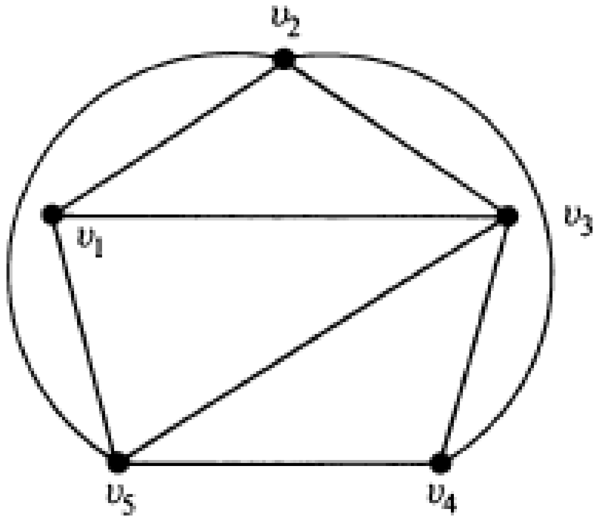
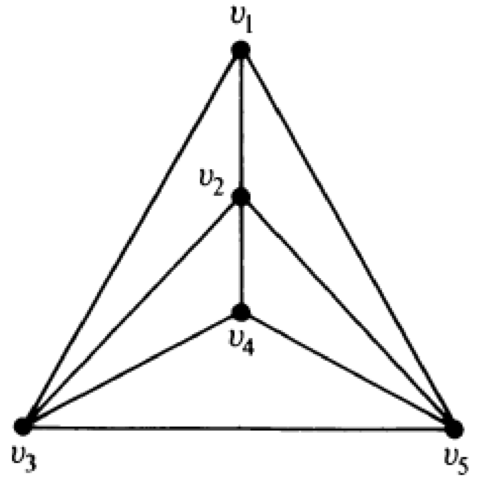
GRAPH THEORY (Module 4)

DIFFERENT REPRESENTATIONS OF PLANAR GRAPH

Theorem

Any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.

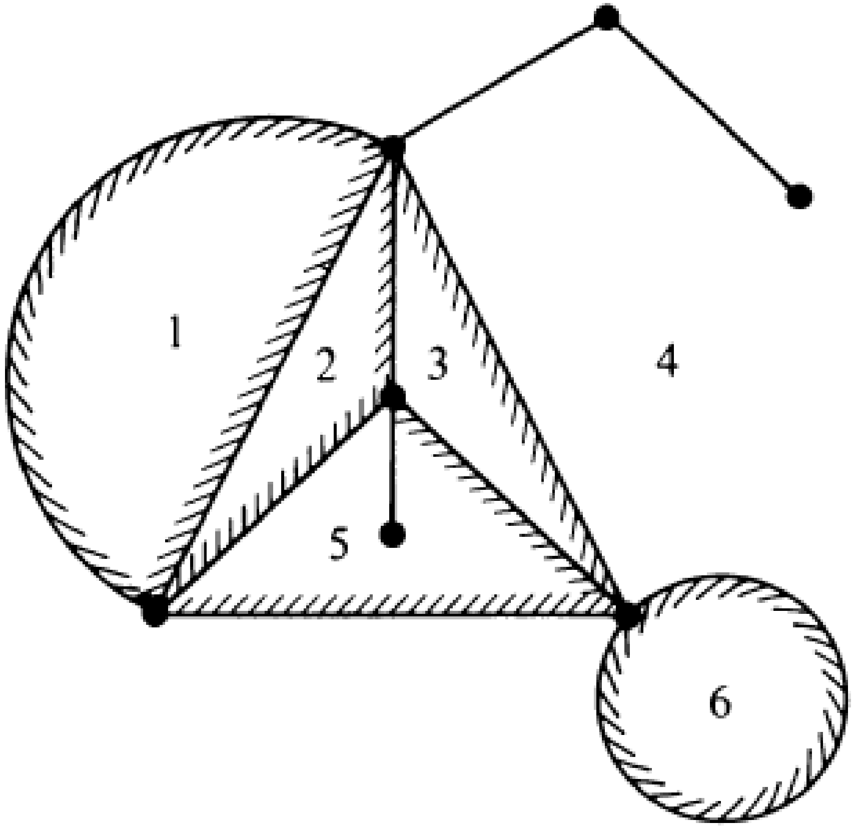




In this theorem, it is necessary for the graph to be simple because self-loop or one of two parallel edges cannot be drawn by a straight line segment.

