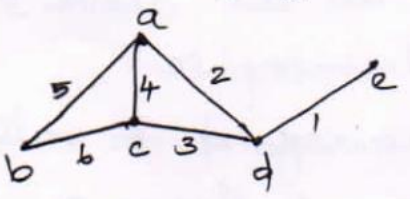
→ In geometry two figures are said to be equivalent if they have identical behaviour in terms of geometric properties.

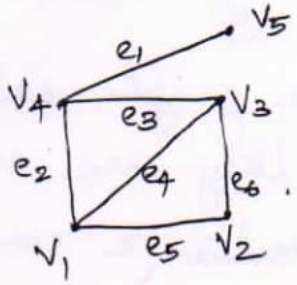
→ Similarly two graphs are said to be equivalent or isomorphic if they have identical behavior in terms of graph-theoretic properties.

→ i.e. Two graphs G & G' are said to be isomorphic if there is a one-to one correspondence b/w their vertices and between their edges such that the incidence relationship is preserved.

→ Suppose edge e is incident on vertices v_1 & v_2 in G. Then the corresponding edge e' in G' must be incident on the vertices v_1' & v_2' that correspond to v_1 & v_2 resptly.

eg: 1

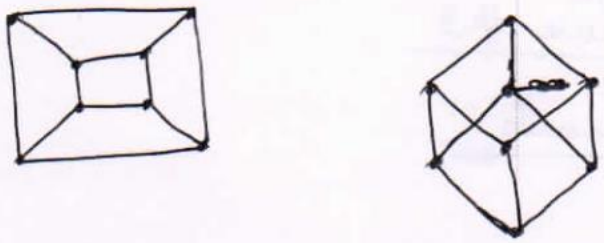




Vertices a, b, c, d & e correspond to v_1, v_2, v_3, v_4 & v_5 resptly

Edges 1, 2, 3, 4, 5 & 6 correspond to e_1, e_2, e_3, e_4, e_5 & e_6 resptly

eg: 2



eg: 3

→ So by definition, two isomorphic graphs must have

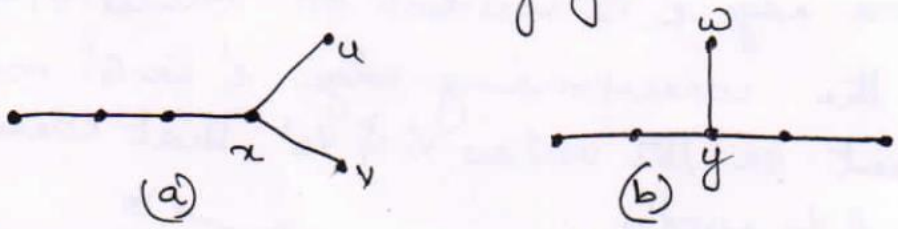
(a) Same no: of vertices

(b) Same no: of edges

(c) Equal no: of vertices with a given degree.

→ But in certain cases these conditions are not sufficient.

for eg: Consider the two graphs



→ The two graphs satisfy all the above 3 conditions. but they are not isomorphic.

→ In (a) vertex x must correspond to vertex y, bcoz there are no other vertices of degree 3.

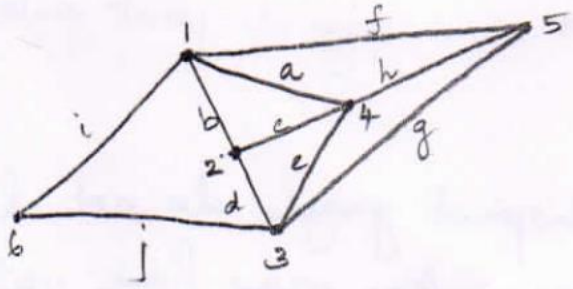
→ In (b) there is only one pendant vertex, w adjacent to y. while in (a) there are two pendant vertices u and v adjacent to x.

Subgraphs

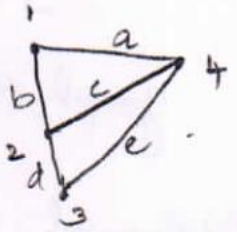
→ A graph g is said to be a subgraph G if all the vertices and all the edges of g are in G.

→ Each edge of g has the same end vertices in g as in G.

eg:



graph G



subgraph g

→ A subgraph can be thought of as being contained in another graph.

→ Denoted by $g \subset G \Rightarrow g$ is a subgraph of G.

→ "Every graph is its own subgraph".

→ "A subgraph of a subgraph of G is a subgraph of G".

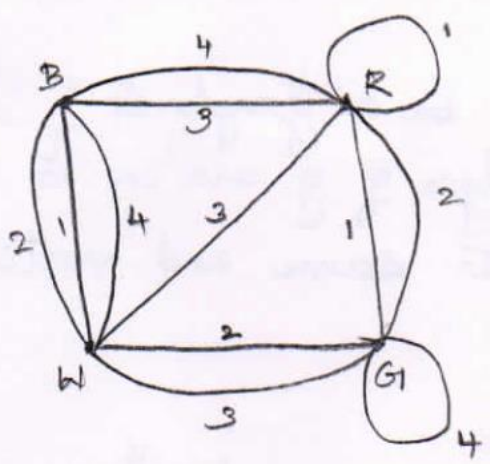
→ "A single vertex in a graph G is a subgraph of G"

→ "A single edge in G, together with its end vertices is also a subgraph of G".

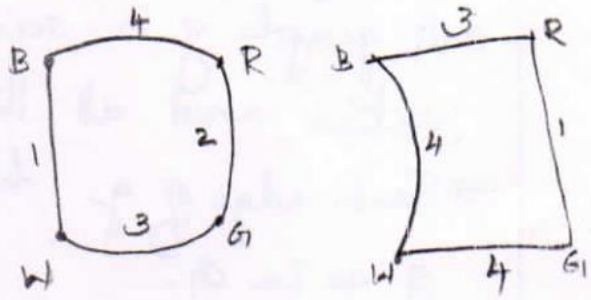
Edge Disjoint Subgraphs

→ Two (or more) subgraphs g_1 & g_2 of a graph G are said to be edge disjoint if g_1 & g_2 do not have any edges in common.

eg:



graph G



edge disjoint subgraphs g_1 & g_2

→ Although edge disjoint graphs do not have any edge in common, they may have vertices in common.

