Module 2

Euler Graph

- Euler Line: is a closed walk running through every edge of G exactly once

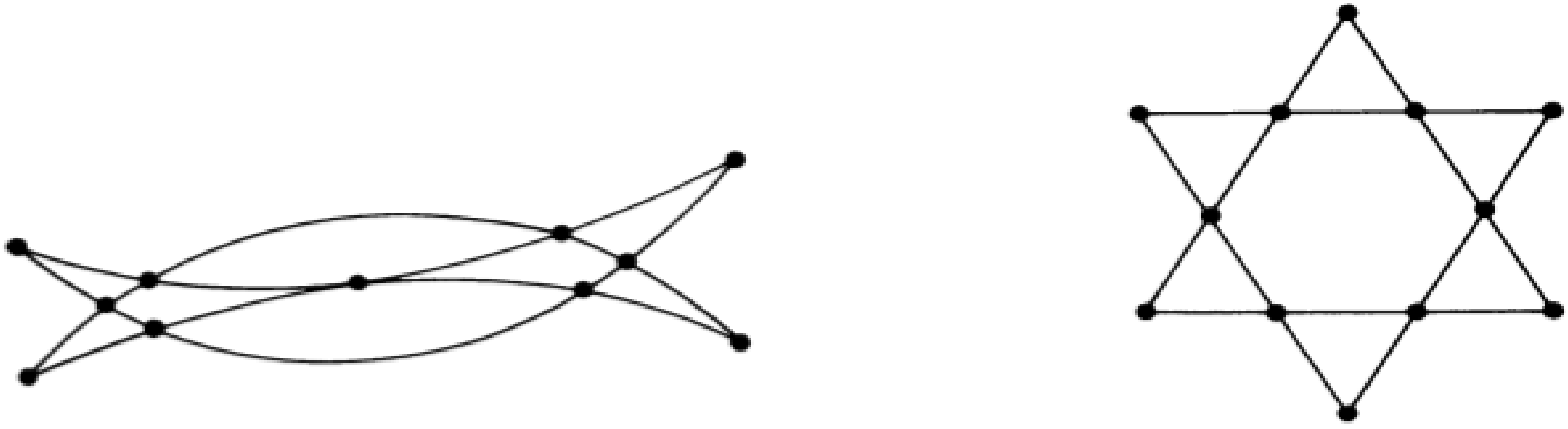
- Euler Graph: A graph which consists of Euler line

Theorem-1

- *A given connected graph G is an Euler graph iff all vertices of G are of even degree*

- Konigsberg Bridge Problem:

  - All vertices are not in even degree
  - So not an Euler graph
  - Hence its not possible to walk over each of 7 bridges exactly once and return to starting point.



(a) (b)

Fig. 2-12 Two Euler graphs.

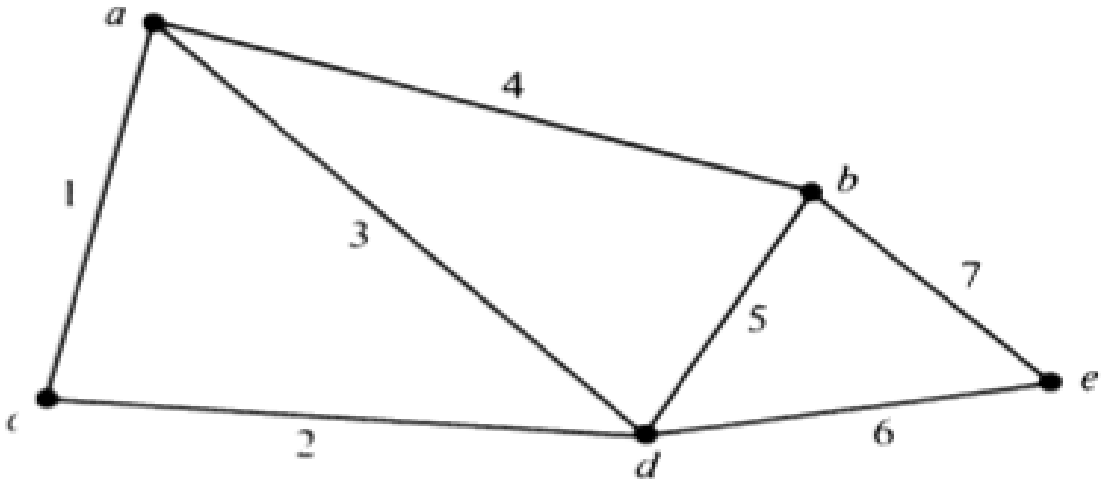


Fig. 2-13 Unicursal graph.

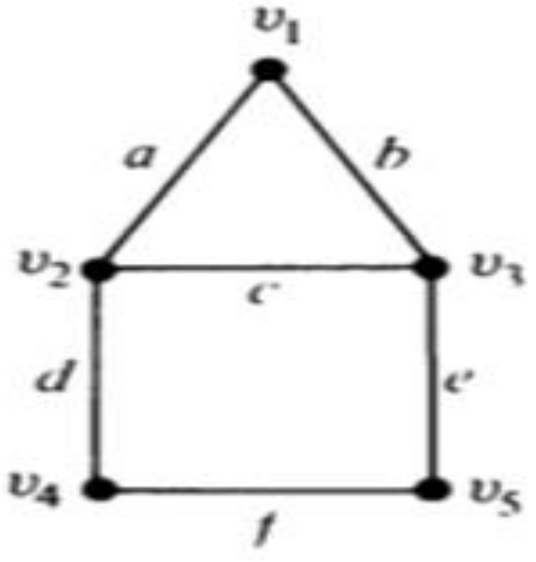
Unicursal Graph

- Fig 2-13: It includes all edges of the graph, and does not retrace any edge, is not closed.
- Initial vertex is a and final vertex is b.

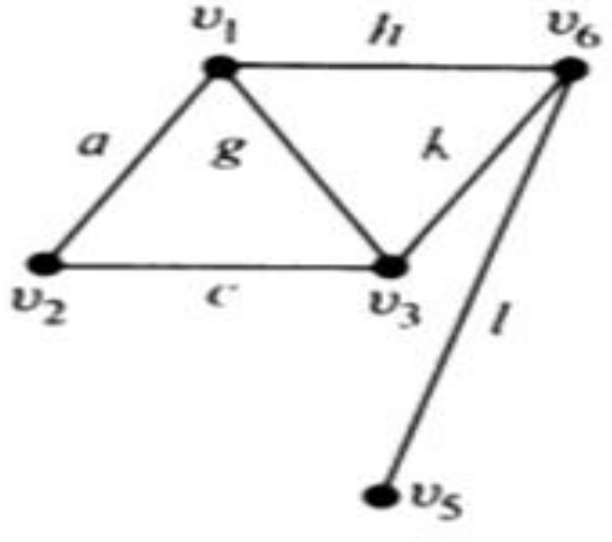
- Unicursal Line *is an open walk that includes all edges of a graph without retracing any edge*
- *Also called open Euler line*
- *A graph with unicursal line is called unicursal graph*