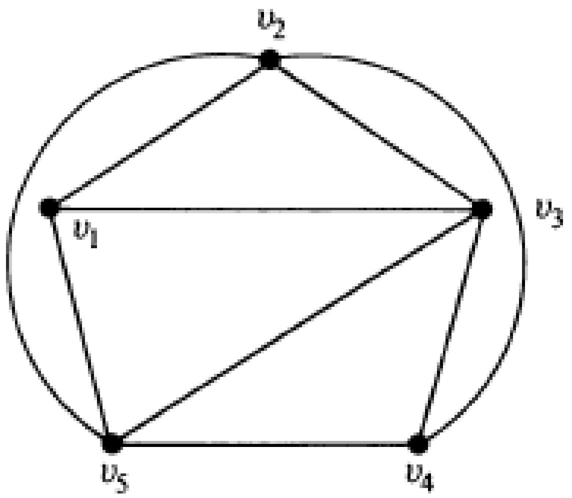
Region

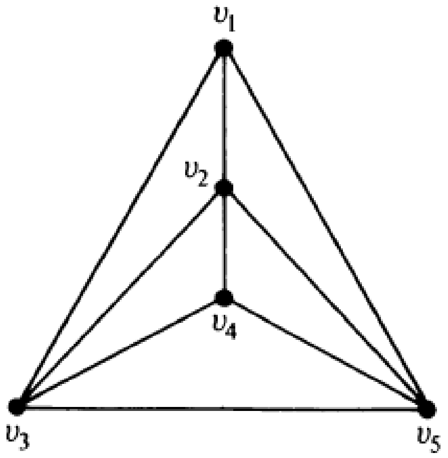
- Also called *windows, faces* or *meshes.*
- A plane representation of a graph divides the plane into *regions.*
- A region is characterized by the set of edges (or the set of vertices) forming its *boundary.*
- A region is not defined in a nonplanar graph or even in a planar graph not embedded in a plane.
- Thus a region is a property of the specific plane representation of a graph and not of an abstract graph.



Infinite Region

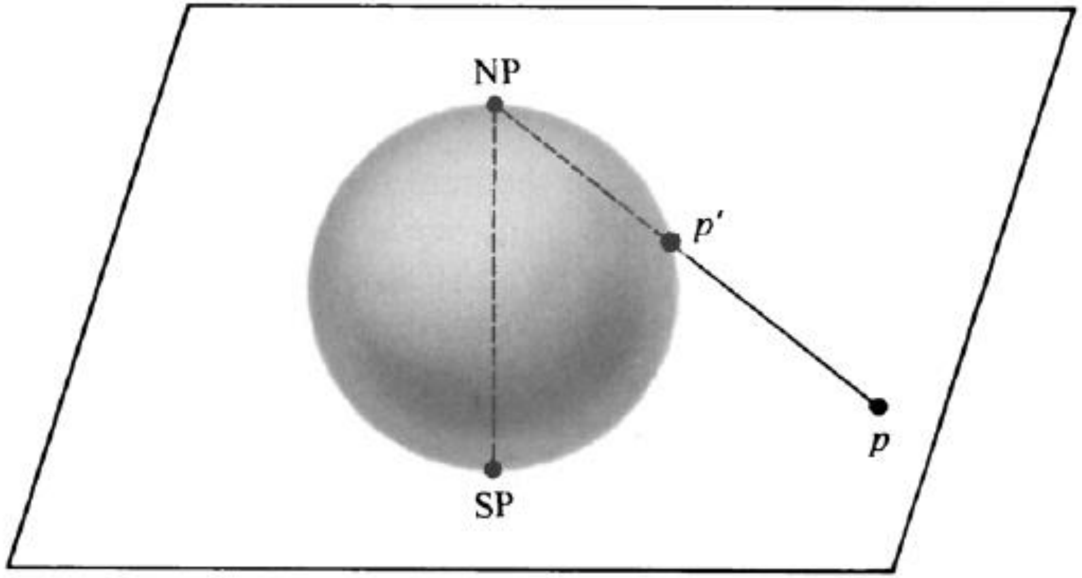
- The portion of the plane lying outside a graph embedded in a plane is infinite in its extent.
- Such a region is called the *infinite, unbounded, outer,* or *exterior* region for that particular plane representation.
- Infinite region is also characterized by a set of edges (or vertices).
- Any region can be made the infinite region by proper embedding.





