Mod 5

Matrix representation of Graphs

- Used for better computer processing .
- G → n vertices, e edges & no self loop

Matrix will be n by e matrix, where n rows correspond to n vertices & e columns correspond to e edges.

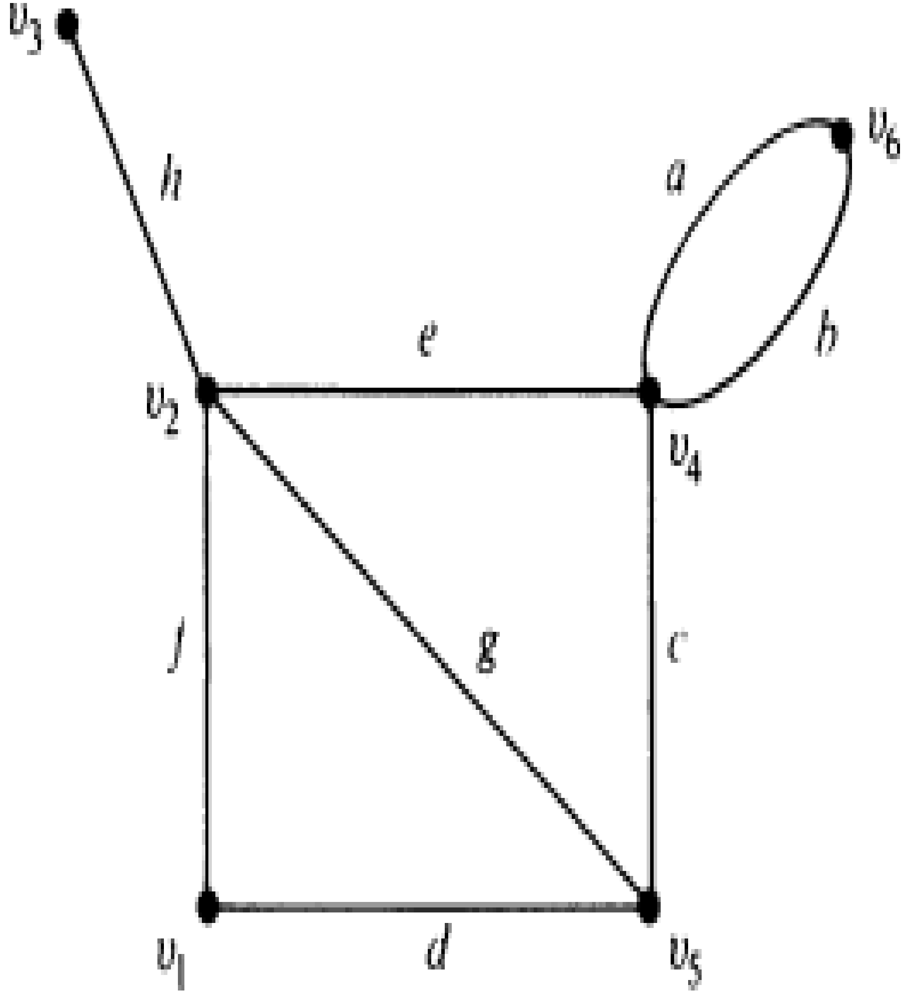
Incidence matrix

A_{ij} = 1, if j^{th} edge is incident on i^{th} vertex v_i

= 0 , otherwise

Such a matrix A is called ***vertex-edge incidence matrix or incidence matrix.***

- Denoted as A(G)
- Incidence matrix contains only 0 or 1. Such a matrix is called ***binary matrix or (0, 1) matrix***
- Rank of incidence matrix is **n-1**



$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Properties of incidence matrix

1. Since every edge is incident on exactly two vertices, each column of A has exactly two 1's.
2. The number of 1's in each row equals the degree of the corresponding vertex.
3. A row with all 0's, therefore, represents an isolated vertex.
4. Parallel edges in a graph produce identical columns in its incidence matrix,
5. If a graph G is disconnected and consists of two components g_1 and g_2, the incidence matrix $A(G)$ of graph G can be written in a block-diagonal form as

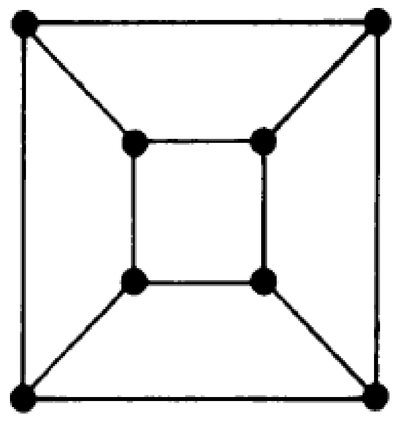
$$A(G) = \left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array}\right], \tag{7-1}$$

