Problem F1

Consider a system consisting of two indistinguishable bosons, each of which can be in three quantum states: Two with energy $-\epsilon$ and one with energy ϵ, where $\epsilon > 0$, see figure 1. The system is in equilibrium with a reservoir having the temperature T.

Question a:

Determine the possible states of the 2-particle system, and write down its partition function Z.

We now take into account that the two bosons interact with energy U, when they are in the same quantum state. Consequently the energy is $2\epsilon + U$, when both particles are in the upper quantum state and $-2\epsilon + U$, when one of the lower states contain both particles.

Question b:

Write down the partition function for the system. Give an expression for the energy $E(T)$ as a function of the temperature.

Question c:

Give an expression for the entropy $S(T)$ of the system as a function of the temperature. Compute $\lim_{T\to\infty} S(T)$ and $\lim_{T\to 0} S(T)$ for $U < 0$ and for $U > 0$. Give a physical interpretation.

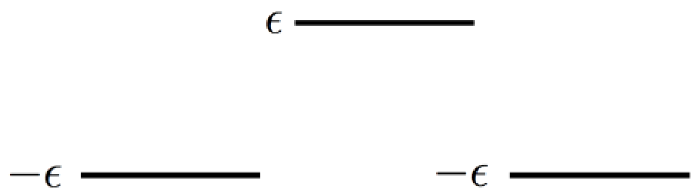

Figure 1: The three quantum states for the bosons.

Problem F2

An electron gas at temperature T and chemical potential μ is in equilibrium with a surface, where the electrons can bind to a number of binding points. The electron gas can be seen as a reservoir for the surface. A binding point can bind maximally one electron, which either has spin $\uparrow$ or spin $\downarrow$. When an electron is bound to a binding point it vibrates, which is described by that it sits in a one-dimensional harmonic oscillator with the energies $\epsilon_j = jhf$, where $j = 0, 1, 2, \ldots$.

Question a:

Show that the grand partition function $\mathcal{Z}$ for a single binding point can be written as

$$\mathcal{Z} = 1 + e^{\beta\mu} g(\beta h f),$$

where $\beta = 1/(kT)$, and specify the function $g(x)$.

Question b:

Give an expression for the average vibrational energy $E(T)$ of bound electrons in a given binding point.
Hint: You can use without proof that $\sum_{j=0}^{\infty} j \cdot x^j = x \cdot (1-x)^{-2}$.

Question c:

Give an expression for the average number of electrons $N_b(T)$ bound to a binding point. Determine N_b for $T = 0$ and for $T \gg \epsilon_F/k$, where $\epsilon_F = h^2(3n/\pi)^{2/3}/(8m)$ is the Fermi energy for the electron gas with density $n > 0$, and m is the mass of the electron.

Problem F3

In this problem, we consider a quantum well with four different single particle states: One state has energy 0, two states have energy ϵ, and one state has energy 2ϵ. The energy spectrum is illustrated in Fig. 1.

Question a:

Write down the partition function Z_1 for a single particle sitting in this quantum well. Calculate the average energy E_1 and sketch it as a function of the temperature T. Calculate $\lim_{T\to 0} E_1$ and $\lim_{T\to\infty} E_1$, and interpret the results physically.

Question b:

Now consider two identical non-interacting fermions in the quantum well. Find the allowed quantum states and their energies for these two particles, and write down their partition function.

Question c:

Calculate the average energy E_2 of the two particles and sketch it as a function of T. Calculate $\lim_{T\to 0} E_2$ and $\lim_{T\to\infty} E_2$, and interpret the results physically.

Question d:

Calculate the entropy S_2 of the two particles and sketch it as a function of T. Calculate $\lim_{T\to 0} S_2$ and $\lim_{T\to\infty} S_2$, and interpret the results physically.

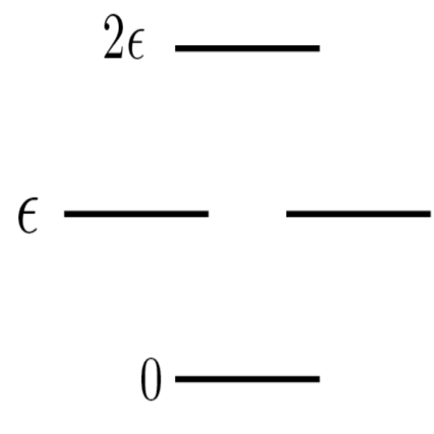

Fig. 1: The single particle spectrum of the quantum well.

Hand-in F

Consider a system described by two single particle levels with energy ϵ and $-\epsilon$. The system is in equilibrium with a reservoir with which it can exchange both energy and particles. The temperature of the reservoir is T and its chemical potential is μ.

Consider first the case where the particles are fermions.

Question a:

Write down the grand partition function for the system.

Question b:

Give an expression for the average number of particles N in the system. Sketch N as a function of μ for fixed temperature $T > 0$.

Consider next the case where the particles are bosons.

Question c:

Write down the grand partition function for the system.

Question d:

Give an expression for the average number of particles N in the system. Sketch N as a function of μ for fixed temperature $T > 0$. What is the maximum number of particles in the system and for which value of μ is it obtained?