Embedding on a Sphere

- To eliminate the distinction between finite and infinite regions, a planar graph is often embedded in the surface of a sphere.
- It is accomplished by stereographic projection of a sphere on a plane.
- Any graph that can be embedded in a plane (i.e., drawn on a plane such that its edges do not intersect) can also be embedded in the surface of the sphere, and vice versa.



Theorem

A graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane.

Proof

Consider the stereographic projection of a sphere on the plane. Put the sphere on the plane and call the point of contact as SP (south-pole). At point SP, draw a straight line perpendicular to the plane, and let the point where this line intersects the surface of the sphere be called NP (north-pole).

Now, corresponding to any point p on the plane, there exists a unique point p’ on the sphere, and vice versa, where p’ is the point where the straight line from point p to point NP intersects the surface of the sphere.

Thus there is a one-one correspondence between the points of the sphere and the finite points on the plane, and points at infinity in the plane corresponding to the point NP on the sphere.

Therefore from this construction, it is clear that any graph that can be embedded in a plane can also be embedded on the surface of the sphere, and vice versa.

Theorem

A planar graph may be embedded in a plane such that any specified region (i.e., specified by the edges forming it) can be made the infinite region.

Proof

A planar graph embedded on the surface of the sphere divides the surface of the sphere into different regions.

Each region of the sphere is finite, the infinite region on the plane having been mapped onto the region containing the point NP.

Clearly, by suitably rotating the sphere, we can make any specified region map onto the infinite region on the plane.

Hence the result.

Euler's Formula (Euler Theorem)

A connected planar graph with n vertices and e edges has ***e-n+2*** regions.

Proof. Since any simple planar graph can have a plane representation such that each edge is a straight line, any planar graph can be drawn such that each region is a polygon. (polygon net). Let the polygonal net representing the given graph consist of f regions and let k_p be the number of p-sided regions.

Since each edge is on the boundary of exactly two regions

$$3 \cdot k_3 + 4 \cdot k_4 + 5 \cdot k_5 + \cdots + r \cdot k_r = 2 \cdot e, \Longrightarrow \sum_{p=3}^{r} p\, k_p = 2e$$

where k_r is the number of polygons, with maximum edges.

Also, $k_3 + k_4 + \ldots k_r = f.$

The sum of all angles subtended at each vertex in the polygonal net is $2\pi n$.

A polygon of r sides has $\pi(r - 2)$ as the sum of its interior angles and $\pi(r + 2)$ as the sum of its external angles.

Hence for $r = 3, 4, \ldots f - 1$. We have $\sum_{r=3}^{f-1} \pi(r-2)k_r$ as the sum of interior angles.

Sum of the exterior angles of the polygon defining infinite region $= \pi(r - 2)k_r + 4\pi$.

Hence the sum of interior and exterior angles are

$$\pi(3 - 2)k_3 + \pi(4 - 2)k_4 + \ldots + \pi(r - 2)k_r + 4\pi = \pi(2e - 2f) + 4\pi$$

This is equal to $2\pi n$.

So,
$$\pi(2e - 2f) + 4\pi = 2\pi n.$$
$$2\pi(e - f) + 4\pi = 2\pi n.$$
$$2\pi(e - f + 2) = 2\pi n.$$

So,
$$n = e - f + 2 \Rightarrow f = e - n + 2$$

Therefore, the number of regions is $e - n + 2$.

