Answers B

Problem 2.2

a)

$$2^{20} = 1048576$$

b)

$$\frac{1}{2^{20}} = \frac{1}{1048576} \approx 9.54 \cdot 10^{-7}$$

c)

$$\frac{\frac{20!}{12!8!}}{2^{20}} = \frac{125970}{1048576} \approx 0.120$$

Problem 2.8

a) There are 21 macrostates, since the number of energy units in Einstein solid A can be any integer from 0 to 20.

b) The number of microstates is

$$\frac{(q+N-1)!}{q!(N-1)!} = \frac{39!}{20!\,19!} \approx 6.89 \cdot 10^{10},$$

where $N = 20$ is the total number of oscillators and $q = 20$ is the total number of energy units.

c) The number of microstates, which have all energy units in Einstein solid A, is

$$\frac{(20+10-1)!}{20!(10-1)!} \cdot \frac{(0+10-1)!}{0!(10-1)!} = \frac{29!}{20!\,9!} \approx 1.00 \cdot 10^{7}.$$

The probability to have all energy units in Einstein solid A is hence

$$\frac{\frac{29!}{20!\,9!}}{\frac{39!}{20!\,19!}} = \frac{29!\,19!}{39!\,9!} \approx 1.45 \cdot 10^{-4}$$

d) The number of microstates, which have 10 energy units in Einstein solid A, is

$$\frac{(10+10-1)!}{10!(10-1)!} \cdot \frac{(10+10-1)!}{10!(10-1)!} = \frac{19!}{10!\,9!} \cdot \frac{19!}{10!\,9!} \approx 8.53 \cdot 10^{9}.$$

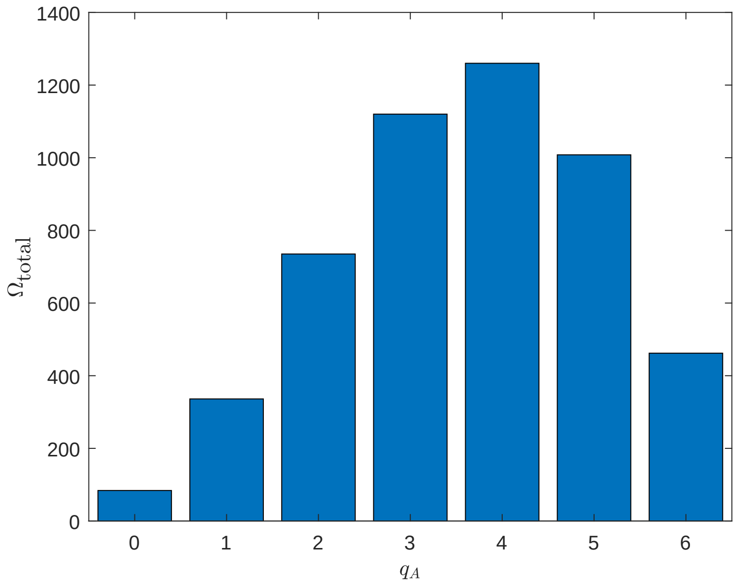
The probability to have 10 energy units in Einstein solid A is hence

$$\frac{\frac{19!}{10!\,9!} \cdot \frac{19!}{10!\,9!}}{\frac{39!}{20!\,19!}} \approx 0.124.$$

e) The system would exhibit irreversible behavior if it is initiated in a state that does not have approximately half of the energy units in solid A.

Problem 2.9

q_A	Ω_A	q_B	Ω_B	$\Omega_{\text{total}} = \Omega_A \Omega_B$
0	1	6	84	84
1	6	5	56	336
2	21	4	35	735
3	56	3	20	1120
4	126	2	10	1260
5	252	1	4	1008
6	462	0	1	462
				5005



The most probable macrostate is the one with $q_A = 4$ and it has probability 1260/5005. The least probable macrostate is the one with $q_A = 0$ and it has probability 84/5005.

Problem 2.13

a) $e^{a\ln(b)} = e^{\ln(b^a)} = b^a$
b) $\ln(a+b) = \ln(a) + \ln(1 + b/a) \approx \ln(a) + b/a$, since $\ln(1+x) \approx x$ for $|x| \ll 1$.