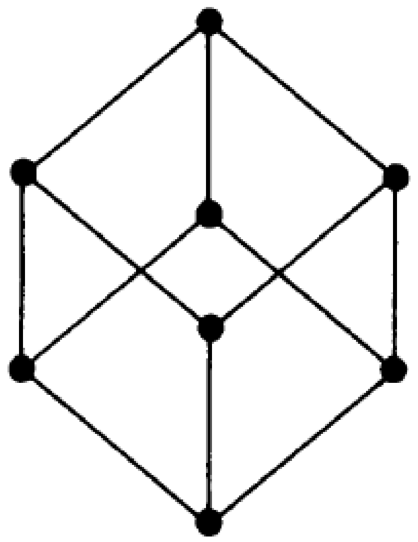
6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph.

Theorem

Two graphs G1 & G2 are isomorphic iff their matrices A(G1) & A(G2) differ only permutations of rows and columns

Proof:





Theorem

If A(G) is an incidence matrix of a connected graph G with n vertices, the rank of A(G) is n-1

Proof:

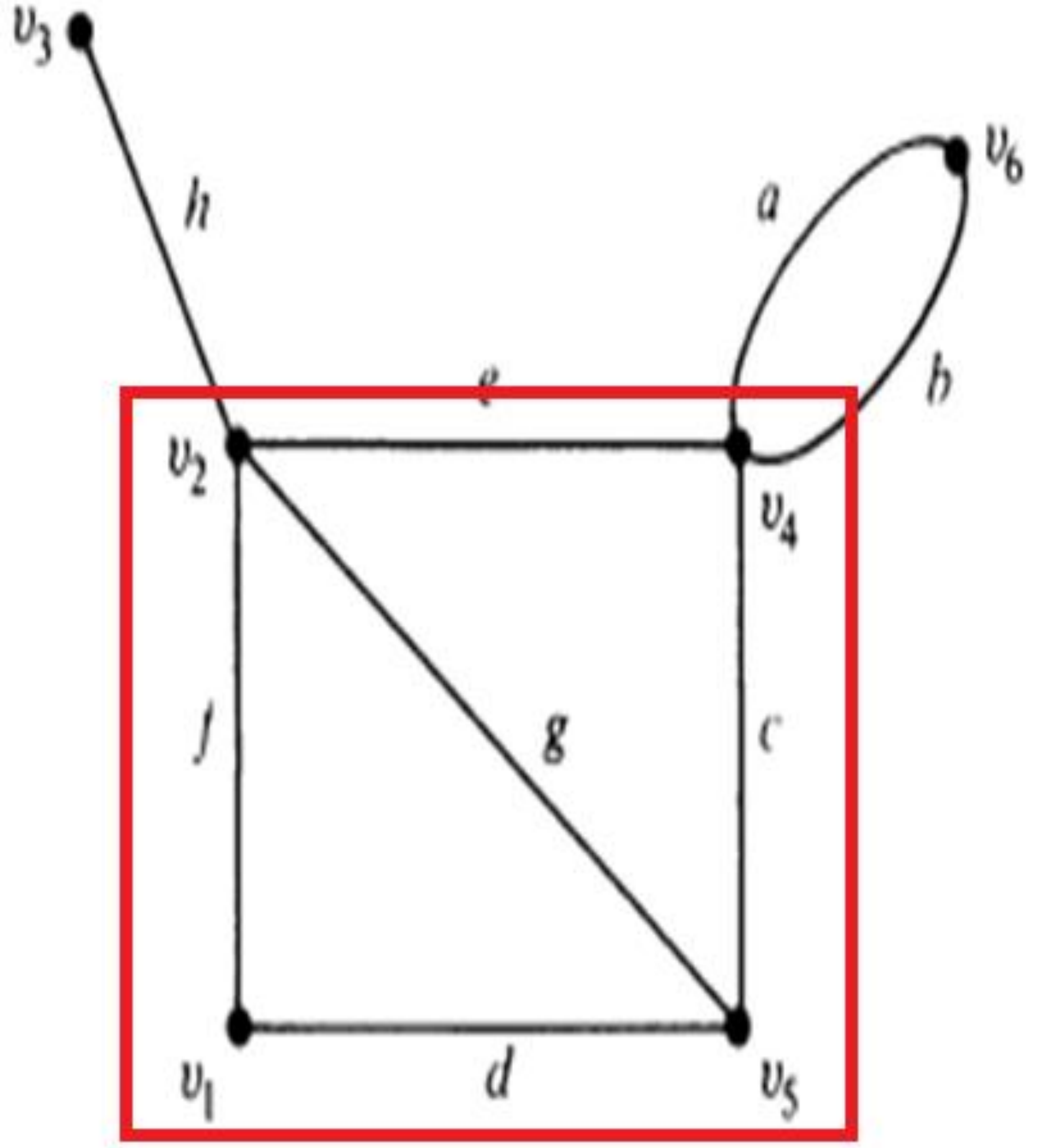
Use ex. to prove

- If we remove a row from incidence matrix A, the matrix will be **(n-1) by e** matrix.
- Such a submatrix of A_f of A → **reduced incidence matrix .**
- The vertex corresponding to the deleted row in A_f of A → r**eference vertex.**

- A tree in a connected graph with n vertices and n-1 edges, its reduced incidence matrix is a square matrix of order and rank n-1

Submatrices

- What will be submatrix of the Subgraph????



Circuit Matrix

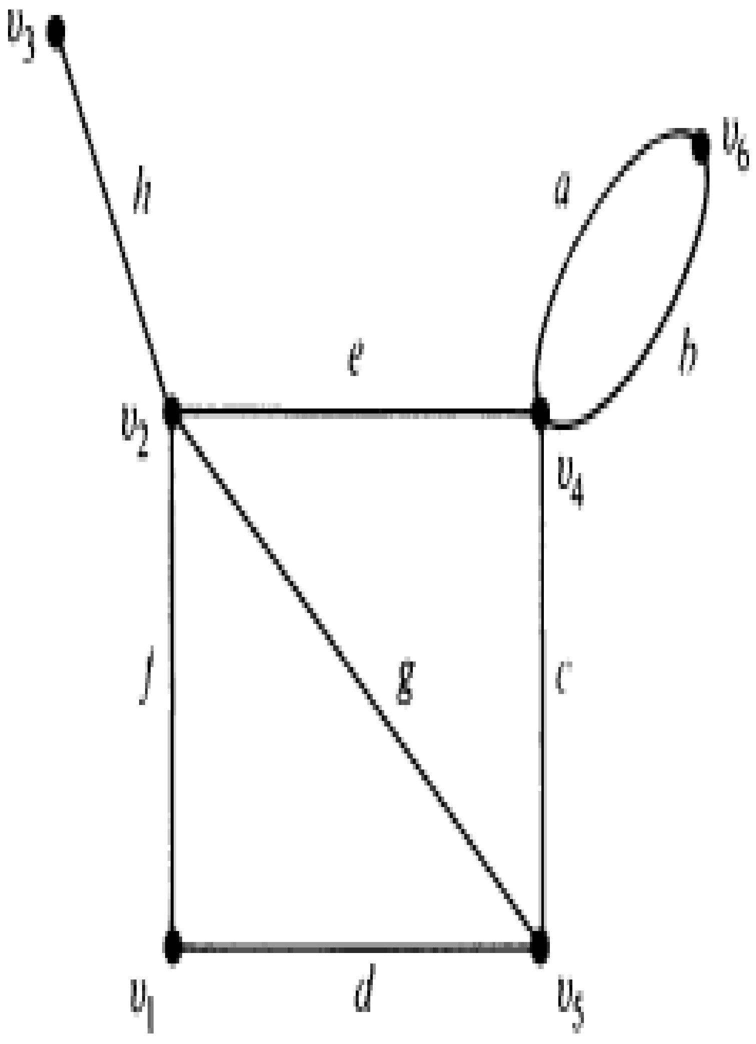
Let the number of different circuits in a graph G be q and the number of edges in G be e. Then a *circuit matrix* $\mathsf{B} = [b_{ij}]$ of G is a q by e, (0, 1)-matrix defined as follows:

$$
\begin{aligned}
b_{ij} &= 1, && \text{if } i\text{th circuit includes } j\text{th edge, and} \\
&= 0, && \text{otherwise.}
\end{aligned}
$$

- Denoted as B(G)

$\{a, b\}, \{c, e, g\}, \{d, f, g\}$, and $\{c, d, f, e\}$. Therefore, its circuit matrix is a 4 by 8, (0, 1)-matrix as shown:

$$B(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$



Properties of Circuit matrix

1. A column of all zeros corresponds to a noncircuit edge (i.e., an edge that does not belong to any circuit).
2. Each row of $B(G)$ is a circuit vector.
3. Unlike the incidence matrix, a circuit matrix is capable of representing a self-loop—the corresponding row will have a single 1.
4. The number of 1's in a row is equal to the number of edges in the corresponding circuit.
5. If graph G is separable (or disconnected) and consists of two blocks (or components) g_1 and g_2, the circuit matrix $B(G)$ can be written in a block-diagonal form as

$$B(G) = \left[\begin{array}{c|c} B(g_1) & 0 \\ \hline 0 & B(g_2) \end{array}\right],$$