→ Subgraphs that do not have vertices in common are said to be <u>vertex disjoint</u>.

Walks, Paths & Circuits

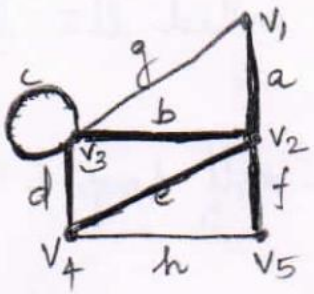
Walk

→ Defined as a finite alternating sequence of vertices and edges beginning & ending with vertices, such that each edge is incident with the vertices preceding & following it.

→ No edge appears more than once in a walk.

→ A vertex may appear more than once.

eg:



$v_1 a v_2 b v_3 c v_3 d v_4 e v_2 f v_5$ is a walk

→ A walk is also referred to as an <u>edge train</u> or a <u>chain</u>.

→ The set of vertices & edges constituting a given walk in a graph G is a <u>subgraph</u> of G.

→ Vertices with which a walk begins and ends are called its <u>terminal vertices</u>.

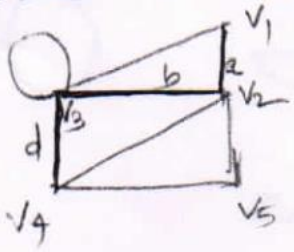
eg: vertices v_1 & v_5.

→ A walk begin and end at the same vertex is called a closed ~~path~~ walk:

→ A walk that is not closed (ie the terminal vertices are distinct) is called an <u>open walk</u>.

→ An open walk in which no vertex appears more than once is called a path.

eg:



$v_1 a v_2 b v_3 d v_4$ is a path → length → 3.

where $v_1 a v_2 b v_3 c v_3 d v_4 e v_2 f v_5$ is not a path

→ The no: of edges in a path is called the <u>length of a path</u>.

→ An edge which is not a self loop is a path of length one.

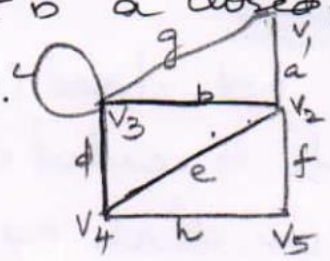
→ The <u>terminal vertices</u> of a path are of <u>degree one</u> and the rest of the vertices are of <u>degree two</u>

→ The vertices in a path other than terminal vertices are known as <u>intermediate vertices</u>.

→ <u>Circuit</u>

→ Closed walk in which no vertex (except the initial and final vertex) appears more than once is called a <u>circuit</u>.

→ ie Circuit is a closed, non intersecting walk.



eg: $v_2\, b\, v_3\, d\, v_4\, e\, v_2$.

Egs of circuits

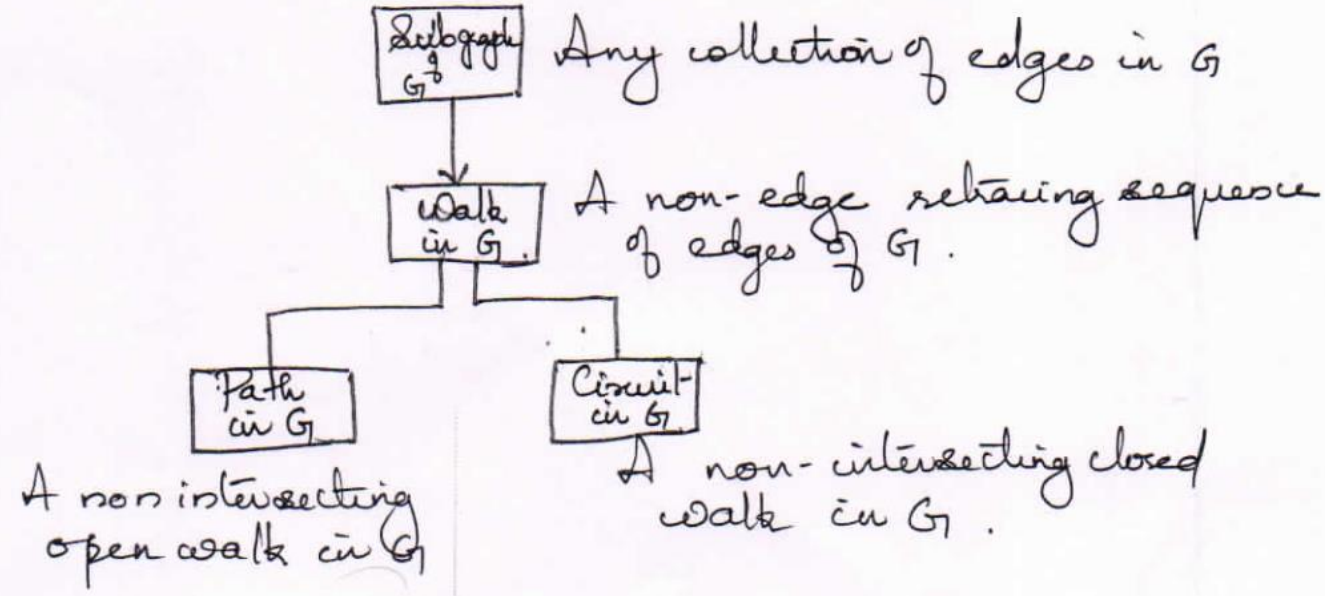
→ Every vertex in a circuit is of degree two.

→ If the circuit is a subgraph of another graph, one must count degrees contributed by the edges in the circuit only.