Problem 2.14

$$e^{10^{23}} = 10^{\log\left(e^{10^{23}}\right)} = 10^{10^{23}\log(e)}$$

Problem 2.18

Stirling's approximation says that

$$N! \approx N^N e^{-N}\sqrt{2\pi N} \quad \text{for} \quad N \gg 1.$$

We assume $N \gg 1$ and $q \gg 1$. Then

$$\Omega = \frac{(q+N-1)!}{q!(N-1)!} = \frac{N}{q+N} \cdot \frac{(q+N)!}{q!N!} \approx \frac{N}{q+N} \cdot \frac{(q+N)^{q+N} e^{-(q+N)} \sqrt{2\pi(q+N)}}{q^q e^{-q} \sqrt{2\pi q} N^N e^{-N} \sqrt{2\pi N}}$$
$$= \frac{N}{q+N} \cdot \frac{(q+N)^{q+N} \sqrt{q+N}}{q^q N^N \sqrt{2\pi q N}} = \frac{(q+N)^{q+N}}{q^q N^N \sqrt{2\pi q(q+N)/N}} = \frac{\left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{N}\right)^N}{\sqrt{2\pi q(q+N)/N}}.$$

Problem 2.22

a) The number of macrostates is $2N+1$.

b) The total number of microstates is

$$\Omega(2N, 2N) \approx \frac{\left(\frac{2N+2N}{2N}\right)^{2N} \left(\frac{2N+2N}{2N}\right)^{2N}}{\sqrt{2\pi \cdot 2N(2N+2N)/(2N)}} = \frac{2^{2N} \cdot 2^{2N}}{\sqrt{2\pi \cdot 4N}} = \frac{2^{4N}}{\sqrt{8\pi N}}.$$

c) The multiplicity of the macrostate for which the energy units are shared equally between the two solids is

$$(\Omega(N, N))^2 \approx \frac{\left(\frac{N+N}{N}\right)^{2N} \left(\frac{N+N}{N}\right)^{2N}}{2\pi N(N+N)/N} = \frac{2^{4N}}{4\pi N}.$$

d) The estimation of the width of the peak is

$$\frac{\Omega(2N, 2N)}{(\Omega(N, N))^2} \approx \frac{4\pi N}{\sqrt{8\pi N}} = \sqrt{2\pi N}.$$

The fraction of the macrostates having a reasonably large probability is

$$\frac{\sqrt{2\pi N}}{2N+1} \approx \sqrt{\pi/(2N)}.$$

For $N = 10^{23}$ this number evaluates to $4.0 \cdot 10^{-12}$.

Problem 2.24

a) The height of the peak in the multiplicity function is

$$\Omega\left(N_\uparrow = \frac{N}{2}\right) = \frac{N!}{\left(\frac{N}{2}\right)! \left(\frac{N}{2}\right)!} \approx \frac{N^N e^{-N} \sqrt{2\pi N}}{\left(\left(\frac{N}{2}\right)^{N/2} e^{-N/2} \sqrt{\pi N}\right)^2} = 2^N \cdot \sqrt{\frac{2}{\pi N}}$$

b) Introduce

$$x \equiv N_\uparrow - \frac{N}{2}$$

and consider values of $N_\uparrow$ close to $N/2$ such that $x \ll N$. Then

$$\begin{aligned}
\Omega &= \frac{N!}{\left(\frac{N}{2}+x\right)! \left(\frac{N}{2}-x\right)!} \\
&\approx \frac{N^N e^{-N} \sqrt{2\pi N}}{\left(\frac{N}{2}+x\right)^{N/2+x} e^{-N/2-x} \sqrt{\pi N + 2\pi x} \left(\frac{N}{2}-x\right)^{N/2-x} e^{-N/2+x} \sqrt{\pi N - 2\pi x}} \\
&= \sqrt{\frac{2}{\pi N}} \cdot \frac{1}{\sqrt{1 - \frac{4x^2}{N^2}}} \cdot \frac{N^N}{\left(\frac{N}{2}+x\right)^{N/2+x} \left(\frac{N}{2}-x\right)^{N/2-x}}
\end{aligned}$$

and

$$
\begin{aligned}
&\ln\left(\sqrt{\frac{\pi N}{2}}\Omega\right) \\
\approx\;& -\frac{1}{2}\ln\left(1-\frac{4x^2}{N^2}\right)+N\ln(N)-\left(\frac{N}{2}+x\right)\ln\left(\frac{N}{2}+x\right)-\left(\frac{N}{2}-x\right)\ln\left(\frac{N}{2}-x\right) \\
\approx\;& N\ln(N)-\left(\frac{N}{2}+x\right)\ln\left(\frac{N}{2}+x\right)-\left(\frac{N}{2}-x\right)\ln\left(\frac{N}{2}-x\right) \\
=\;& N\ln(N)-N\ln\left(\frac{N}{2}\right)-\left(\frac{N}{2}+x\right)\ln\left(1+\frac{2x}{N}\right)-\left(\frac{N}{2}-x\right)\ln\left(1-\frac{2x}{N}\right) \\
\approx\;& N\ln(2)-\left(\frac{N}{2}+x\right)\left(\frac{2x}{N}-\frac{2x^2}{N^2}\right)-\left(\frac{N}{2}-x\right)\left(-\frac{2x}{N}-\frac{2x^2}{N^2}\right) \\
\approx\;& N\ln(2)-\frac{2x^2}{N}.
\end{aligned}
$$