6. Permutation of any two rows or columns in a circuit matrix simply corresponds to relabeling the circuits and edges.

Theorem

Let B and A be, respectively, the circuit matrix and the incidence matrix (of a self-loop-free graph) whose columns are arranged using the same order of edges. Then every row of B is orthogonal to every row A; that is,

$$A \cdot B^T = B \cdot A^T = 0 \qquad (\text{mod } 2),$$

where superscript T denotes the transposed matrix.

Proof: *Consider v and a circuit Ƭ in G. Either v will be in Ƭ or not in Ƭ .*

If v is not in Ƭ , then there is no edge in Ƭ that is incident on v.

If v in Ƭ , then there will be 2 edges that are incident on v.

Consider i^{th} row of A & j^{th} row of B. Non zero entries occur only in positions if the particular edge is incident on the i^{th} vertex & is also in j^{th} circuit. If i^{th} vertex not in j^{th} circuit, there is no such non zero entry and dot product of the two rows is zero. If i^{th} vertex is in j^{th} row , then there will be exactly two 1's in the sum of the products of individual entries . Since 1+1 = 0 (mod 2), the dot product of the two arbitrary rows – one from A and the other from B – is zero. Hence proved.

Ex:

$$A \cdot B^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \pmod 2.$$

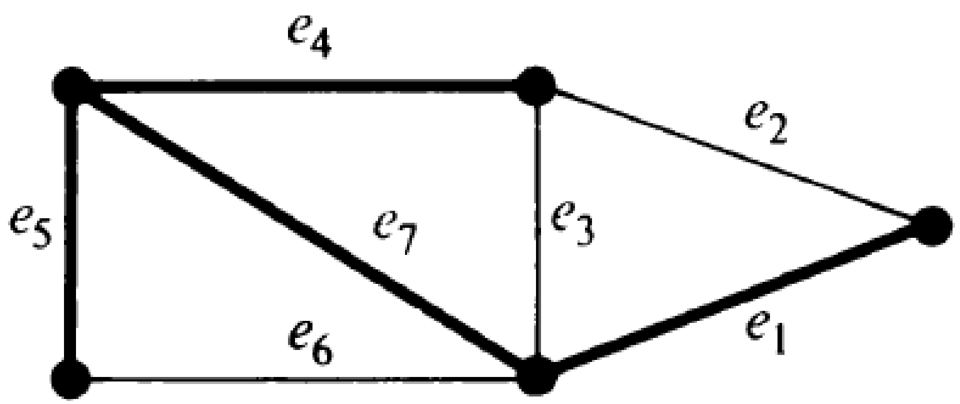
Fundamental Circuit Matrix

- A submatrix of a circuit matrix in which all rows correspond to a set of fundamental circuits is called a ***fundamental circuit matrix of B_f***

- If a connected graph has n vertices and e edges, then B_f is (e-n+1) by e matrix, because the number of fundamental circuits is e-n+1.
- Arrange columns of B_f such that all the e-n+1 chords correspond to the first e-n+1 columns.
- A matrix bf thus arranged can be written as :

Where $I_\mu \rightarrow$ identity matrix of order μ = e-n+1

$$B_f = [I_\mu \mid B_t],$$



$$
\begin{array}{c}
\begin{array}{ccc:cccc} e_2 & e_3 & e_6 & e_1 & e_4 & e_5 & e_7 \end{array} \\
\left[\begin{array}{ccc:cccc}
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1
\end{array}\right]
\end{array}
$$

Theorem

If B is a circuit matrix of a connected graph G with e edges and n vertices, rank of B = e-n+1

Proof: If A is incidence matrix of G, we have ***A. B^T =0 (mod 2)***

Therefore according to already proven theorem,

Rank of A + rank of B < = e

ie, rank of B< = e - rank of A ………………….eq 1

Since we know rank of A is n-1; substitute in eq 1

rank of B< = e – n+1

But rank of B cannot be less than e – n+1

So we **have rank of B = e-n+1.** Hence proved.

Cut Set Matrix

Matrix in which rows represent the cut sets and columns represent edges.