→ Cycle, Elementary cycle, circular path & polygon
→ Other names for circuit.

→ In electrical engg → circuit is referred to as loop.

→ The definitions are summarized as follows



Questions

1) What is walk?
2) Define circuit.

Lecture #4

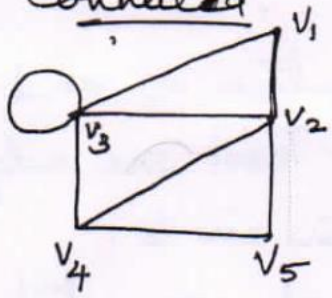
Connected Graphs & Disconnected Graphs

→ A graph is connected if we can reach any vertex from any other vertex by traveling along the edges.

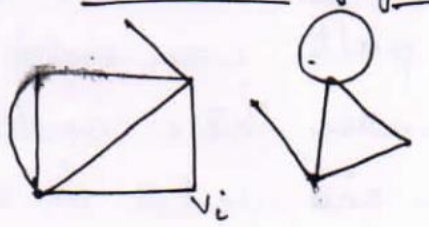
→ ie A graph G is said to be connected if there is at least one path b/w every pair of vertices in G

→ Otherwise the graph G is disconnected.

eg: Connected



Disconnected graph with 2 components



→ A null graph of more than one vertex is disconnected.

→ A disconnected graph consists of two or more ~~component~~ connected graphs.

→ Each of these connected subgraphs is called a component.

→ Consider a vertex v_i in a disconnected graph G.

→ By definition not all vertices of G are joined by paths to v_i.

→ Vertex v_i and all the vertices of G that have paths to v_i, together with all the edges incident on them, form a component.

→ A component itself is a graph.

Theorem :-1

"A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 & V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2".

Proof.

① → Suppose that such a partitioning exists.
→ Consider two arbitrary vertices a and b of G such that $a \in V_1$ & $b \in V_2$.
→ No path can exist b/w vertices a and b;
→ Otherwise there could be at least one edge whose one end vertex in V_1 & other in V_2.
→ Hence partition exists, G is not connected.

② → Conversely, let G be a disconnected graph.
→ Consider a vertex a in G.
→ Let V_1 be the set of all vertices that are joined by paths to a.
→ Since G is disconnected V_1 does not include all the vertices of G.
→ The remaining vertices will form a set V_2.
→ No vertex in V_1 is joined to any in V_2 by an edge.
→ Hence the partition:

Theorem - 2.

"If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices".

Proof.

→ Let G be a graph with all even vertices (vertices with even degree) except vertices v_1 & v_2, which are odd.

→ From the theorem proved earlier.

"The no: of vertices of odd degree in a graph is always even".

→ which holds for every graph and therefore for every component of a disconnected graph, no graph can have an odd no: of odd vertices.

→ Therefore in graph G, v_1 & v_2 must belong to the same component.

→ Hence ~~they~~ must ~~a~~ & have a path b/w them.

→ Hence the theorem

ie $\sum_{i=1}^{k}(n_i-1)^2 + 2\sum_{i,j=1, i\neq j}^{k}(n_i-1)(n_j-1) = n^2+k^2-2nk$.

ie $\sum_{i=1}^{k}\left[n_i^2-2n_i+1\right] + 2\sum_{i,j=1, i\neq j}^{k}(n_i-1)(n_j-1) = n^2+k^2-2nk$.

ie $\sum_{i=1}^{k} n_i^2 + k - 2\sum_{i=1}^{k} n_i + 2\sum_{i,j=1, i\neq j}^{k}(n_i-1)(n_j-1) = n^2+k^2-2nk$

ie $\sum_{i=1}^{k} n_i^2 + k - 2n + 2\sum_{i,j=1, i\neq j}^{k}(n_i-1)(n_j-1) = n^2+k^2-2nk$.

$$\sum_{i=1}^{k} n_i^2 + 2\sum_{i,j=1, i\neq j}^{k}(n_i-1)(n_j-1) = n^2+k^2-2nk-k+2n.$$

$$= n^2 - k(2n-k) + (2n-k)$$

$$= n^2 + \cancel{2nk}(2n-k)(-k+1)$$

$$= \underline{\underline{n^2 - (k-1)(2n-k)}}.$$

ie $\sum_{i=1}^{k} n_i^2 \leq n^2-(k-1)(2n-k)$ [Algebraic Inequality]

Max no: of edges in a graph with n vertices

$$= \frac{n(n-1)}{2}.$$

$\therefore$ Max no: of edges in i^{th} component $= \frac{n_i(n_i-1)}{2}$.

$\therefore$ Total no: of edges $= \frac{1}{2}\sum_{i=1}^{k} n_i(n_i-1)$

$$= \frac{1}{2}\sum_{i=1}^{k}(n_i^2 - n_i)$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k}(n_i)^2 - \sum_{i=1}^{k} n_i\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k}(n_i)^2 - n\right]$$

$$= \frac{1}{2}\left[n^2 - (k-1)(2n-k) - n\right]$$

$$= \frac{1}{2}\left[n^2 - 2nk + k^2 + 2n - k - n\right]$$

$$= \frac{1}{2}\left[n^2 + k^2 - 2nk - k + n\right]$$

$$= \frac{1}{2}\left[(n-k)^2 + (n-k)\right]$$

$$= \frac{1}{2}\left[(n-k)(n-k+1)\right]$$

Hence the proof.

Question

1) Prove that a simple graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Lecture #5

Euler Graphs.

→ Graph theory was born in 1736 with Euler's famous paper in which he solved the Konigsberg bridge problem.

→ Solved by closed walk

→ Euler proposed that → "If some closed walk in a graph contains all the edges of the graph, then the walk is called as Euler line and the graph is called and Euler graph.

→ Euler graph do not have any isolated vertices and are therefore connected.

Theorem -1

" A given connected graph G is an Euler graph if and only if all vertices of G are of even degree".