Operations in Graphs

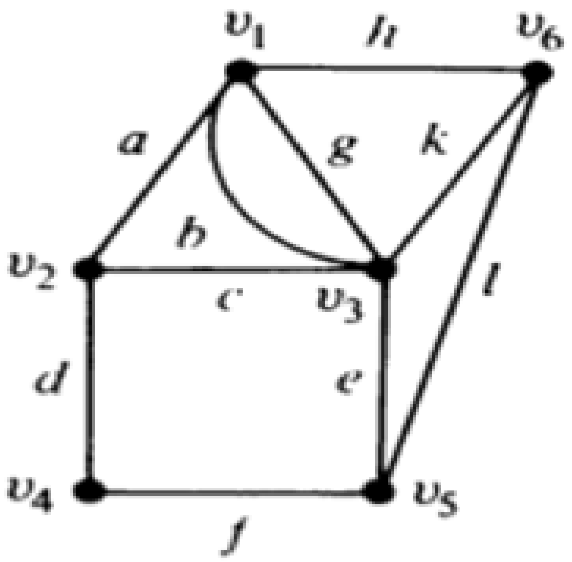
- Union
  - $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$
  - $G_3=G_1 \cup G_2 \rightarrow V_3=V_1 \cup V_2$ and $E_3= E_1 \cup E_2$
- Intersection
  - $G_3 = G_1 \cap G_2$
- Ring sum $G_1 \oplus G_2$
  - It is the graph G_3 , consisting of vertex set $V_1 \cup V_2$ and edges that either in G_1 or G_2, but not in both.



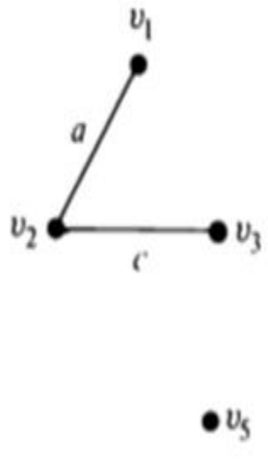
G_1



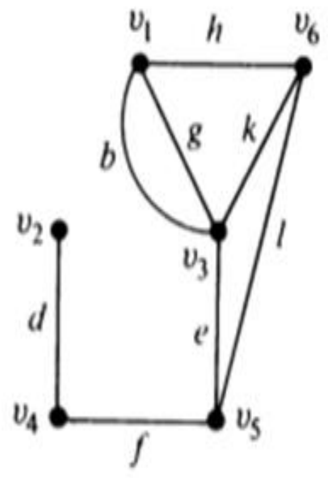
G_2



$G_1 \cup G_2$



$G_1 \cap G_2$



$G_1 \oplus G_2$

- Commutative operations
    - $G_2 \cup G_1 = G_1 \cup G_2$
    - $G_2 \cap G_1 = G_1 \cap G_2$
    - $G_2 \oplus G_1 = G_1 \oplus G_2$

- If G_1 and G_2 are edge disjoint, then $G_1 \cap G_2$ is a null graph and

$G_1 \oplus G_2 = G_1 \cup G_2$

- If G_1 and G_2 are vertex disjoint, then $G_1 \cap G_2$ is empty.

- For any graph G,
    - $G \cup G = G \cap G = G$
    - $G \oplus G$ = a null graph

- If g is a subgraph of G, then G $\oplus$ g, subgraph of G which remains after all edges in g have been performed from G.

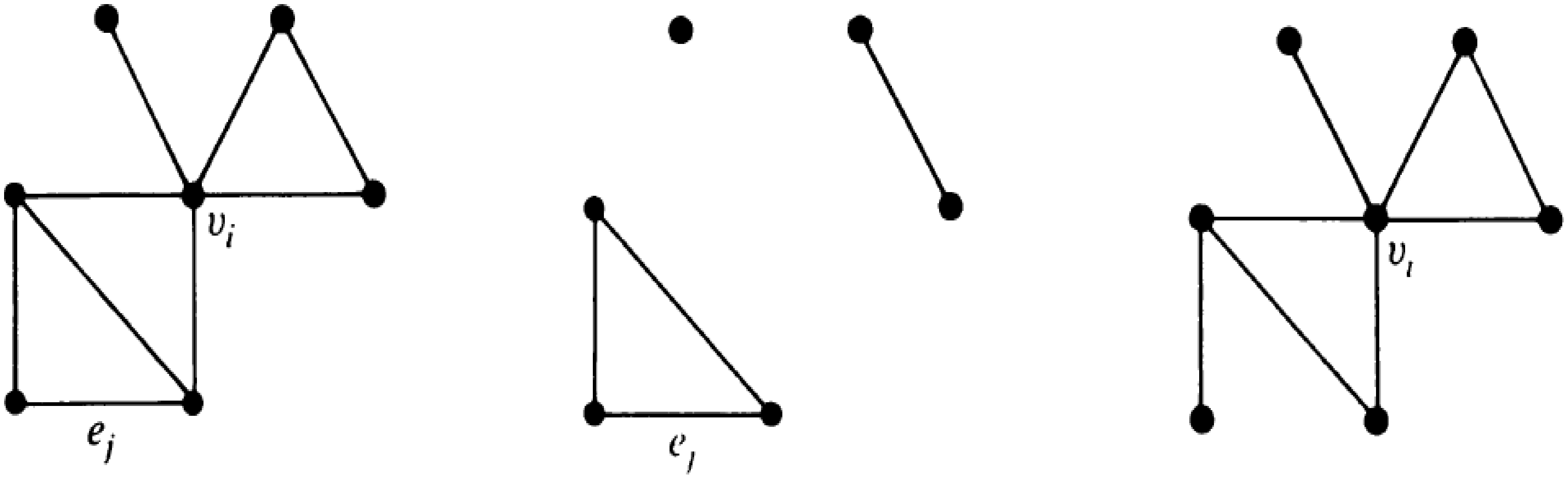
- G $\oplus$ g = G – g is called complement of g in G

- *Decomposition* : A graph G is said to have been decomposed into two subgraphs g_1 and g_2 if

  - $g_1 \cup g_2 = G$
  - $g_1 \cap g_2$ = null graph

- A graph can be decomposed into more than two subgraphs – subgraphs that are edge disjoint and collectively include every edge in G.

