# Answers F

**Problem 7.28**

a) We consider a free Fermi gas of spin-1/2 particles in two dimensions. The single particle states have energies

$$\epsilon = \frac{h^2}{8mA}\left(n_x^2 + n_y^2\right), \qquad n_x \in \{1, 2, 3, \ldots\}, \qquad n_y \in \{1, 2, 3, \ldots\}.$$

The number of particles is

$$N = 2\int_0^{\pi/2} d\phi \int_0^{n_{\max}} ndn = \frac{\pi}{2}n_{\max}^2 \qquad \text{so} \qquad n_{\max}^2 = \frac{2N}{\pi}.$$

The Fermi energy is hence

$$\epsilon_F = \frac{h^2}{8mA}n_{\max}^2 = \frac{h^2 N}{4\pi mA},$$

and the average energy per particle is

$$\frac{U}{N} = \frac{2}{N}\int_0^{\pi/2} d\phi \int_0^{n_{\max}} n\epsilon dn = \frac{\pi h^2}{8mAN}\int_0^{n_{\max}} n^3 dn = \frac{\pi h^2 n_{\max}^4}{32mAN} = \frac{h^2 N}{8\pi mA} = \frac{\epsilon_F}{2}.$$

b) Note that

$$N = \int_0^{\epsilon_F} g(\epsilon)d\epsilon = 2\int_0^{\pi/2} d\phi \int_0^{n_{\max}} ndn = \pi\int_0^{n_{\max}} ndn.$$

We change variables utilizing

$$\epsilon = \frac{h^2 n^2}{8mA} \quad \Leftrightarrow \quad n = \sqrt{\frac{8mA\epsilon}{h^2}} \quad \Rightarrow \quad dn = \sqrt{\frac{8mA}{h^2}}\frac{1}{2\sqrt{\epsilon}}d\epsilon$$

and obtain

$$N = \int_0^{\epsilon_F} \frac{4\pi mA}{h^2}d\epsilon.$$

We hence conclude that

$$g(\epsilon) = \begin{cases} \frac{4\pi mA}{h^2} & \text{for } \epsilon > 0, \\ 0 & \text{otherwise.} \end{cases}$$

c) Let us first note that

$$\frac{1}{e^{\frac{\epsilon-\mu}{kT}} + 1} = \frac{e^{-\frac{\epsilon-\mu}{kT}}}{1 + e^{-\frac{\epsilon-\mu}{kT}}} = \frac{1 + e^{-\frac{\epsilon-\mu}{kT}}}{1 + e^{-\frac{\epsilon-\mu}{kT}}} - \frac{1}{1 + e^{-\frac{\epsilon-\mu}{kT}}} = 1 - \frac{1}{1 + e^{-\frac{\epsilon-\mu}{kT}}}.$$

This result means that the probability for a state at energy $\epsilon = \mu + \Delta$ to be occupied by a particle is the same as the probability for a state at energy $\epsilon = \mu - \Delta$ to be unoccupied.

As long as the temperature is small enough that the probability for a state at $\epsilon = 0$ to be occupied by a particle is practically unity, it follows that the number of particles we remove at energies below $\epsilon_F$ is equal to the number of particles we add at energies above $\epsilon_F$, if we keep $\mu = \epsilon_F$ also at nonzero temperature. Therefore the chemical potential will remain unchanged and is hence independent of $T$.

When $T$ gets sufficiently large, however, we begin to see that $g(\epsilon)$ is zero for negative $\epsilon$. If we keep $\mu$ at $\epsilon_F$, we will remove fewer particles below $\epsilon_F$ than we add above $\epsilon_F$, giving an increase in the number of particles. To keep the number of particles fixed, we need the chemical potential to be smaller than $\epsilon_F$ for high $T$. The effect is larger, the larger the temperature is, and therfore the chemical potential decreases with temperature for high $T$.

d) We have

$$N = \int_0^\infty g(\epsilon)\frac{1}{e^{\beta(\epsilon-\mu)}+1}d\epsilon = \frac{4\pi mA}{h^2}\int_0^\infty \frac{1}{e^{\beta(\epsilon-\mu)}+1}d\epsilon = \frac{4\pi mkTA}{h^2}\int_{-\frac{\mu}{kT}}^\infty \frac{1}{e^x+1}dx.$$