① ~~Prove~~ Assume G is an Euler graph.

Prove ~~G~~ every vertex in G is even.

Proof

→ Suppose G is an Eulergraph

→ It therefore contains an Euler line (which is a closed walk).

→ In tracing the walk → we observe that every time the walk meets a vertex v it goes through to new edges incident on v

→ One that is entered v and the other exited.

→ For terminal vertices we exited and entered the same vertex at the beginning and end of the walk respectively.

→ Thus if G is an Euler graph, the degree of every vertex is even.

(2) Assume all vertices in G are of even degree. Prove G is an Euler graph.

<u>Proof</u>.

→ Construct a walk starting at an arbitrary vertex v and going through the edges of G such that no edge is traced more than once.

→ Since every vertex is of even degree, we can exit from every vertex we enter.

→ Stop tracing at the starting vertex ~~G~~ v.

→ Name this closed walk as 'h'.

→ If h includes all the edges of G, <u>G is an Euler graph</u>

→ If not, remove from G all the edges in h and obtain a subgraph h' of G formed by the remaining edges.

→ Since both G and h have all their vertices of even degree, the degrees of vertices of h' are even.

→ Moreover h' must touch h at least at one vertex a, bcoz G is connected.

→ Starting from a, we can again construct a new walk in graph h'.

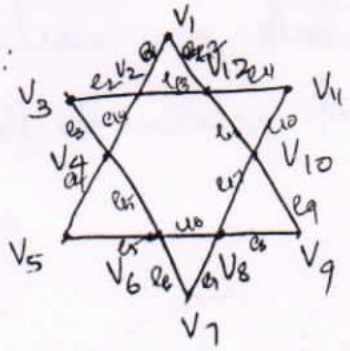
→ Since all vertices are of even degree, the walk in h' must terminate at a.

→ The walk in h' can be combined with h to form a new walk → which starts and ends at vertex v and has more edges than h.

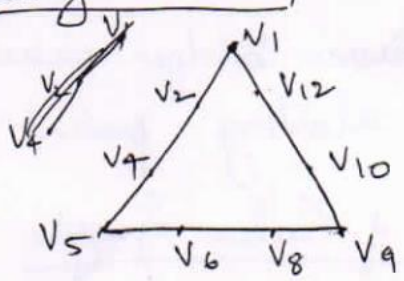
→ This process is repeated until we obtain a closed walk that traverses all the edges of G.

→ Thus G is an Euler graph.

eg:



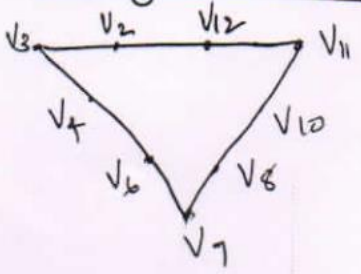
Starting vertex v_1



h.

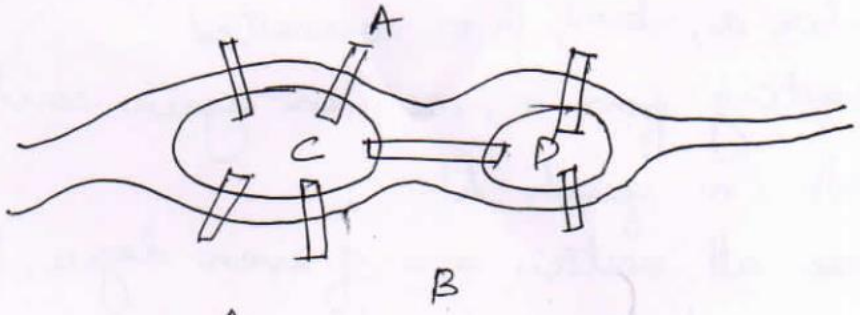
h'

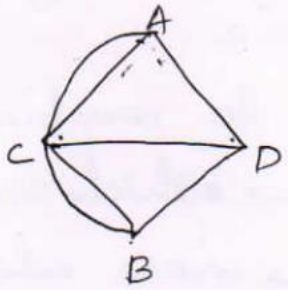
starting vertex v_2



$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_6$
$e_6 v_7 e_7 v_8 e_8 v_9 e_9 v_{10} e_{10} v_{11}$
$e_{11} v_{12} e_{13} v_2 e_{14} v_4 e_{15} v_6$
$e_{16} v_8 e_{17} v_{10} e_{18} v_{12} e_{19} v_1$

Konigsberg Bridge Problem

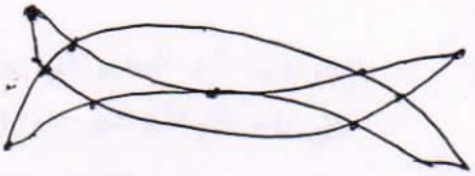




→ In the graph of Konigsberg briges
  → Not all of its vertices are of even degree.
→ Hence it is not an Euler graph.
→ Thus it is not possible to walk over each of the seven bridges exactly once and return to the starting point.

Egs: of Euler Graphs

① Mohammed's schimitars → believed to have come from Arab



② Star of David

Unicursal Graph

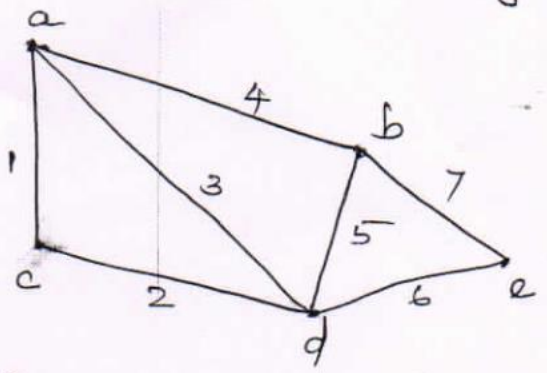
→ Open walk.

→ Includes all the edges of the graph and does not retrace any edge, and is not closed.

→ Such an open walk is called a unicursal line or an open Euler line.