Hence

$$\Omega \approx 2^N\sqrt{\frac{2}{\pi N}}\,e^{-2x^2/N}.$$

This result agrees with the result above for $x=0$.

c) The multiplicity Ω is a factor e^{-1} smaller than the maximum value when

$$\frac{2x^2}{N}=1 \quad\Leftrightarrow\quad x=\pm\sqrt{\frac{N}{2}}.$$

The width of the peak can hence be estimated as

$$\text{width of peak } = \sqrt{\frac{N}{2}}-\left(-\sqrt{\frac{N}{2}}\right)=\sqrt{2N}.$$

d) The probability $P(x)$ to get a given value of x is the multiplicity for that value of x divided by the total number of microstates, which is 2^N. Hence

$$P(x)\approx\sqrt{\frac{2}{\pi N}}e^{-2x^2/N}.$$

For $N=10^6$, we find

$$
\begin{aligned}
P(10^3) &\approx 1.08\cdot 10^{-4}, \\
P(10^6) &\approx 1.10\cdot 10^{-90}.
\end{aligned}
$$

Hence, $x=10^3$ is rather unlikely, while $x=10^4$ is extremely unlikely.

Problem Hand-in B

a) The mean value is defined as

$$\overline{q_A}=\sum_{q_a=0}^{q} q_A P(q_A),$$

where $P(q_A)$ is the probability to have q_A energy units in Einstein solid A, and we have utilized that q_A is an integer in the range from 0 to q. The fundamental assumption of statistical mechanics says that all accessible microstates are equally probable. Therefore

$$P(q_A) = \frac{\Omega(q_A)}{\sum_{q_a=0}^{q} \Omega(q_A)}.$$

Inserting this, we obtain

$$\overline{q_A} = \frac{\sum_{q_a=0}^{q} q_A \Omega(q_A)}{\sum_{q_a=0}^{q} \Omega(q_A)}.$$

Now define

$$x = q_A - \frac{q}{2}.$$

Then

$$\overline{q_A} = \frac{\sum_{q_a=0}^{q} q_A \Omega(q_A)}{\sum_{q_a=0}^{q} \Omega(q_A)} = \frac{\sum_{x=-q/2}^{q/2} (q/2 + x)\Omega(q/2 + x)}{\sum_{x=-q/2}^{q/2} \Omega(q/2 + x)} = \frac{q}{2} + \frac{\sum_{x=-q/2}^{q/2} x\Omega(q/2 + x)}{\sum_{x=-q/2}^{q/2} \Omega(q/2 + x)} = \frac{q}{2}.$$

The last equality follows because

$$\Omega(q/2 + x) = \frac{(q/2 + x + N - 1)!}{(q/2 + x)!(N - 1)!} \cdot \frac{(q/2 - x + N - 1)!}{(q/2 - x)!(N - 1)!} = \Omega(q/2 - x),$$

which means that $x\Omega(q/2 + x)$ is an odd function of x.

b) When q is large, there are many terms contributing to the sum, and we can replace the sums by integrals. Note that the distance between two allowed values of x is 1, which means that $dx = 1$. We should hence replace $\sum_{x=-q/2}^{q/2}$ by $\int_{-q/2}^{q/2} dx$. Note that the multiplicity function Ω is a factor of e^{-N} smaller for $x = \pm q/2$ compared to its value for $x = 0$. The multiplicity function is hence already so small for $x = \pm q/2$ that it makes no difference to extend the integrals to minus and plus infinity. This leads to

$$\overline{q_A} = \frac{\int_{-\infty}^{\infty} (\frac{q}{2} + x)\Omega dx}{\int_{-\infty}^{\infty} \Omega dx} = \frac{\int_{-\infty}^{\infty} (\frac{q}{2} + x)e^{-N(2x/q)^2} dx}{\int_{-\infty}^{\infty} e^{-N(2x/q)^2} dx} = \frac{q}{2}.$$

The last equality follows because $xe^{-N(2x/q)^2}$ is an odd function of x.

c)

$$\sigma^2 = \overline{\left(q_A - \frac{q}{2}\right)^2} = \overline{x^2} = \frac{\int_{-\infty}^{\infty} x^2 e^{-N(2x/q)^2} dx}{\int_{-\infty}^{\infty} e^{-N(2x/q)^2} dx} = \frac{q^2}{8N} \quad \Leftrightarrow \quad \sigma = \frac{q}{2^{3/2}\sqrt{N}}$$

d) The ratio

$$\frac{\sigma}{\overline{q_A}} = \frac{1}{\sqrt{2N}}$$

is very small when N is macroscopic.