Note that

$$\begin{aligned}
\int_{-\frac{\mu}{kT}}^\infty \frac{1}{e^x+1}dx &= \int_{-\frac{\mu}{kT}}^\infty \frac{e^{-x}}{1+e^{-x}}dx = \sum_{n=0}^\infty \int_{-\frac{\mu}{kT}}^\infty e^{-x}(-e^{-x})^n dx \\
&= \sum_{n=0}^\infty (-1)^n \int_{-\frac{\mu}{kT}}^\infty e^{-(n+1)x}dx = \sum_{n=0}^\infty (-1)^n \left[-\frac{1}{n+1}e^{-(n+1)x}\right]_{-\frac{\mu}{kT}}^\infty \\
&= \sum_{n=0}^\infty (-1)^n \frac{1}{n+1}\left(e^{\frac{\mu}{kT}}\right)^{n+1} = \sum_{n=1}^\infty (-1)^{n+1}\frac{1}{n}\left(e^{\frac{\mu}{kT}}\right)^n = \ln\left(1+e^{\frac{\mu}{kT}}\right)
\end{aligned}$$

Inserting this into the expression for $N$, we obtain

$$N = \frac{4\pi mkTA}{h^2}\ln\left(1+e^{\frac{\mu}{kT}}\right),$$

which we rearrange into

$$\mu = kT\ln\left(e^{\frac{Nh^2}{4\pi mkTA}}-1\right) = kT\ln\left(e^{\epsilon_F/(kT)}-1\right).$$

For $kT \ll \epsilon_F$, the exponential is large, and we can neglect the $-1$. We hence obtain

$$\mu \approx \frac{Nh^2}{4\pi mA} = \epsilon_F.$$

The chemical potential is hence independent of $T$ and equals the Fermi energy as expected.

For $kT \gg \epsilon_F$, we can Taylor expand the exponential to first order, which gives

$$\mu \approx kT\ln\left(\frac{Nh^2}{4\pi mkTA}\right) = kT\ln\left(\frac{\epsilon_F}{kT}\right).$$

When $kT \gg \epsilon_F$, the argument of the logarithm is less than one and the logarithm is therefore negative. Increasing $T$ in the high temperature limit, therefore makes $\mu$ more and more negative as expected.

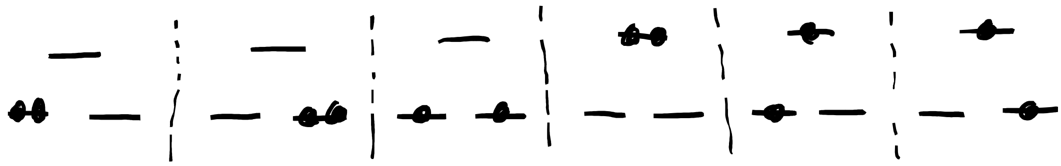
e) Adapting equation (6.93) in Schroeder to two dimensions gives

$$\mu_{\text{ideal gas}} = -kT \ln\left(\frac{2\pi mkTA}{h^2 N} Z_{\text{int}}\right).$$

In our case the particles have two different spin states with the same energy. Therefore $Z_{\text{int}} = 2$ and we see that the result agrees with the chemical potential in the high temperature limit found above.

**Problem F1**

a) The possible system states are



The first three states have energy $-2\epsilon$, the fourth state has energy $2\epsilon$, and the last two states have energy 0, so the partition function is

$$Z = 3e^{2\beta\epsilon} + 2 + e^{-2\beta\epsilon}, \qquad \beta = \frac{1}{kT}.$$

b) The partition function is now

$$Z = e^{2\beta\epsilon} + 2e^{-\beta(U-2\epsilon)} + 2 + e^{-\beta(U+2\epsilon)},$$

and the average energy is

$$E(T) = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{-2\epsilon e^{2\beta\epsilon} + 2(U-2\epsilon)e^{-\beta(U-2\epsilon)} + (2\epsilon+U)e^{-\beta(U+2\epsilon)}}{e^{2\beta\epsilon} + 2e^{-\beta(U-2\epsilon)} + 2 + e^{-\beta(U+2\epsilon)}}.$$

c) The entropy is

$$\begin{aligned} S(T) = \frac{E-F}{T} = \frac{E}{T} + k\ln(Z) = k&\frac{-2\beta\epsilon e^{2\beta\epsilon} + 2\beta(U-2\epsilon)e^{-\beta(U-2\epsilon)} + \beta(U+2\epsilon)e^{-\beta(U+2\epsilon)}}{e^{2\beta\epsilon} + 2e^{-\beta(U-2\epsilon)} + 2 + e^{-\beta(U+2\epsilon)}} \\ &+ k\ln\left(e^{2\beta\epsilon} + 2e^{-\beta(U-2\epsilon)} + 2 + e^{-\beta(U+2\epsilon)}\right). \end{aligned}$$

In the high temperature limit, we obtain

$$\lim_{T\to\infty} S(T) = \lim_{\beta\to 0} S(T) = k\frac{0}{1+2+2+1} + k\ln(1+2+2+1) = k\ln(6).$$

In the high temperature limit, all six states are equally probable, since this maximizes the entropy of the system. The entropy is hence $k\ln(6)$.