→ A connected graph that has a unicursal line will be called a unicursal graph.

eg:



Unicursal line :- a1c2d3a4b5d6e7b.

Questions

1) Explain Euler & Unicursal graph with examples.

Lecture # 6

Operations on Graphs.

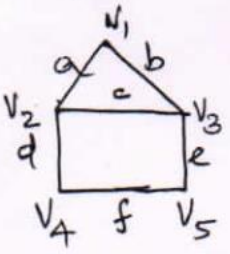
Union.

→ The union of two graphs $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ is another graph G_3

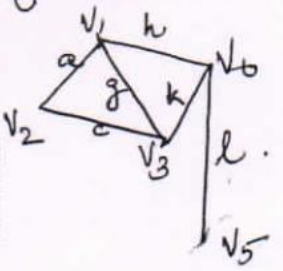
→ $G_3 = G_1 \cup G_2$

→ Vertex set of G_3 is $V_3 = V_1 \cup V_2$.

→ Edge set of G_3 is $E_3 = E_1 \cup E_2$.

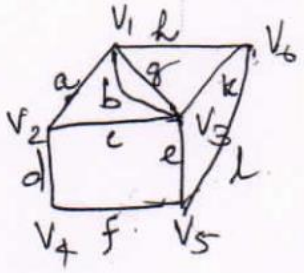


G_1



G_2

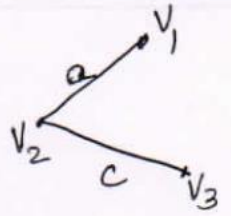
$G_1 \cup G_2 \Rightarrow$



Intersection

→ The intersection $G_1 \cap G_2$ of graphs G_1 & G_2 is a graph consisting of the ~~vertex set~~ only those vertices and edges that are in both G_1 & G_2.

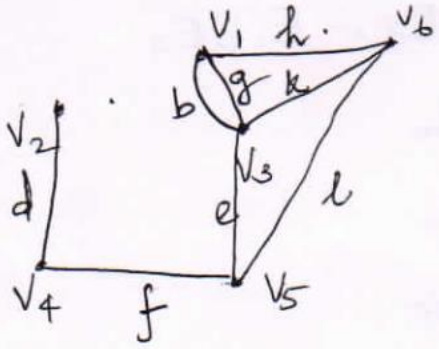
eg:



Ring Sum

→ The ring sum of two graphs G_1 & G_2 (written as $G_1 \oplus G_2$) is a graph consisting of the vertex set $V_1 \cup V_2$ and of the edges that are either in G_1 or G_2, but not in both.

eg:



Properties

Commutative Property

$G_1 \cup G_2 = G_2 \cup G_1$

$G_1 \cap G_2 = G_2 \cap G_1$

$G_1 \oplus G_2 = G_2 \oplus G_1$

Null graph

→ If G_1 & G_2 are edge disjoint then

$G_1 \cap G_2$ = null graph.

→ If G_1 & G_2 are edge disjoint, then

$$G_1 \oplus G_2 = G_1 \cup G_2.$$

→ If G_1 & G_2 are vertex disjoint,

then $G_1 \cap G_2$ is empty.

→ For any graph G

$$G \cup G = G \cap G = G$$

$$G \oplus G = \text{a null graph}.$$

→ If g is a subgraph of G, then

$G \oplus g$ is → subgraph of G which remains after all the edges in g have been removed from G.

$\therefore$ $G \oplus g = G - g$, whenever $g \subseteq G$.

<u>Complement</u>

$G \oplus g = G - g$ is called the complement of g in G.

<u>Decomposition</u>

→ A graph G is said to have been decomposed into two graphs g_1 & g_2 if

$$g_1 \cup g_2 = G$$

& $$g_1 \cap g_2 = \text{a null graph}.$$

→ ie Every edge of G occurs either in g_1 or in g_2 but not in both.

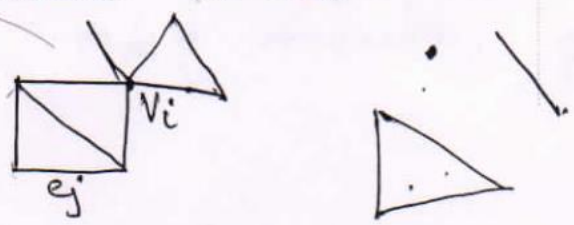
→ Some of the vertices, may occur in both g_1 & g_2.

→ A graph containing m edges $\{e_1, e_2, \ldots e_m\}$ can be decomposed in $2^{m-1}-1$ diff ways into pairs of subgraph.

<u>Deletion</u>

→ If V_i is a vertex in graph G, then $G - V_i$ denotes a subgraph of G obtained by deleting V_i from G.

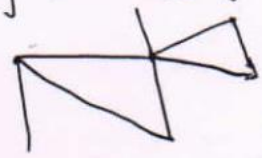
→ Deletion of a vertex ⇒ deletion of all edges incident on that vertex.



G $\qquad$ $(G - V_i)$

→ If e_j is an edge in G, then $G - e_j$ is a subgraph of G obtained by deleting e_j from G.

→ $G - e_j = G \oplus e_j$

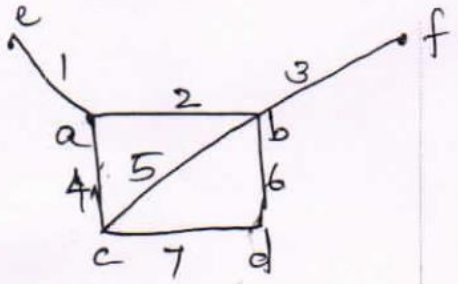


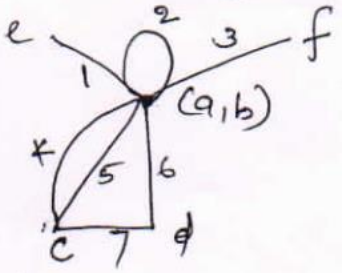
Fusion

→ A pair of vertices a, b in a graph are said to be fused (merged or identified)

→ if the two vertices are replaced by a single new vertex such that every edge that was incident on either a or b or on both is incident on the new vertex.