- ***Deletion:*** If v_i is a vertex in graph G, then G- v_i denotes a subgraph of G obtained by deleting v_i from G.
- Deletion of a vertex always implies the deletion of all edges incident on that vertex.
- If e_j is an edge in G, the G - e_j is a subgraph of G obtained by deleting e_j from G.
- Deletion of an edge does not imply deletion of its end vertices.
- $G - e_j = G \oplus e_j$

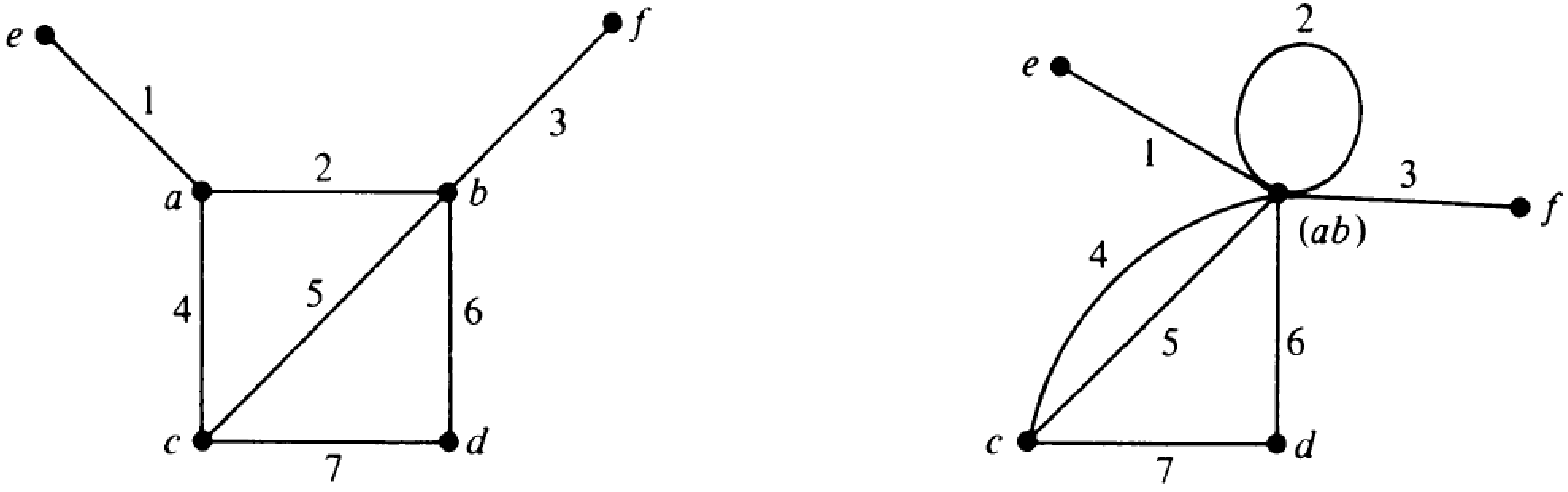


G $\qquad$ $(G - v_i)$ $\qquad$ $(G - e_j)$

- *Fusion:* *A pair of vertices a, b in a graph are said to be fused(merged or identified), if the two vertices are replaced by a single new vertex such that every edge that was incident on either a or b or both is incident on the new vertex.*

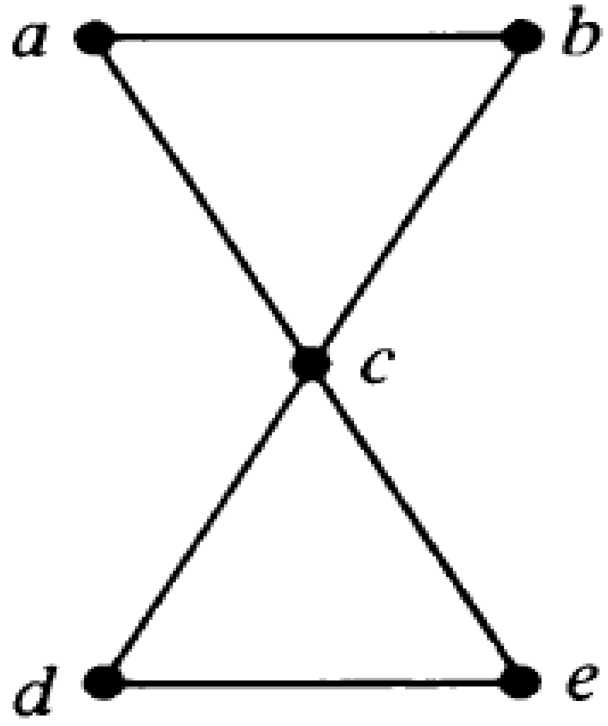
- Fusing do not alter number of edges

- Fusing reduces number of vertices

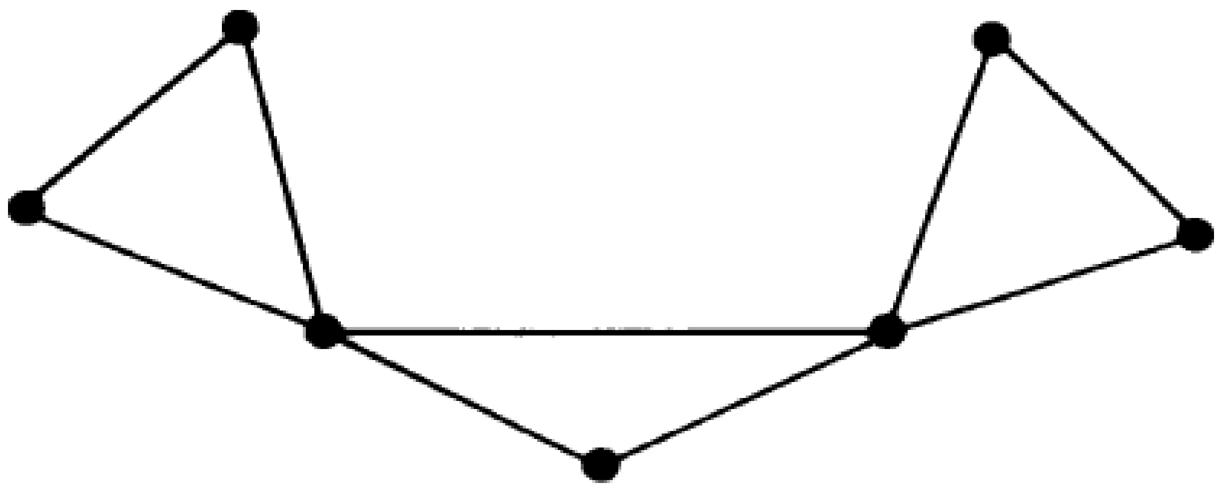


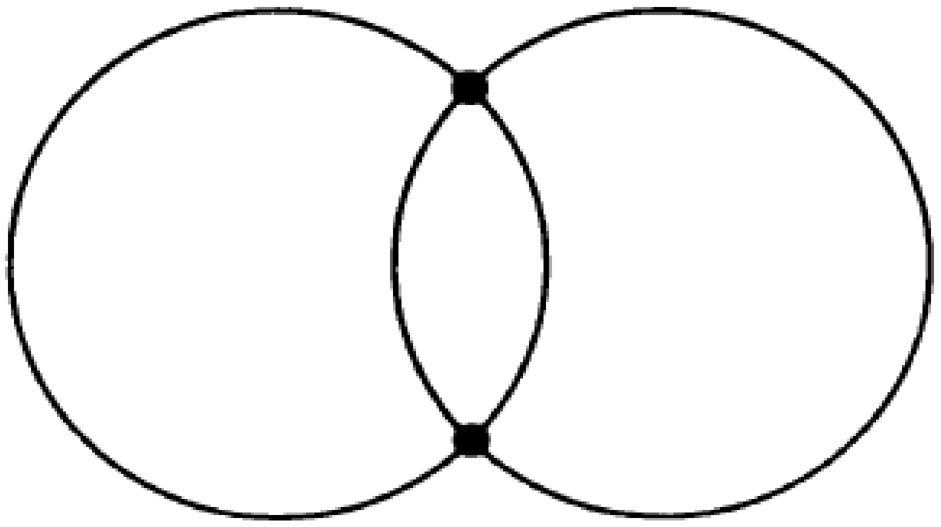
Fusion of vertices a and b.