Denoted as C(G)

$$
\begin{aligned}
c_{ij} &= 1, \qquad \text{if } i\text{th cut-set contains } j\text{th edge, and} \\
&= 0, \qquad \text{otherwise.}
\end{aligned}
$$

Properties of cut set matrix

1. As in the case of the incidence matrix, a permutation of rows or columns in a cut-set matrix corresponds simply to a renaming of the cut-sets and edges, respectively.
2. Each row in $\mathsf{C}(G)$ is a cut-set vector.
3. A column with all 0's corresponds to an edge forming a self-loop.
4. Parallel edges produce identical columns in the cut-set matrix

5. In a nonseparable graph, every set of edges incident on a vertex is a cut-set Therefore, every row of incidence matrix $\mathsf{A}(G)$ is included as a row in the cut-set matrix $\mathsf{C}(G)$. In other words, for a nonseparable graph G, $\mathsf{C}(G)$ contains $\mathsf{A}(G)$. For a separable graph, the incidence matrix of each block is contained in the cut-set matrix.

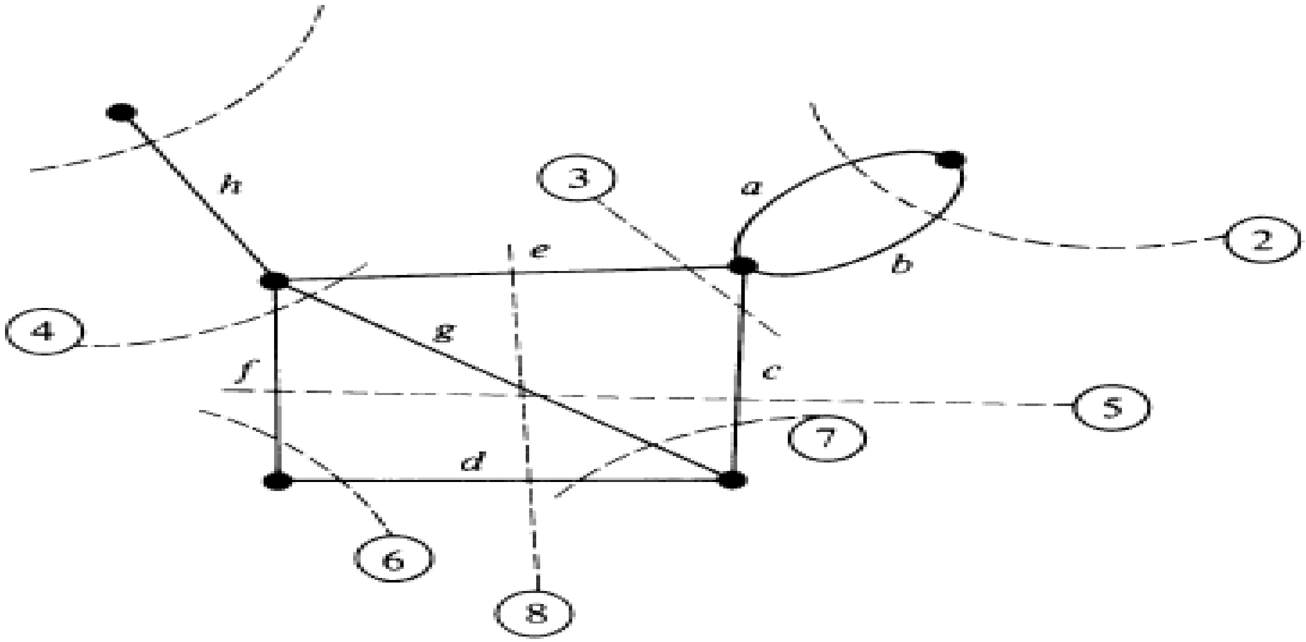
6. rank of $\mathsf{C}(G) \geq$ rank of $\mathsf{A}(G)$.

Hence, for a connected graph of n vertices,

$$\text{rank of } \mathsf{C}(G) \geq n - 1.$$

7. Since the number of edges common to a cut-set and a circuit is always even, every row in C is orthogonal to every row in B, provided the edges in both B and C are arranged in the same order.

$$\mathsf{B} \cdot \mathsf{C}^T = \mathsf{C} \cdot \mathsf{B}^T = 0 \qquad (\text{mod } 2).$$



$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Theorem

The rank of cut matrix C(G) is equal to the rank of the incidence matrix A(G), which equals to the rank of G.