→ Reduces the no: of vertices by one.





Questions

1) What are the diff operations in graph?

7e

Lecture # 7

More on Euler Graphs

Theorem 2

"A connected graph G is an Euler graph if and only if it can be decomposed into circuits.

Proof

① Assume G can be decomposed into circuits.
Prove G is an Euler graph.

Proof

→ G can be decomposed into circuits
→ ie G is a union of edge disjoint circuits.
→ Since the degree of vertex in a circuit is two, the degree of every vertex in G is even.
→ Hence G is an Euler graph.

Conversely

② Assume G be an Euler graph
Proove G can be decomposed into circuits.

Proof

→ Consider a vertex V_1.
→ There are atleast two edges incident at V_1.
→ Let one of the edges be b/w V_1 & V_2.

→ Since vertex v_2 is also of even degree, it must have at least another edge, b/w v_2 & v_3.

→ Proceeding in this way, we arrive at a vertex that has previously been traversed, thus forming a circuit Γ.

→ Remove Γ from G.

→ All vertices in the remaining graph must also be of even degree.

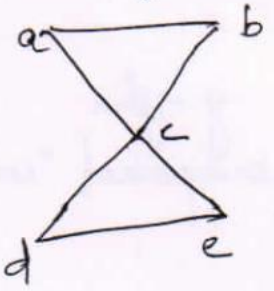
→ From the remaining graph remove another circuit in exactly the same way as we removed Γ from G.

→ Continue this process until no edges are left.

→ Hence the theorem.

Arbitrarily Traceable Graphs

→ Consider the following graph



→ The graph is an Euler graph.

→ Start from vertex a and trace the path a, b, c.

Hamiltonian Paths and Circuits

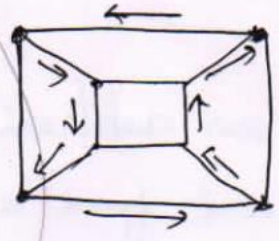
Hamiltonian Circuit

→ A hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly once → except the starting vertex.

→ A circuit in a connected graph G is said to be Hamiltonian if it includes every vertex of G.

→ Thus a hamiltonian circuit in a graph of n vertices consists of exactly n edge.

eg:



Hamiltonian Path

→ If we remove any one edge from a Hamiltonian circuit, we are left with a path

→ This path is called a Hamiltonian path.

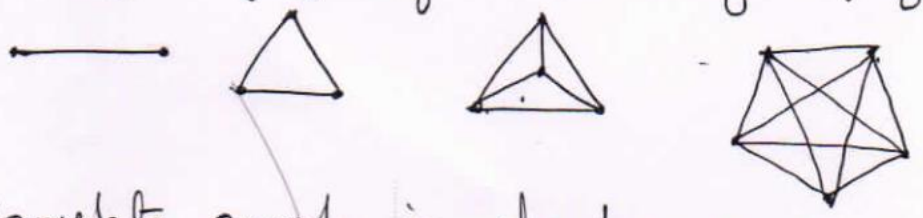
→ A hamiltonian path in a graph G traverses every vertex of G.

Lecture # 8

Complete Graphs.

→ A simple graph in which there exists an edge b/w every pair of vertices is called a complete graph

→ eg: complete graphs of two, three, four & five vertices



→ Complete graph is also known as universal graph or clique.

→ The degree of every vertex is $n-1$ in a complete graph G of n vertices.

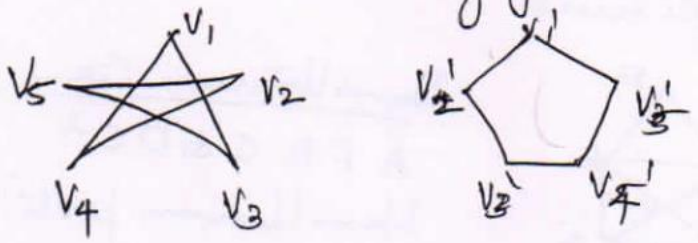
→ Total no: of edges in G is $\frac{n(n-1)}{2}$.

No: of hamiltonian circuits in a graph.

Theorem

" In a complete graph with n vertices there are $(n-1)/2$ edge-disjoint hamiltonian circuits, if n is an odd number ≥ 3."

1) Prove that the two graphs are isomorphic



$V = \{v_1, v_2, v_3, v_4, v_5\}$

$V_1' = \{v_1', v_2', v_3', v_4', v_5'\}$

$E = \{(v_1 v_2), (v_2 v_3), (v_3, v_4), (v_4, v_5), (v_5, v_1)\}$.

$E' = \{(v_1' v_2'), (v_2' v_3'), (v_3' v_4'), (v_4' v_5'), (v_5' v_1')\}$.

No: of vertices = 5

No: of edges = 5

Degree of vertices = all of degree 2

Connectedness - Each is fully connected.

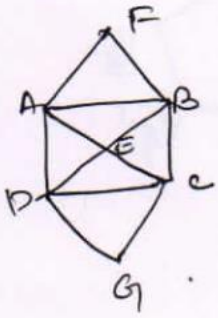
Pairs of connected vertices : All correspond

No. of loops = 0

No: of parallel edges = 0.

Everything equal. so the graphs are isomorphic

2) Draw a graph which has Euler circuit and has Hamiltonian circuits



Hamiltonian Circuit.

A F B C G D E A

Hamiltonian path

A F B C G D E

Travelling Salesman Problem

Problem

A salesman is required to visit a no: of cities during a trip. Given the distances b/w the cities, in what order should he travel so as to visit every city precisely once & return home with the min mileage traveled?.

Soln

$4!$
$1 \times 2 \times 3 = 6$

→ Represent the cities by vertices and roads b/w them by edges.