Arbitrarily traceable graph



Arbitrarily traceable graph from c.





Theorem

A connected graph G is an Euler graph iff it can be decomposed into circuits

Proof: G decomposed into circuits.ie, G is the union of edge disjoint circuits.

Since degree of each vertex is 2 in circuit, degree of G will be even.

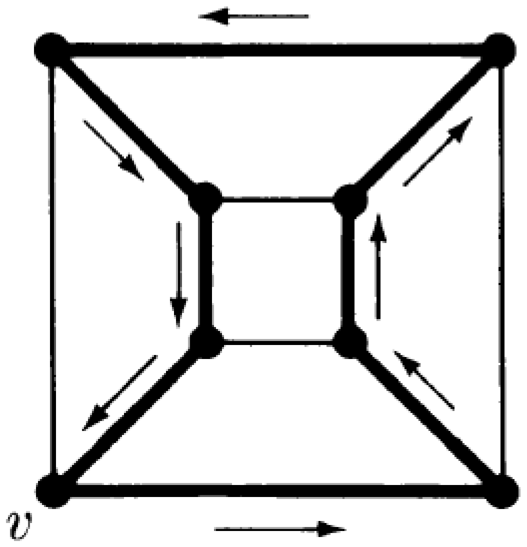
Hence G is an Euler graph

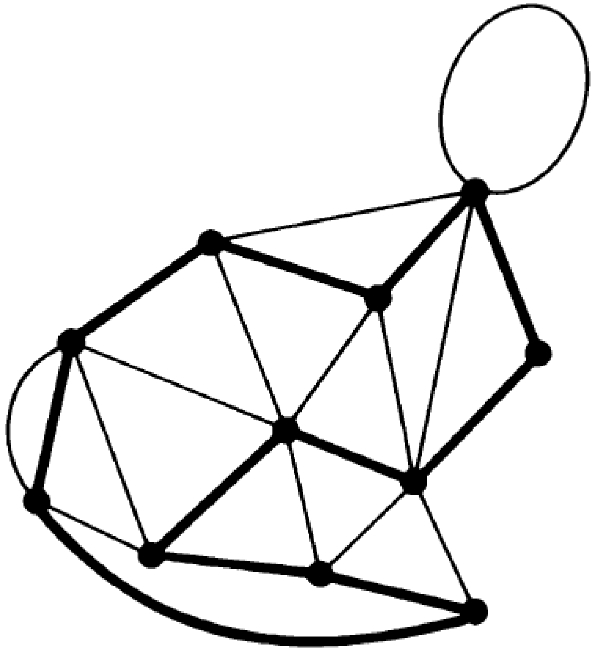
Theorem

An Euler graph G is an arbitrarily traceable from a vertex v in G iff every circuit in G contains v.

Hamiltonian Circuits

- *Hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly once, except of the course of starting vertex at which that walk also terminates.*





- *A circuit in a connected graph G is said to be Hamiltonian if it includes every vertex of G.*

- *Hence a Hamiltonian circuit in a graph with n vertices contains exactly n edges.*

Hamiltonian path

- If we remove any one edge from a Hamiltonian circuit, we get a path. This path is called *Hamiltonian path* .

- *Hamiltonian path in a graph G traverses every vertex of G.*

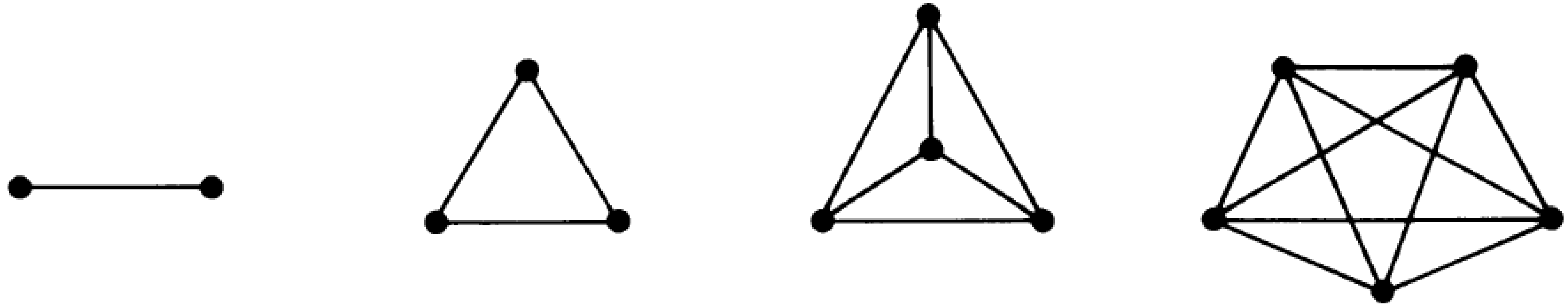
- Since Hamiltonian path is a subgraph of Hamiltonian circuit .

- Every graph that has a Hamiltonian circuit also has a Hamiltonian path.

- *The length of a Hamiltonian path in a connected graph of n vertices is n-1*

Complete graph

- A simple graph in which there exists an edge between every pair of vertices is called a complete graph.
- Also called universal graph or clique



Complete graphs of two, three, four, and five vertices.

- Since every vertex is joined with every other vertex through one edge, degree of every vertex is n-1 in a complete graph G of n vertices.

- Total number of edges in G is n(n-1)/2

Theorem

In complete graph with n vertices, there are (n-1)/2 edge disjoint Hamiltonian circuits, if n is an odd number >= 3

Proof :

- No. of edges in a complete graph G of n vertices = n(n-1)/2

- No. of edges in Hamiltonian circuit of G consists = n

- Therefore, the number of edge-disjoint Hamiltonian circuit in G cannot exceed (n-1)/2.

- ie, there are (n-1)/2 edge disjoint Hamiltonian circuits, when n is odd.