Corollary

In any simple, connected planar graph with *f* regions, *n* vertices, *e* edges (*e>2*), the following inequalities must hold :

$$e \geq \frac{3}{2} f$$

$$e \leq 3n - 6$$

Proof: Since each region is bounded by at least three edges and each edge belongs to exactly two regions,

$$2e \geq 3f$$

or

$$e \geq \frac{3}{2}f.$$

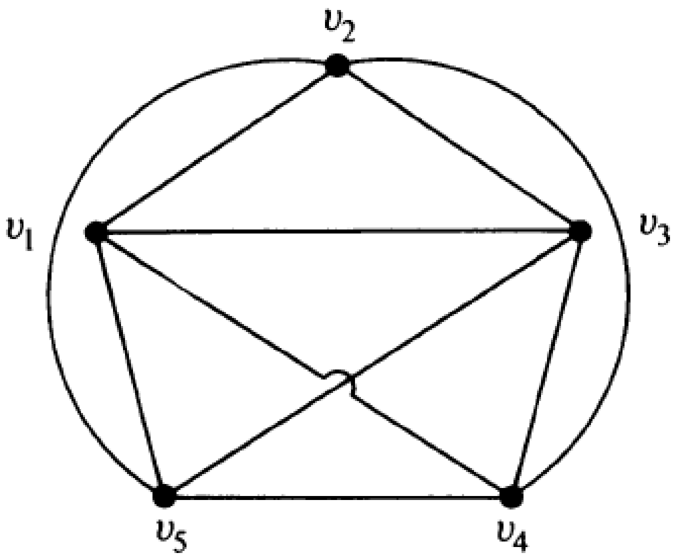
Substituting for f from Euler's formula in inequality (5-5),

$$e \geq \frac{3}{2}(e - n + 2)$$

or

$$e \leq 3n - 6. \quad \blacksquare$$

The inequality is only a necessary, but ***not*** a sufficient, condition for the planarity of a graph.

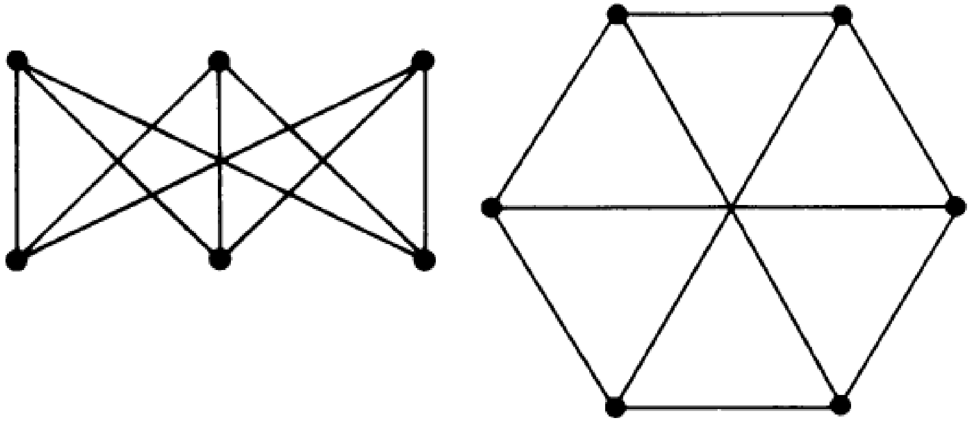


Here n=5

e=10

3n-6=9 < e

The graph violates the inequality and hence it is not planar.



Here n=6

e=9

3n-6=12 >e

The graph satisfies the inequality but it is not planar.

To prove the nonplanarity of Kuratowski's second graph, we make use of the additional fact that no region in this graph can be bounded with fewer than four edges. Hence, if this graph were planar, we would have

$$2e \geq 4f,$$

and, substituting for f from Euler's formula,

$$2e \geq 4(e - n + 2),$$

or

$$2 \cdot 9 \geq 4(9 - 6 + 2),$$

or

$$18 \geq 20, \qquad \text{a contradiction.}$$

Hence the graph cannot be planar.