→ With every edge e_i, there is associated ~~with~~ a real no: :- $w(e_i)$

→ Such a graph is called a weighted graph

→ Weight of edge $e_i \Rightarrow w(e_i)$.

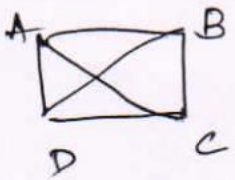
→ If each of the cities has a road to every other city we have a complete weighted graph.

→ This graph contains many Hamiltonian circuits

→ Pick the one that has smallest sum of distances.

→ Total no: of hamiltonian circuit in a complete graph is $\frac{(n-1)!}{2}$.

For eg: Consider the graph with vertices 4.



Hamiltonian circuits

1. ABCDA
2. ABDCA
3. ACBDA
4. ACDBA
5. ADBCA
6. ADCBA

$(n-1)!$

$= \frac{(4-1)!}{2} = 3! = \underline{6}$

1 & 6 → mirror images.
2 & 4 → "
3 & 5 → "

So total no: of hamiltonian circuit = 3.

In general $\frac{(n-1)!}{2}$ circuits.

→ Theoretically the problem of the ~~TSP~~ travelling salesman can be solved by enumerating all $\frac{(n-1)!}{2}$ Hamiltonian circuits.

→ Calculate the distance travelled in each

→ Pick the shortest one.

→ NP hard problem.

Theorem. Lecture #9

" In a complete graph with n vertices there are $(n-1)/2$ edge disjoint Hamiltonian circuits, if n is an odd number ≥ 3 ".

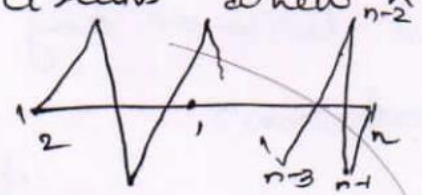
Proof

→ A complete graph of G of n vertices has $n(n-1)/2$ edges

→ A hamiltonian circuit in G consists of n edges.

→ ∴ No: of edge disjoint hamiltonian circuits in G cannot exceed $\frac{n-1}{2}$.

→ i.e. There are $(n-1)/2$ edge disjoint hamiltonian circuits when n is odd.



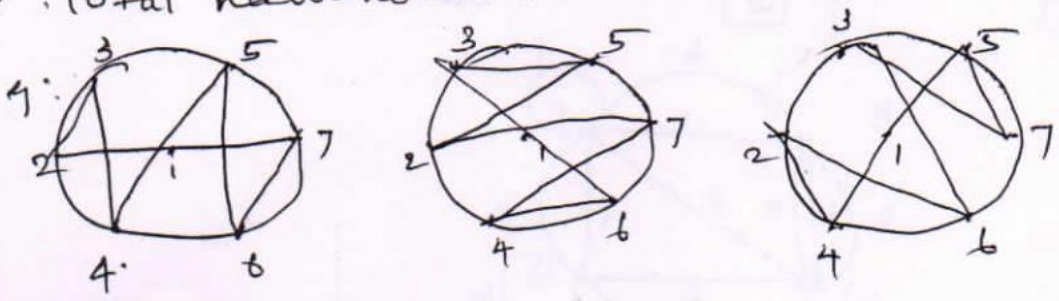
→ The subgraph is a hamiltonian circuit.

→ Keeping the vertices fixed on a circle, rotate the polygonal pattern clockwise by $\frac{360}{n-1}$, $2\cdot\frac{360}{n-1}$, $3\cdot\frac{360}{n-1}$ .. $\left(\frac{n-3}{2}\right)\cdot\frac{360}{n-1}$ degrees.

→ Each rotation produces a hamiltonian circuit that has no edge in common

→ Thus we have $\frac{n-3}{2}$ new cycles

∴ Total hamiltonian circuit $= \frac{n-3}{2}+1 = \frac{n-1}{2}$



A puzzle with multicolored cubes

→ Well known puzzle available in toy stores
→ Known as Instant Insanity.

Problem

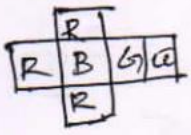
→ Gives four cubes.
→ The six faces of every cube are variously colored blue, green, red or white.
→ Is it possible to stack the cubes one on top of another to form a column such that no color appears twice on any of the 4 sides of this column?

Solution

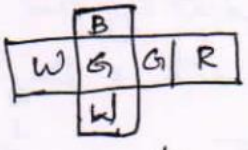
Step 1 :-

→ Draw a graph with 4 vertices B, G, R & W.
→ One for each color.

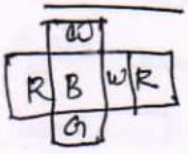
Cube 1



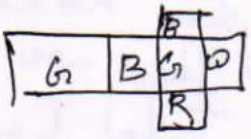
Cube 2

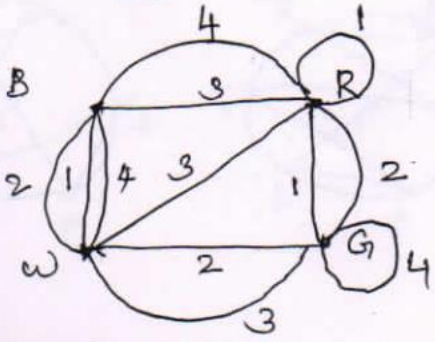


Cube 3



Cube 4





Step 2

→ Consider the graph

→ The degree of each vertex is the total no: of faces with the corresponding color.

→ We have 5 blue faces, six green, seven red and six white.

→ Consider 2 opposite vertical sides of the desired column of 4 cubes, say facing north and south

→ A subgraph with 4 edges will represent these 8 faces → four facing south and 4 north.

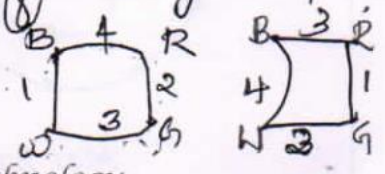
→ Each of the 4 edges in this subgraph will have diff label 1, 2, 3, 4

→ Moreover no color occurs twice on either the north side or south side of the column if and only if every vertex in the subgraph is of degree two.