Seating arrangement in a round table problem can be solved using this theorem

Dirac’s theorem for Hamiltonicity

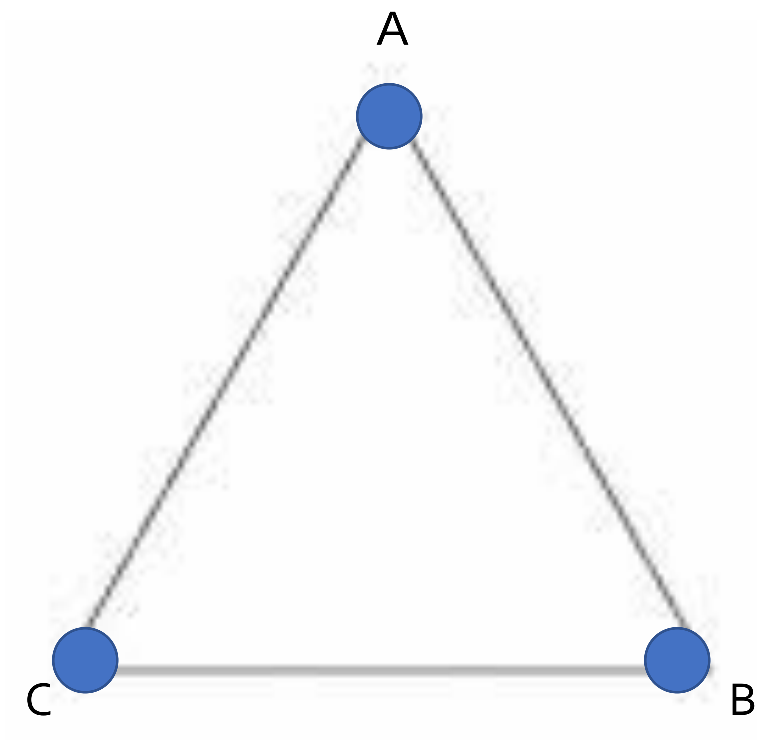
A sufficient condition for a simple graph G to have Hamiltonian circuit is that the degree of every vertex in G be at least n/2 , where n is the number of vertices in G

Traveling - Salesman Problem

- A salesman is required to visit a number of cities during a trip. Given the distances between the cities, in what order should he travel so as to visit every city precisely once and return home, with the minimum mileage travelled ?

- Cities as vertices
- Roads as edges
- Each edge e_i associated with a real number(miles), $w(e_i)$. Such a graph is called a *weighted graph.*
- If each vertex has a edge to every other vertex with weights on each edge, its is called *complete weighted graph.*

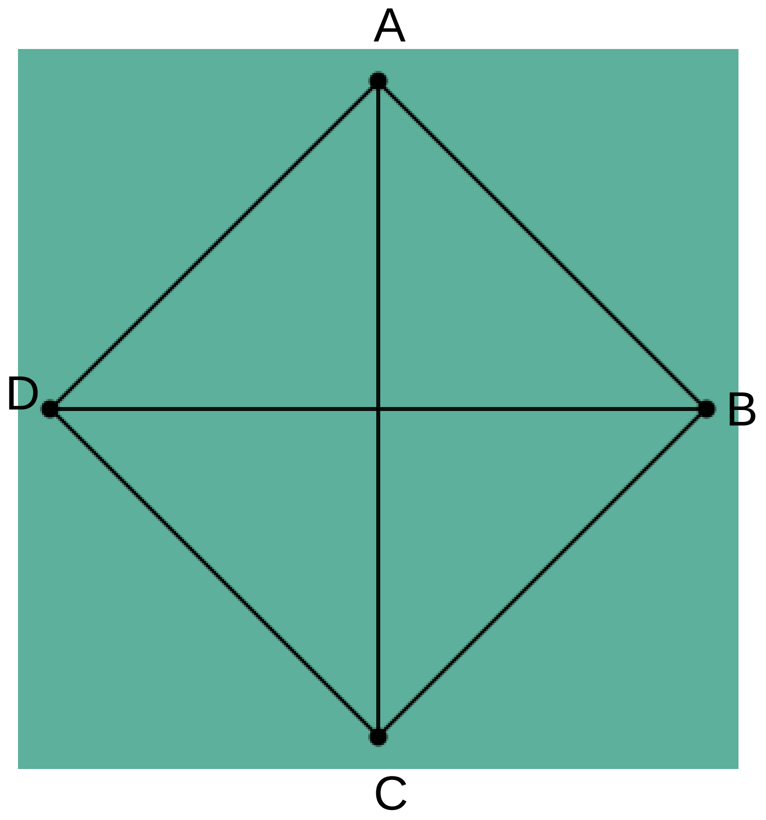
Complete Graph on 3 Vertices



- Starting at vertex A, I have 2 choices: B,C
- Choose B; now I have 1 choice: C
- Return to A and multiply together :2 x 1 = 2
- I have N = 3 vertices; the number of Hamilton circuits for a graph with 3 vertices is:

  $(3 - 1)! = 2! = 2 \times 1 = 2$

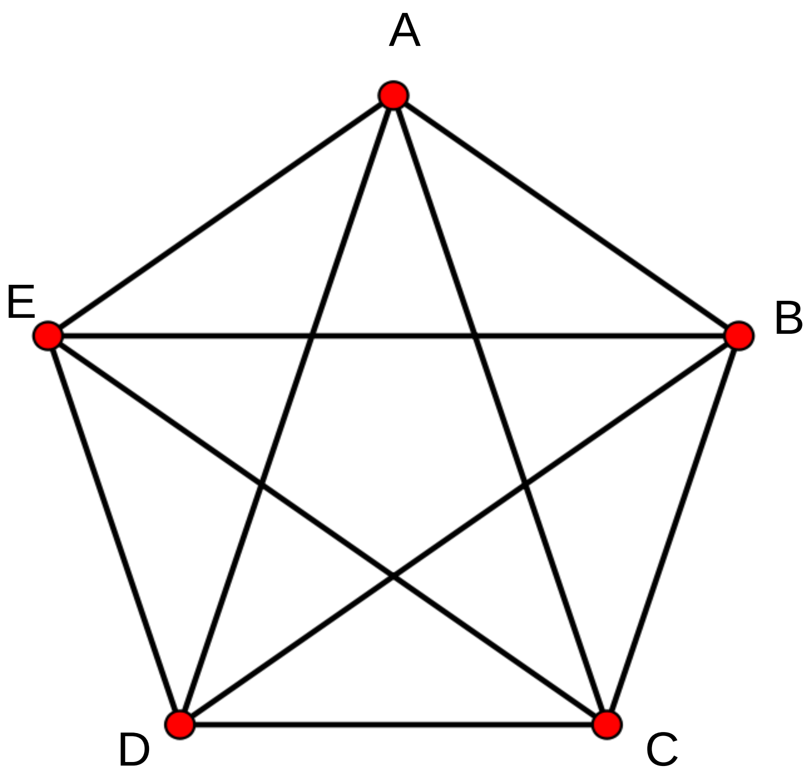
Complete Graph on 4 Vertices



- Starting at vertex A, I have 3 choices: B,C,D
- Choose C; now I have 2 choices: B,D
- Choose B; now I have 1 choice: D
- Return to A and multiply together: 3 x 2 x 1=6

▪ I have N= 4 vertices; the number of Hamilton circuits for a graph with 4 vertices is (4-1)! = 3! = 3 x 2 x 1 = 6