Plane Representation and Connectivity

- In a disconnected graph, the embedding of each component can be considered independently.
- Therefore, a disconnected graph is planar if and only if each of its components is planar.
- Similarly, in a separable (or 1-connected) graph, the embedding of each block (i.e., maximal non-separable subgraph) can be considered independently.
- Hence, a separable graph is a planar if and only if each of its blocks is planar.

Does a non-separable planar graph G have a unique embedding on a sphere?

Unique embedding :

- Two embeddings of a planar graph on spheres are not distinct if the embeddings can be made to coincide by suitably rotating one sphere w.r.t the other and possibly distorting regions (without letting a vertex cross an edge).
- If of all possible embeddings on a sphere no two are distinct, the graph is said to have a *unique embedding* on a sphere (or a unique plane representation).

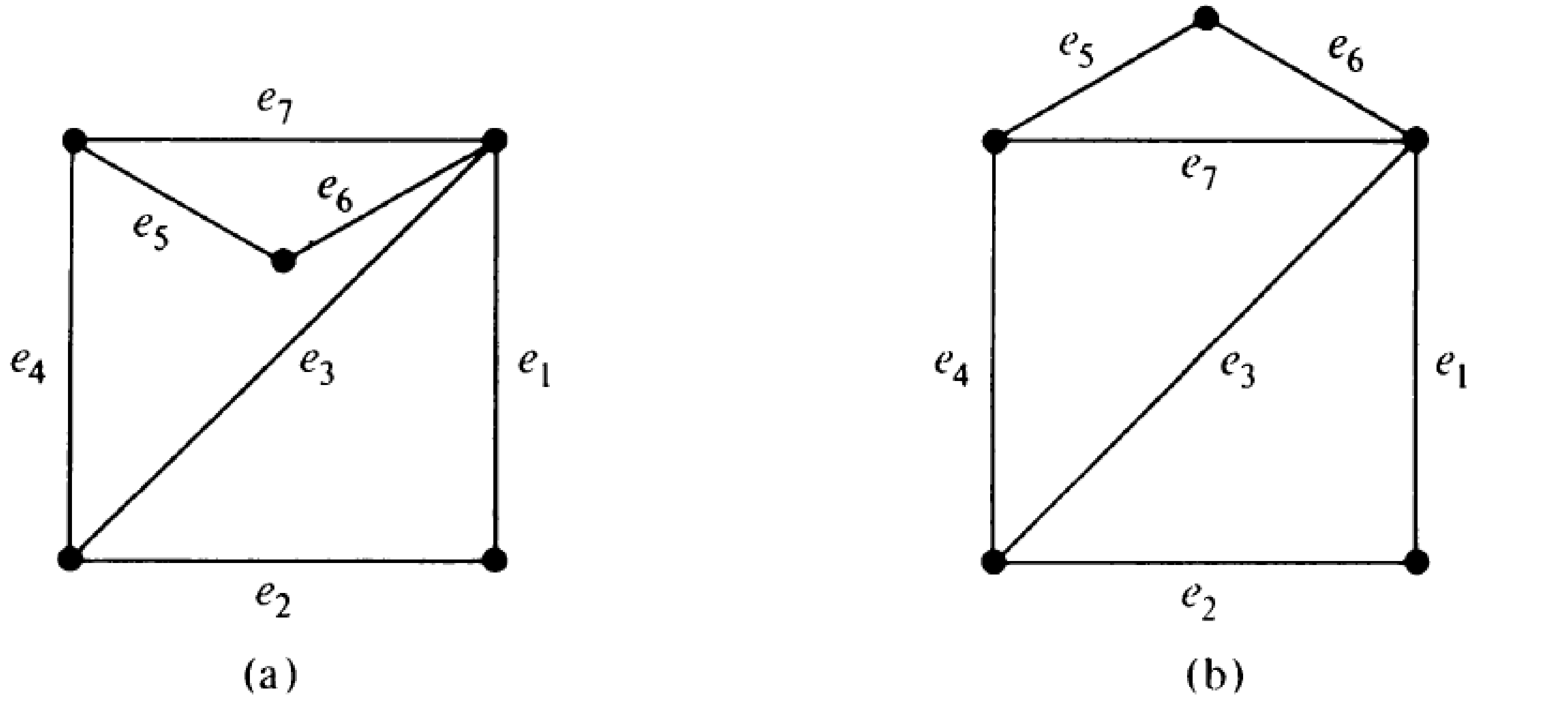
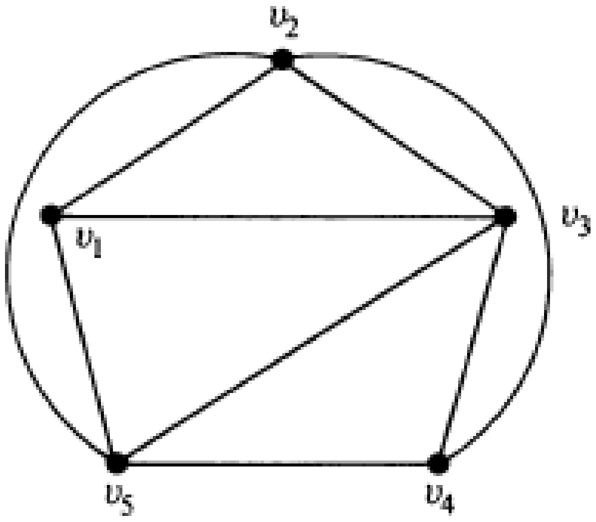