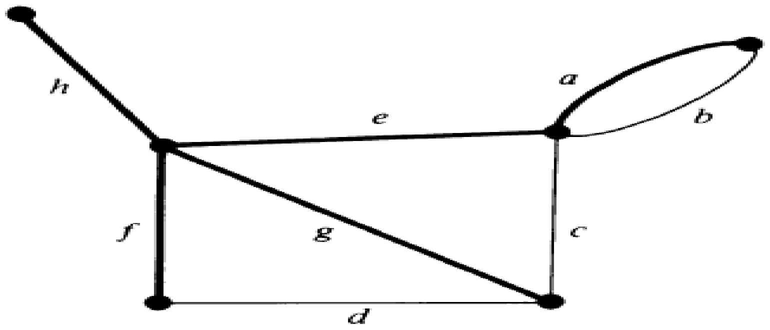
Proof:

Already proven rank of A(G) is n-1.

There fore, rank C(G) = rank A(G) = rank of G = n=1

Connected graph & fundamental cut set matrix w.r.t a spanning tree

A fundamental cut set matrix C_f is n-1 by e matrix submatrix of C. Fundamental cut set matrix is partitioned into two submatrices, one of which as identity matrix I_{n-1} of order n-1.



$$\begin{array}{c} \\ C_f = \end{array}\begin{array}{c} \begin{array}{cccccccc} b & c & d & a & e & f & g & h \end{array} \\ \left[\begin{array}{ccc|ccccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\mathbf{C}_f = [\mathbf{C}_c \,|\, \mathbf{I}_{n-1}],$$

where the last $n - 1$ columns forming the identity matrix correspond to the $n - 1$ branches of the spanning tree, and the first $e - n + 1$ columns forming $\mathbf{C}_c$ correspond to the chords.

Path Matrix

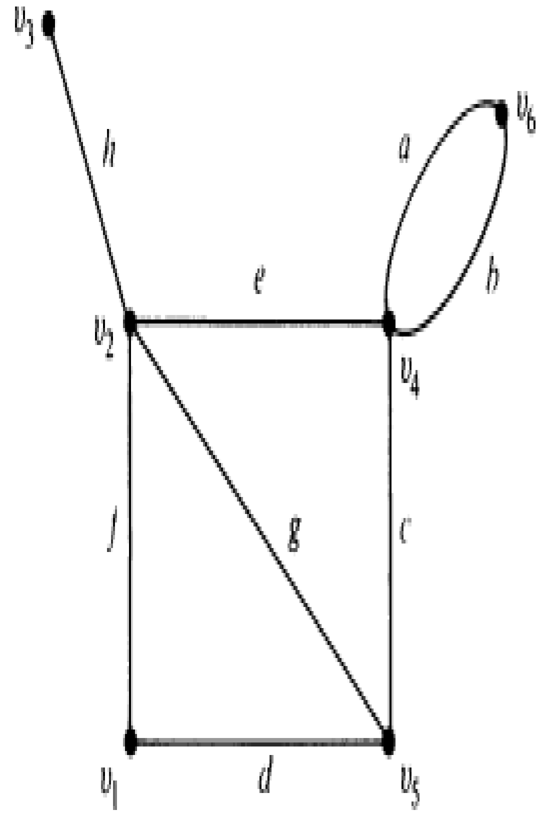
- Defined for a pair of vertices in a graph, (x, y)
- Denoted as P(x, y)
- Rows correspond to different paths between vertices x & y, and the column corresponds to edges in G.
- Path matrix P(x, y) = [p_{ij}]

where p_{ij} = 1 , if j^{th} edge lies in i^{th} path

= 0, otherwise

- Consider v_3 & v_4
- 3 different paths: {h, e}, {h, g, c}, {h, f ,d, c}

$$P(v_3, v_4) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$



Properties of path matrix

1. A column of all 0's corresponds to an edge that does not lie in any path between x and y.
2. A column of all 1's corresponds to an edge that lies in every path between x and y.
3. There is no row with all 0's.
4. The ring sum of any two rows in $P(x, y)$ corresponds to a circuit or an edge-disjoint union of circuits.

Theorem

If the edges of a connected graph are arranged in the same order for the columns of the incidence matrix A and the path matrix P(x, y), then the product (mod 2)

$$A \cdot P^T(x, y) = M,$$

where the matrix M has 1's in two rows x and y, and the rest of the $n - 2$ rows are all 0's.