Complete Graph on 5 Vertices

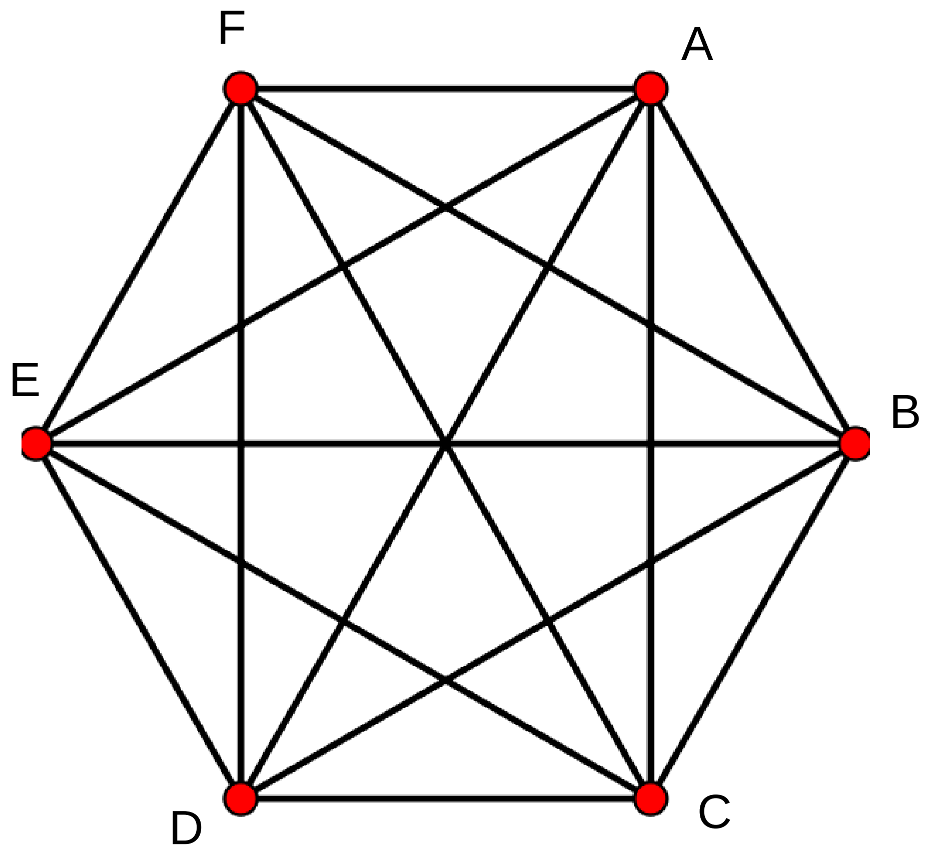


- Starting at vertex A, I have 4 choices : B,C,D,E
- Choose B; now I have 3 choices: C,D,E
- Choose C; now I have 2 choices: D,E
- Choose D; now I have 1 choice: E
- Return to A and multiply together:

  4 x 3 x 2 x 1 = 24
- I have N = 5 vertices; the number of Hamilton circuits for a graph with 5 vertices is:

  (5-1)! = 4! = 4 x 3 x 2 x 1 = 24

Complete Graphs on 6 Vertices

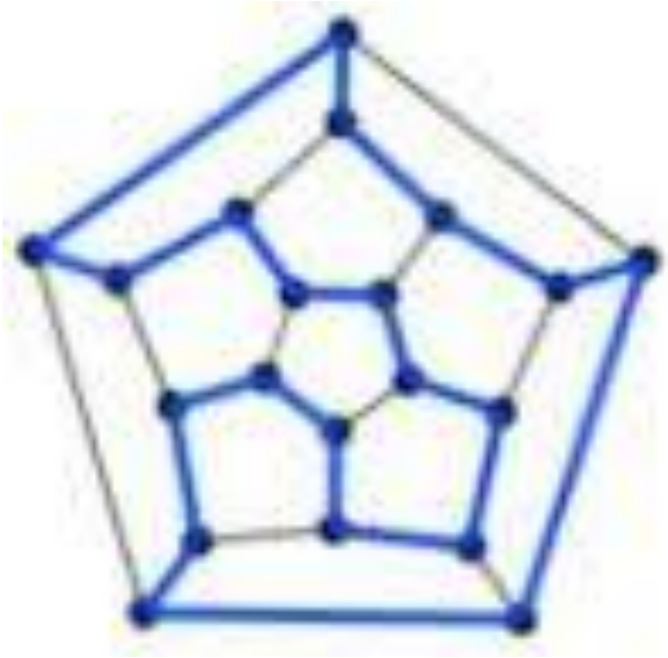


- Starting at vertex A, I have 5 choices: B,C,D,E,F
- Choose C; now I have 4 choices: B,D,E,F
- Choose F; now I have 3 choices: B,D,E
- Choose D; now I have 2 choices: B,E
- Choose B; now I have 1 choice: E
- Return to A and multiply together:

  5 x 4x 3 x 2 x 1 = 120
- I have N = 6 vertices; the number of Hamilton circuits for a graph with 6 vertices is:

  (6-1)! = 5! = 5x 4 x 3 x 2 x 1 = 120

Finding the Total Number of Hamilton Graphs in General



- In general, in a complete graph with N vertices, starting at vertex 1 you have (N – 1) choices of moving to vertex 2; from vertex 2 you have (N – 2) choices of moving to vertex 3; from vertex 3 you have (N – 3) choices of moving to vertex 4; continuing in this manner, returning to vertex 1 and multiplying these choices together we have:

  (N – 1) x (N -2) x (N – 3) x … x 3 x 2 x 1 = (N – 1)!

- (N – 1)! gives the total number of Hamilton circuits in a complete graph.

- *Thus total number of Hamiltonian circuits possible in a complete graph = (N-1)!/2*

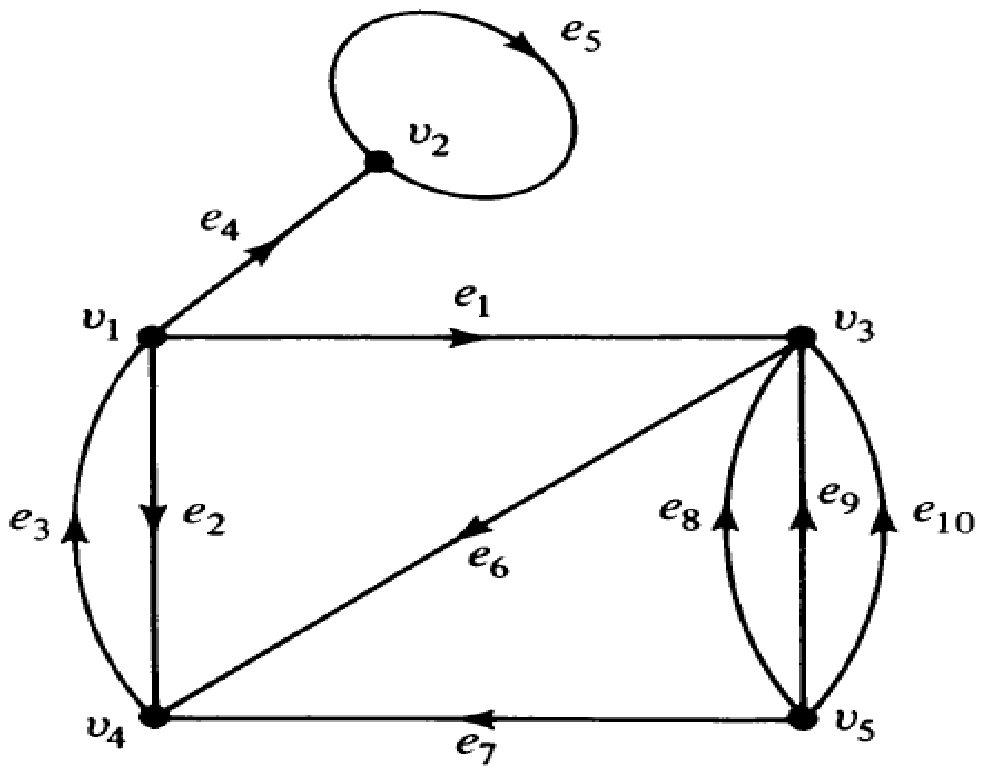
- Dividing by 2 because half of those are the reverse of another.