We now consider the low temperature limit and $U < 0$. In this case the energy of the ground state is $-2\epsilon + U$. We therefore rewrite the entropy into

$$S(T) = k\frac{4\beta\epsilon e^{-4\beta\epsilon} - \beta U e^{\beta U} - 2\beta(U-2\epsilon)e^{\beta(U-2\epsilon)}}{e^{\beta U} + 2 + 2e^{\beta(U-2\epsilon)} + e^{-4\beta\epsilon}} + k\ln\left(e^{\beta U} + 2 + 2e^{\beta(U-2\epsilon)} + e^{-4\beta\epsilon}\right).$$

Since $xe^{-x} \to 0$ for $x \to \infty$, and remembering that $U < 0$, we conclude that

$$\lim_{T\to 0} S(T) = \lim_{\beta\to\infty} S(T) = k\frac{0+0+0}{0+2+0+0} + k\ln(0+2+0+0) = k\ln(2).$$

We obtain this limit, since the system is in the ground state for $T \to 0$, and the ground state has degeneracy 2 for $U < 0$.

Next we consider the low temperature limit, when $U > 0$. In this case, the ground state is the state with energy $-2\epsilon$. We hence rewrite the entropy into

$$S(T) = k\frac{2\beta U e^{-\beta U} + 4\beta\epsilon e^{-2\beta\epsilon} + \beta(U+4\epsilon)e^{-\beta(U+4\epsilon)}}{1 + 2e^{-\beta U} + 2e^{-2\beta\epsilon} + e^{-\beta(U+4\epsilon)}} + k\ln\left(1 + 2e^{-\beta U} + 2e^{-2\beta\epsilon} + e^{-\beta(U+4\epsilon)}\right).$$

From this we conclude that

$$\lim_{T\to 0} S(T) = \lim_{\beta\to\infty} S(T) = k\frac{0+0+0}{1+0+0+0} + k\ln(1+0+0+0) = 0.$$

We obtain this limit, since the system is in the ground state for $T \to 0$, and the ground state is nondegenerate for $U > 0$. This gives the entropy $k\ln(1) = 0$.

Summarizing,

$$\lim_{T\to 0} S(T) = \begin{cases} k\ln(2) & \text{for } U < 0, \\ 0 & \text{for } U > 0. \end{cases}$$

**Problem F2**

a) The grand partition function for a single binding point is

$$\mathcal{Z} = 1 + 2\sum_{j=0}^{\infty} e^{-\beta(jhf-\mu)} = 1 + \frac{2e^{\beta\mu}}{1-e^{-\beta hf}} \quad \Rightarrow \quad g(x) = \frac{2}{1-e^{-x}}.$$

b) The average vibrational energy of bound electrons in a given binding point is

$$E(T) = \frac{2}{\mathcal{Z}}\sum_{j=0}^{\infty} jhfe^{-\beta(jhf-\mu)} = \frac{2hfe^{-\beta(hf-\mu)}}{\mathcal{Z}(1-e^{-\beta hf})^2} = \frac{2hfe^{-\beta(hf-\mu)}}{(1-e^{-\beta hf})(1-e^{-\beta hf}+2e^{\beta\mu})}.$$

c) The average number of electrons bound to the binding point is

$$N_b = \frac{2e^{\beta\mu}\frac{1}{1-e^{-\beta hf}}}{1+2e^{\beta\mu}\frac{1}{1-e^{-\beta hf}}} = \frac{2e^{\beta\mu}}{1-e^{-\beta hf}+2e^{\beta\mu}}.$$

When $T$ goes to zero, the chemical potential approaches the Fermi energy, which is positive. The low temperature limit is therefore

$$\lim_{T\to 0} N_b = 1.$$

When $T$ is large compared to the Fermi temperature, the chemical potential is given by (6.93) in Schroeder with $Z_{\text{int}} = 2$. This leads to

$$N_b \approx \frac{nV_Q}{1-e^{-\beta hf}+nV_Q} \text{ for } T \gg T_F, \text{ where } V_Q = \left(\frac{h^2}{2\pi mkT}\right)^{3/2}.$$