- ***Proof:***

$$
\mathsf{A}\cdot\mathsf{P}^T(v_3, v_4) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}
$$

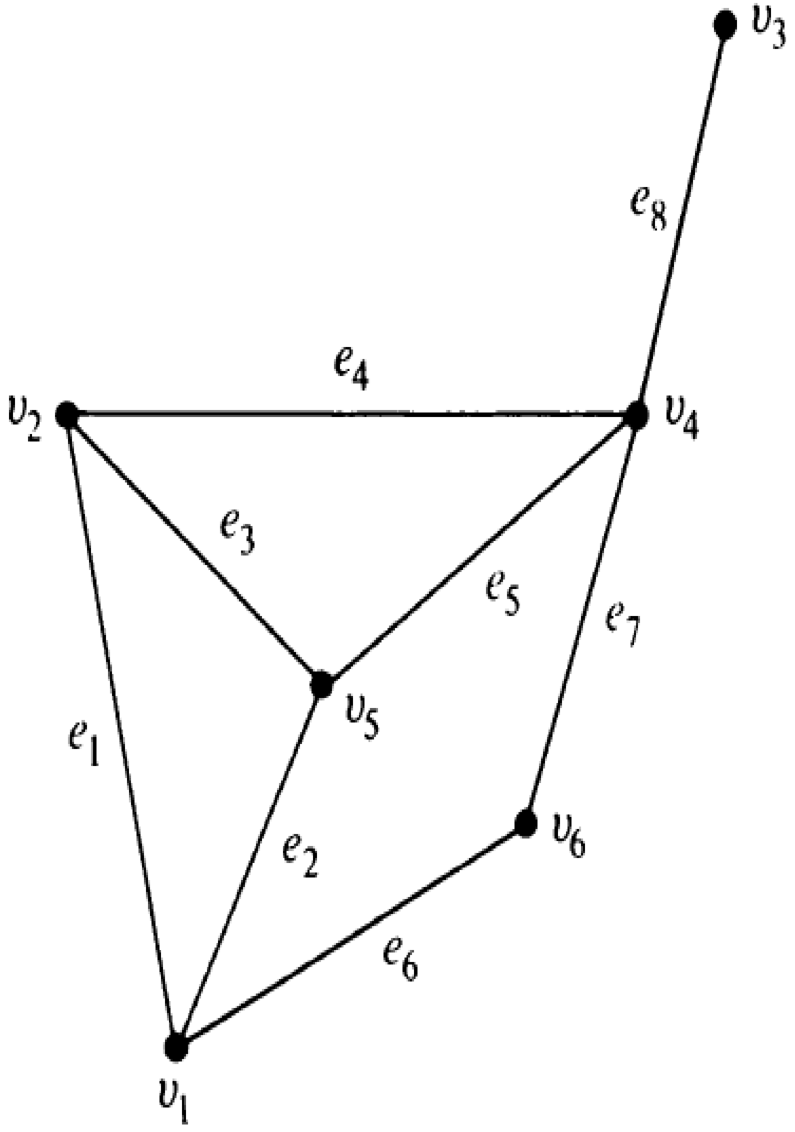
$$
= \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \pmod 2.
$$

Adjacency Matrix

- Adjacency Matrix of a graph G with n vertices and no parallel edges is n by n symmetric binary matrix $X = [x_{ij}]$, such that;

$$x_{ij} = 1 \text{ if there is an edge between } i^{th} \text{ and } j^{th} \text{ vertices}$$

$$= 0 \text{ otherwise}$$



$$
X = \begin{array}{c c}
 & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} &
\left[\begin{array}{cccccc}
0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
\end{array}
$$

Properties of Adjacency Matrix

1. The entries along the principal diagonal of **X** are all 0's if and only if the graph has no self-loops. A self-loop at the ith vertex corresponds to $x_{ii} = 1$.
2. The definition of adjacency matrix makes no provision for parallel edges. This is why the adjacency matrix **X** was defined for graphs without parallel edges.†

3. If the graph has no self-loops (and no parallel edges, of course), the degree of a vertex equals the number of 1's in the corresponding row or column of X.

4. Permutations of rows and of the corresponding columns imply reordering the vertices. It must be noted, however, that the rows and columns must be arranged in the same order. Thus, if two rows are interchanged in X, the corresponding columns must also be interchanged. Hence two graphs G_1 and G_2 with no parallel edges are isomorphic if and only if their adjacency matrices $X(G_1)$ and $X(G_2)$ are related:

$$X(G_2) = R^{-1} \cdot X(G_1) \cdot R,$$

where R is a permutation matrix.

5. A graph G is disconnected and is in two components g_1 and g_2 if and only if its adjacency matrix $X(G)$ can be partitioned as

$$X(G) = \left[\begin{array}{c|c} X(g_1) & 0 \\ \hline 0 & X(g_2) \end{array}\right],$$

where $X(g_1)$ is the adjacency matrix of the component g_1 and $X(g_2)$ is that of the component g_2.

This partitioning clearly implies that there exists no edge joining any vertex in subgraph g_1 to any vertex in subgraph g_2.