- TSP can be solved by enumerating all (N-1)!/2 Hamiltonian circuits, calculate distance travelled by each and pick shortest path

Directed Graph

A *directed graph* (or a *digraph* for short) G consists of a set of vertices $V = \{v_1, v_2, \ldots\}$, a set of edges $E = \{e_1, e_2, \ldots\}$, and a mapping Ψ that maps every edge onto some *ordered* pair of vertices (v_i, v_j).

- Digraph is also called *oriented graph*



- The vertex v_i which edge e_k is incident out of, is called the initial vertex of e_k

- The vertex v_i which edge e_k is incident into , is called the terminal vertex of e_k

- initial vertex→ v_5
- terminal vertex→v_3

- An edge for which the initial and terminal vertices are of same are called self loop.

- The number of edges incident out of a vertex v_i is called out degree(or out- valance or outward demidegree) of v_i
- Written as $\mathbf{d^+(v_i)}$

- The number of edges incident into of a vertex v_i is called in degree(or in- valance or inward demidegree) of v_i
- Written as $\mathbf{d^-(v_i)}$

- In any digraph G the sum of all in degrees is equal to the sum of all out degrees, each sum being equal to the number of edges

$$\sum_{i=1}^{n} d^{+}(v_i) = \sum_{i=1}^{n} d^{-}(v_i).$$

- The **isolated vertex** is a vertex in which the in degree and the out degree are both equal to zero.

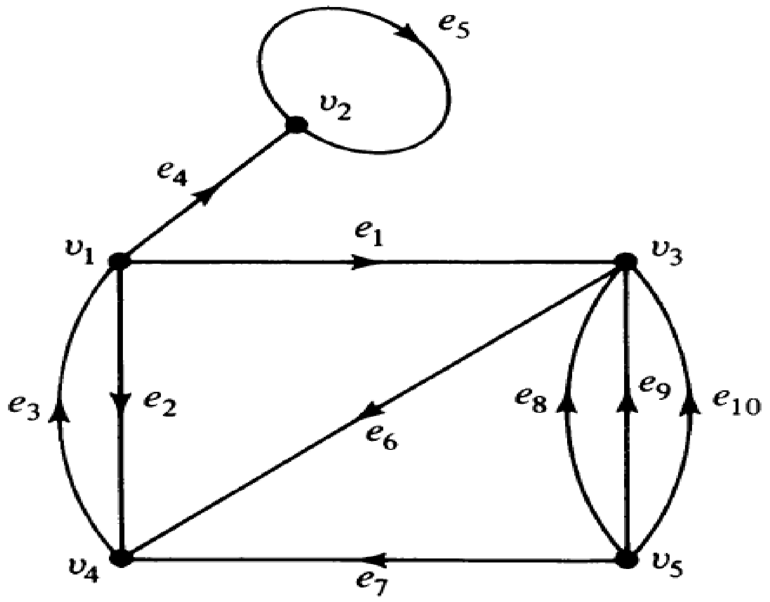
- A vertex v in a digraph is called **pendant** if it is of degree one.

$$d^+(v) + d^-(v) = 1.$$

Two directed edges are said to be *parallel* if they are mapped onto the same ordered pair of vertices. That is, in addition to being parallel in the sense of undirected edges, parallel directed edges must also agree in the direction of their arrows.

edges e_8, e_9, and e_{10} are parallel

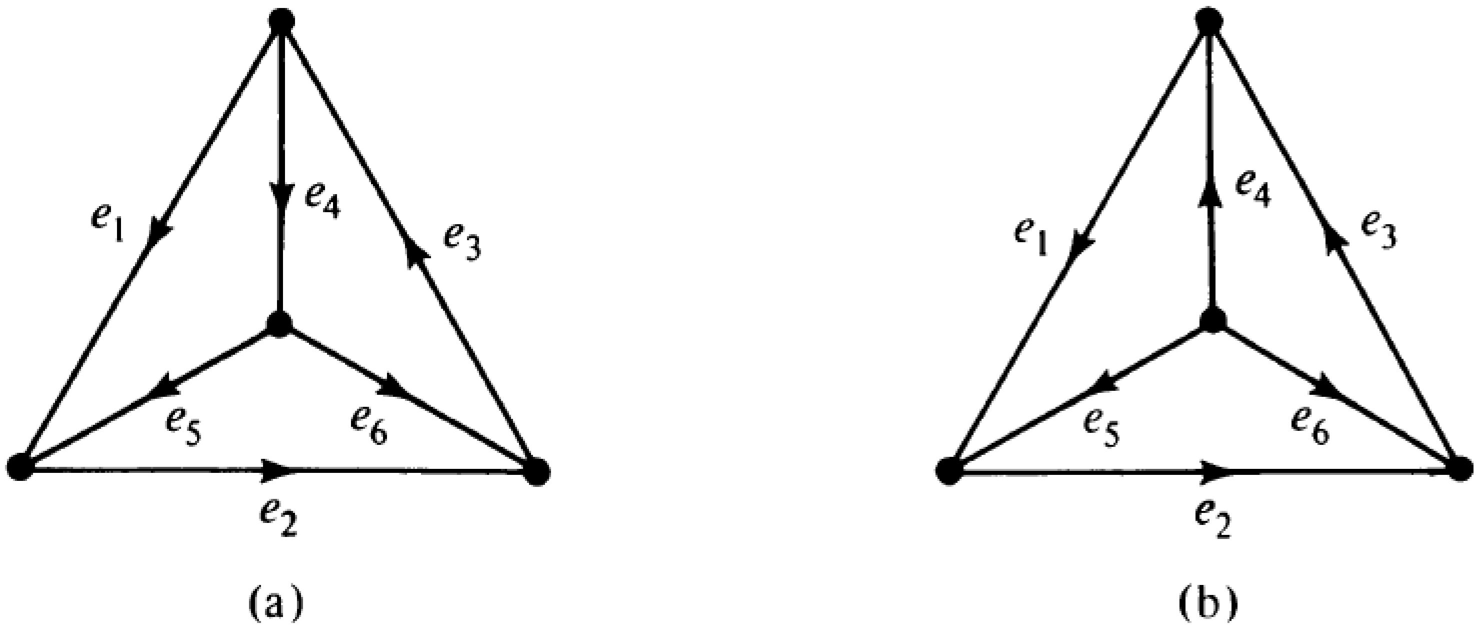
edges e_2 and e_3 are not.



given an undirected graph H, we can assign each edge of H some arbitrary direction. The resulting digraph, designated by $\vec{H}$ is called an *orientation of* H (or a *digraph associated with* H).