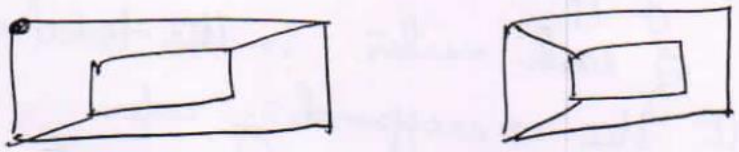
→ Exactly the same argument applies to the other two sides, east & west of the column.

→ Thus the 4 cubes can be arranged if and only if there exist two edge disjoint subgraphs each with 4 edges.

→ Each of the edges labeled differently and each vertex is of degree two.



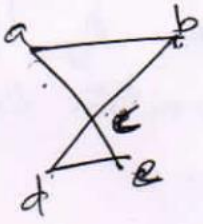
1) Check whether isomorphic or not



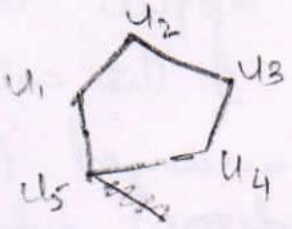
Not isomorphic

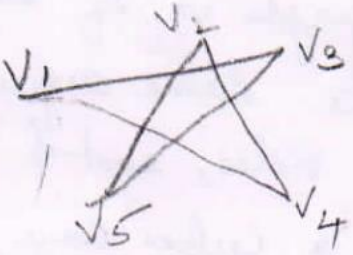
Isomorphic

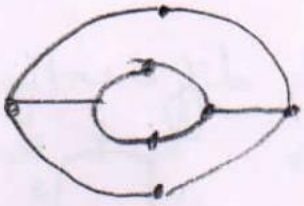
2) Which of these simple graphs has a Hamiltonian circuit, if not a Hamiltonian path

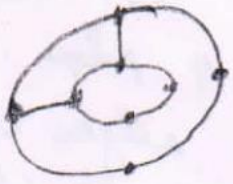


→ circuit abcdea.





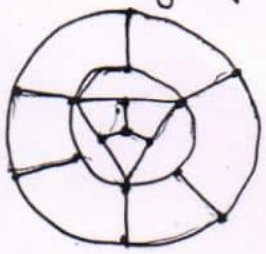




→ A hamiltonian path is a subgraph of hamiltonian circuit

→ Every graph that has a hamiltonian circuit also has a hamiltonian path.

→ The length of a hamiltonian path in a connected graph of n vertices is $n-1$.



→ Graph without hamiltonian circuit.

Questions

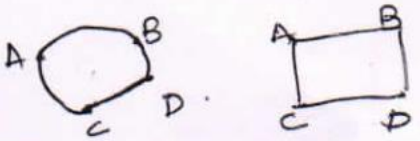
1) Explain Hamiltonian Paths and Circuits.

1. Sketch a connected graph that becomes disconnected when any edge is removed from it.

Soln: Draw a simple graph with n vertices and exactly n-1 edges.

2. Draw a graph that is:

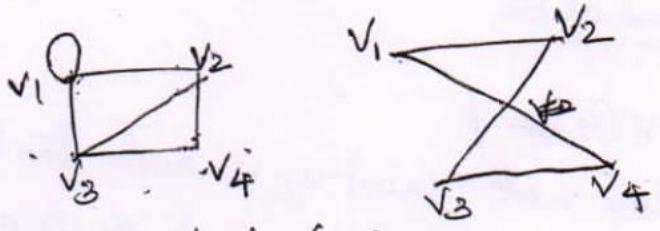
(a) both Euler and Hamiltonian



(b) Euler but not Hamiltonian.

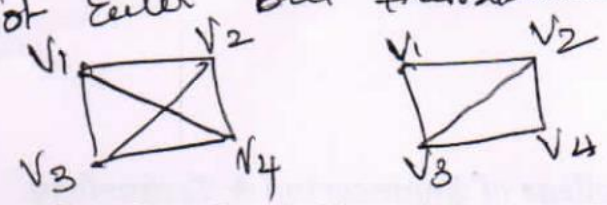
Hamiltonian → Dirac's theorem or ore's theorem.

Dirac's theorem → Let G be a simple graph with n vertices where $n \geq 3$ if $\deg(v) \geq \frac{1}{2}n$ for each vertex

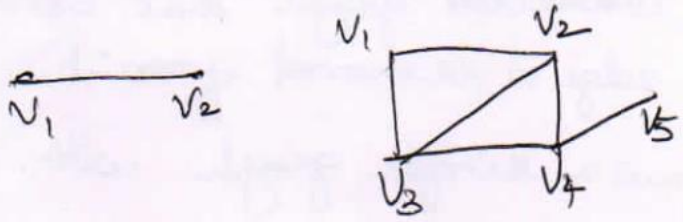


Ore's theorem – Let G be a simple graph with n vertices where $n \geq 2$ if $\deg(v) + \deg(w) \geq n$ for each ~~nor~~ pair of non-adjacent vertices v & w, then G is Hamiltonian.

(c) Not Euler but Hamiltonian



(d) Neither Euler nor Hamiltonian



3. Suppose that a connected graph G is decomposed into 2 graphs G_1 & G_2. P.T G_1 & G_2 must have a common vertex.

Proof

G is connected

$V(G_1) \cup V(G_2) = V(G)$

$E(G_1) \cup E(G_2) = E(G)$.

$E(G_1) \cap E(G_2) = \phi$. [since G_1, G_2 is a decomposition]

Show that $V(G_1) \cap V(G_2) = \phi$.

Proof by contradiction

$V(G_1) \cap V(G_2) = \phi$

ie there does not exist u, v such that

$u \in V(G_1)$, $v \in V(G_2)$ and $(u, v) \in E(G)$