**Problem F3**

a) The partition function for a single particle in the quantum well is

$$Z_1 = 1 + 2e^{-\beta\epsilon} + e^{-2\beta\epsilon}, \qquad \beta = \frac{1}{kT},$$

and the average energy is

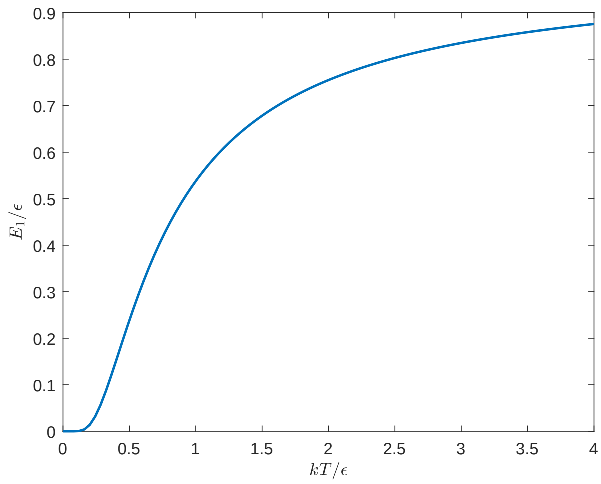
$$E_1 = -\frac{1}{Z_1}\frac{\partial Z_1}{\partial \beta} = 2\epsilon\frac{1+e^{-\beta\epsilon}}{e^{\beta\epsilon}+2+e^{-\beta\epsilon}} = \frac{2\epsilon}{e^{\beta\epsilon}+1}.$$

The limits are

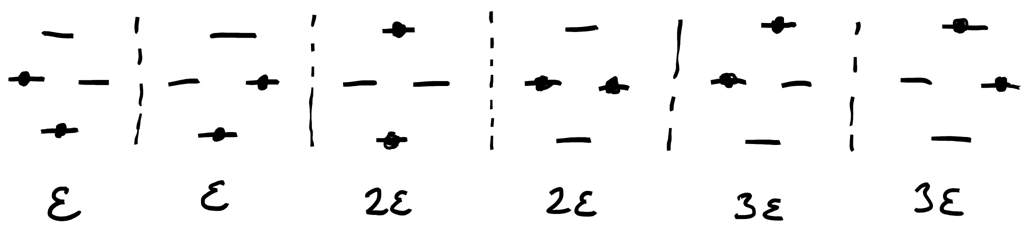
$$\lim_{T\to 0} E_1 = 0 \qquad \text{and} \qquad \lim_{T\to\infty} E_1 = \epsilon.$$

For $T = 0$, the particle sits in the ground state, which has energy 0, and for $T \to \infty$, the particle has equally large probability to sit in all 4 states, so the average energy is the average energy of the 4 states.

Plot of the energy $E_1$ as a function of the temperature




b) The possible system states and their energies are




This gives the partition function

$$Z_2 = 2\left(e^{-\beta\epsilon} + e^{-2\beta\epsilon} + e^{-3\beta\epsilon}\right).$$

c) The average energy is

$$E_2 = -\frac{1}{Z_2}\frac{\partial Z_2}{\partial \beta} = \epsilon\frac{1+2e^{-\beta\epsilon}+3e^{-2\beta\epsilon}}{1+e^{-\beta\epsilon}+e^{-2\beta\epsilon}} = \epsilon + \epsilon\frac{e^{-\beta\epsilon}+2e^{-2\beta\epsilon}}{1+e^{-\beta\epsilon}+e^{-2\beta\epsilon}},$$

and the limits are

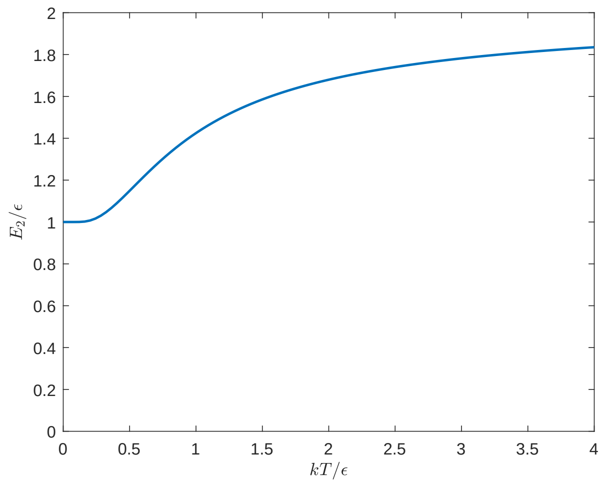
$$\lim_{T\to 0} E_2 = \epsilon + \epsilon\frac{0+0}{1+0+0} = \epsilon$$

and

$$\lim_{T\to \infty} E_2 = \epsilon + \epsilon\frac{1+2}{1+1+1} = 2\epsilon.$$

For $T = 0$, the system is in one of the two ground states with energy $\epsilon$, and for $T \to \infty$, the system has equally large probability to sit in all 6 states, which gives an average energy of $2\epsilon$.