Isomorphic graphs

- For two digraphs to be isomorphic not only must their corresponding undirected graphs be isomorphic, but the directions of the corresponding edges must also agree.



Types of digraphs

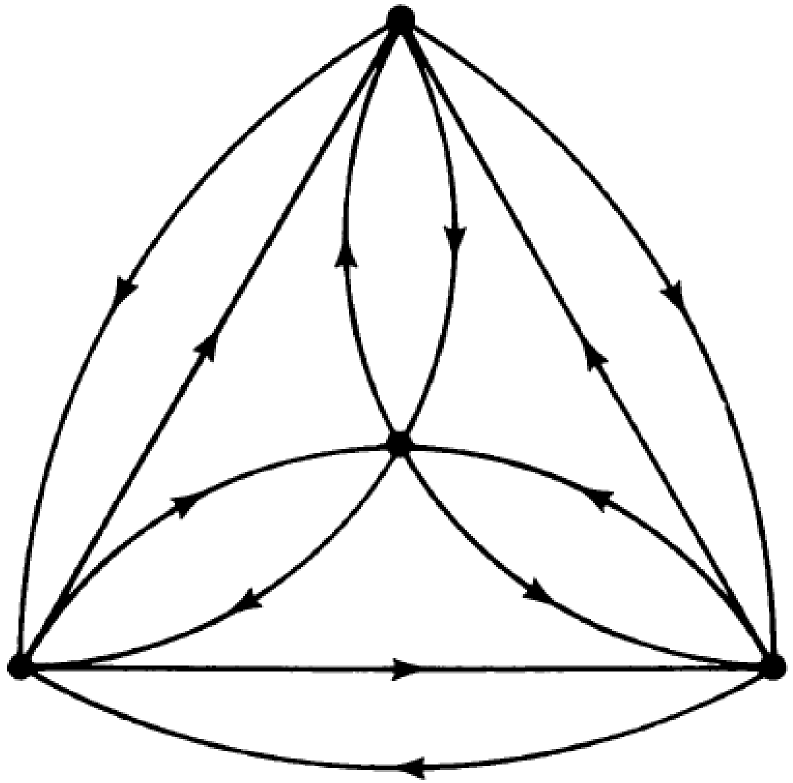
- *Simple Digraph*: A digraph without any self loops and parallel edges
- *Asymmetric digraph*: Digraphs that have at most one directed edges between a pair of vertices, but not allowed to have any self loop are called asymmetric or antisymmetric
- *Symmetric digraph*: Digraphs in which for every edge (a,b)[ie, from vertex a to b], there is also an edge (b,a).

- A digraph which is both simple and symmetric is called *simple symmetric digraph.*

- A digraph which is both simple and asymmetric is called *simple asymmetric digraph.*

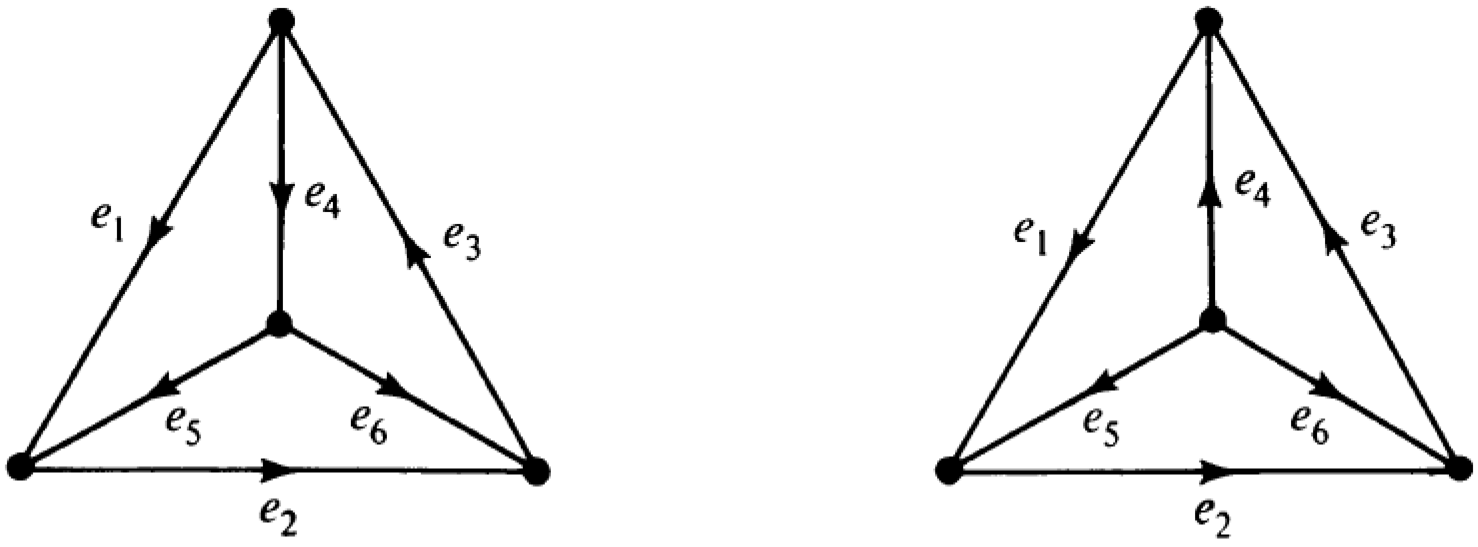
- *Complete digraphs*: Simple graph in which every vertex is joined to every other vertex exactly by one edge.

- *2 types of complete digraphs*
  - *Complete symmetric digraph* : Simple digraph in which there is exactly on edge directed from every vertex to every other vertex



Complete symmetric digraph of four vertices.

- *Complete asymmetric digraph*(tournament or complete tournment): Asymmetric digraph digraph in which there is exactly one edge between every pair of vertices



(a) (b)

- Complete asymmetric digraph of n vertices contains n(n-1)/2 edges

- Complete symmetric digraph of n vertices contains n(n-1) edges.

- A digraph is said to be balanced, if for every pair vertex v_i the in-degree equals to out-degree

$$d^+(v_i) = d^-(v_i)$$

- A balanced digraph is said to be regular, if every vertex has the same in-degree and out-degree as every other vertex

Assignment-1

Digraphs and Binary Relations

Text page no: 198 (Graph Theory – Narsingh Deo)

PART - A

1. (a) Show that there is no graph with 12 vertices and 28 edges where,

   i) the degree of each vertex is either 3 or 4

   ii) The degree of each vertex is either 3 or 6. (6 Marks)

   (b) How many edge disjoint Hamiltonian cycles exist in the complete graph on seven vertices ? Also draw the graph to show these Hamiltonian cycles. (7 Marks)

   (c) Define isomorphism of two graphs. Give an example to show that two graphs need not be isomorphic though they have equal number of edges, equal number of vertices and equal number of vertices with a given degree sequence. (7 Marks)

Thank You