Fig. 5-6 Two distinct plane representations of the same graph.

For example, consider two embeddings of the same graph in Fig. 5-6. The embedding (b) has a region bounded with five edges, but embedding (a) has no region with five edges. Thus, rotating the two spheres on which (a) and (b) are embedded will not make them coincide. Hence the two embeddings are distinct, and the graph has no unique plane representation.



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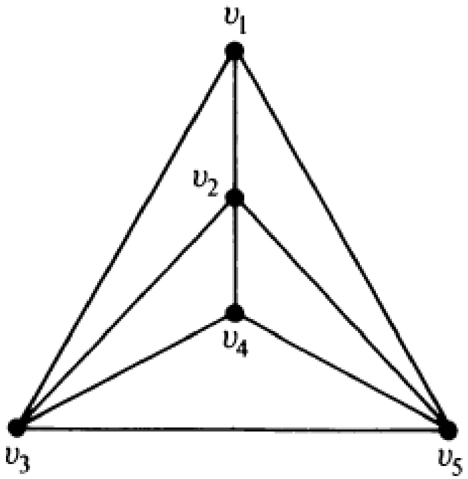

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The embeddings in Fig (1) and (2) when considered on a sphere, can be made to coincide.

Note: Edges can be bent, and in a spherical embedding there is no infinite region.

Theorem

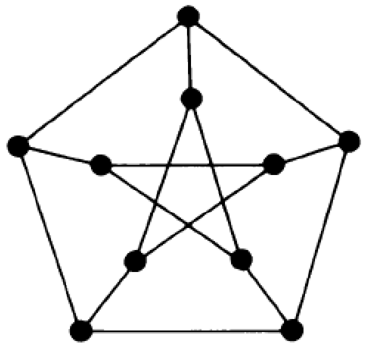
The spherical embedding of every planar 3-connected graph is unique.

This theorem plays a very important role in determining if a graph is planar or not.

The theorem states that a 3-connected graph, if it can be embedded at all, can be embedded in only one way.

Problems:

1. Suppose G is a graph with 1000 vertices and 3000 edges. Is G planar?
2. Determine the number of regions defined by a connected planar graph with 6 vertices and 10 edges. Draw a simple and a non-simple graph.
3. Show that every simple connected planar graph G with less than 12 vertices must have a vertex of degree ≤ 4.
4. Show that the condition $e \leq 3v - 6$ is not a sufficient condition for a connected simple graph with n vertices and m edges to be planar.
5. Find a simple graph G with degree sequence (4,4,3,3,3,3) such that
(a) G is planar
(b) G is non-planar.
6. Prove that Euler's formula fails for disconnected graphs.
7. Show that Petersen graph is non-planar.



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