6. Given any square, symmetric, binary matrix Q of order n, one can always construct a graph G of n vertices (and no parallel edges) such that Q is the adjacency matrix of G.

Powers of X

$$
X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\end{array}
$$

$$
X^2 = \begin{bmatrix}
3 & 1 & 0 & 3 & 1 & 0 \\
1 & 3 & 1 & 1 & 2 & 2 \\
0 & 1 & 1 & 0 & 1 & 1 \\
3 & 1 & 0 & 4 & 1 & 0 \\
1 & 2 & 1 & 1 & 3 & 2 \\
0 & 2 & 1 & 0 & 2 & 2
\end{bmatrix}.
$$

The value of an off-diagonal entry in X^2, that is, ijth entry $(i \neq j)$ in X^2,

= number of 1's in the dot product of ith row and jth column (or jth row) of X.

= number of positions in which both ith and jth rows of X have 1's.

= number of vertices that are adjacent to both ith and jth vertices.

= number of different paths of length two between ith and jth vertices.

Similarly, the ith diagonal entry in X^2 is the number of 1's in the ith row (or column) of matrix X. Thus the value of each diagonal entry in X^2 equals the degree of the corresponding vertex, if the graph has no self-loops.

Since a matrix commutes with matrices that are its own power,

$$X \cdot X^2 = X^2 \cdot X = X^3.$$

$$X^3 = \begin{bmatrix} 2 & 7 & 3 & 2 & 7 & 6 \\ 7 & 4 & 1 & 8 & 5 & 2 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 2 & 8 & 4 & 2 & 8 & 7 \\ 7 & 5 & 1 & 8 & 4 & 2 \\ 6 & 2 & 0 & 7 & 2 & 0 \end{bmatrix}.$$

consider the ijth entry of X^3.

ijth entry of X^3 = dot product of ith row X^2 and jth column (or row) of X.

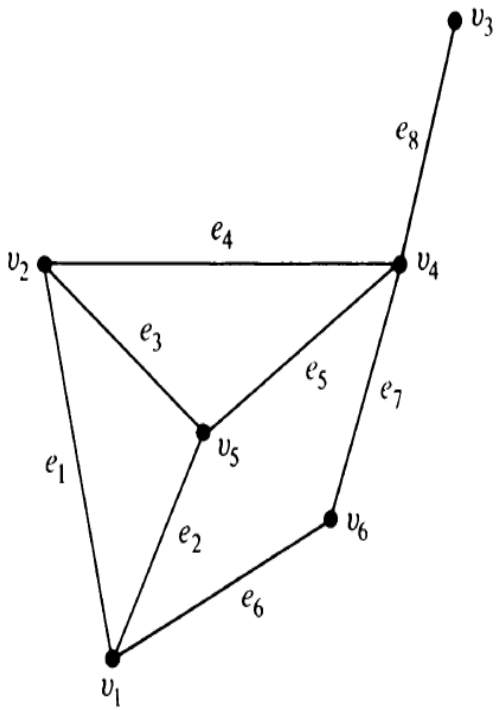
$$= \sum_{k=1}^{n} ik\text{th entry of } X^2 \cdot kj\text{th entry of } X.$$

$$= \sum_{k=1}^{n} \text{number of all different edge sequences}^{\dagger} \text{ of three edges from } i\text{th to } j\text{th vertex via } k\text{th vertex.}$$

= number of different edge sequences of three edges between ith and jth vertices.

- Consider 1, 5 entry of the given graph. It is given by the dot product

$$\text{row 1 of } X^2 \cdot \text{row 5 of } X = (3, 1, 0, 3, 1, 0) \cdot (1, 1, 0, 1, 0, 0)$$
$$= 3 + 1 + 0 + 3 + 0 + 0 = 7.$$



THEOREM

Let X be the adjacency matrix of a simple graph G. Then the ijth entry in X^r is the number of different edge sequences of r edges between vertices v_i and v_j.

Proof: The theorem holds for $r = 1$, and it has been proved for $r = 2$ and 3 also. It can be proved for any positive integer r, by induction.

In other words, assume that it holds for $r - 1$, and then evaluate the ijth entry in X, with the help of the relation

$$X^r = X^{r-1} \cdot X,$$

as was done for X^3.

Relationship between A_f , B_f & C_f

1. Given A or A_f, we can readily construct B_f and C_f, starting from an arbitrary spanning tree and its subgraph A_t in A_f.
2. Given either B_f or C_f, we can construct the other. Thus since B_f determines a graph within 2-isomorphism, so does C_f.
3. Given either B_f or C_f, A_f in general cannot be determined completely.