Plot of the energy $E_2$ as a function of the temperature




d) The entropy can be computed from

$$S_2 = \frac{E_2 - F}{T} = \frac{E_2}{T} + k\ln(Z_2).$$

This gives

$$S_2 = k\beta\epsilon\frac{1+2e^{-\beta\epsilon}+3e^{-2\beta\epsilon}}{1+e^{-\beta\epsilon}+e^{-2\beta\epsilon}} + k\ln\left[2(e^{-\beta\epsilon}+e^{-2\beta\epsilon}+e^{-3\beta\epsilon})\right].$$

We can also write this as

$$S_2 = k\frac{\beta\epsilon e^{-\beta\epsilon}+2\beta\epsilon e^{-2\beta\epsilon}}{1+e^{-\beta\epsilon}+e^{-2\beta\epsilon}} + k\ln(2) + k\ln(1+e^{-\beta\epsilon}+e^{-2\beta\epsilon}).$$

From this we obtain the limits

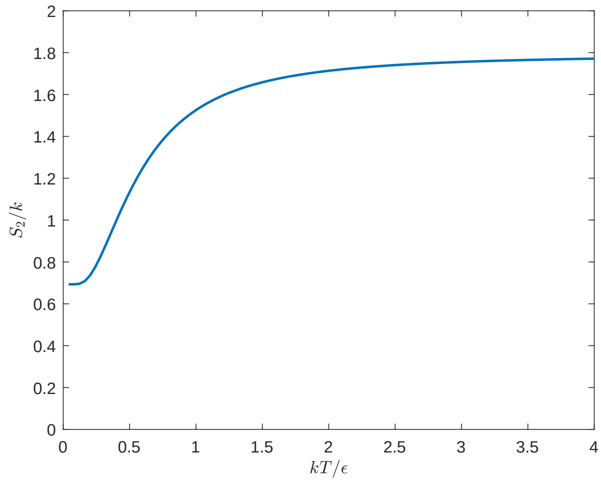
$$\lim_{T\to 0} S_2 = k\frac{0+0}{1+0+0} + k\ln(2) + k\ln(1+0+0) = k\ln(2)$$

and

$$\lim_{T\to \infty} S_2 = k\frac{0+0}{1+1+1} + k\ln(2) + k\ln(1+1+1) = k\ln(6).$$

The limits can be understood from $S = k\ln(\Omega)$. For $T = 0$, there are two ground states accessible for the system, that is $\Omega = 2$, and for $T \to \infty$ all 6 states are equally probably, so $\Omega = 6$.

Plot of the entropy $S_2$ as a function of the temperature




**Problem Hand-in F**

a) The grand partition function for the case of fermions is

$$Z = 1 + e^{\beta(\epsilon+\mu)} + e^{-\beta(\epsilon-\mu)} + e^{2\beta\mu} = \left(e^{\beta(\epsilon+\mu)} + 1\right)\left(e^{-\beta(\epsilon-\mu)} + 1\right), \qquad \beta = \frac{1}{kT}.$$

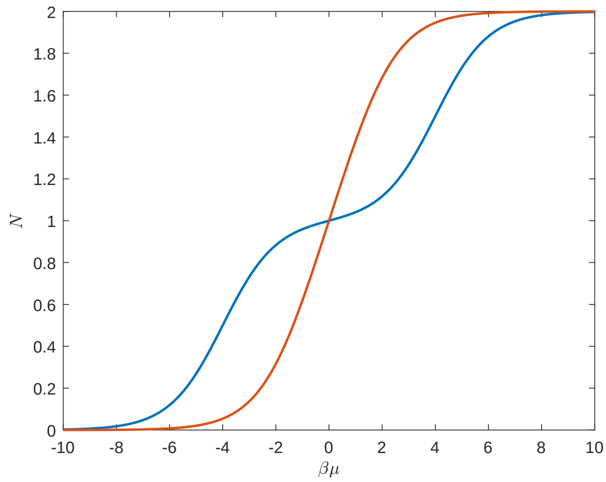
b) The average number of fermions in the system is

$$N = \frac{e^{\beta(\epsilon+\mu)} + e^{-\beta(\epsilon-\mu)} + 2e^{2\beta\mu}}{1 + e^{\beta(\epsilon+\mu)} + e^{-\beta(\epsilon-\mu)} + e^{2\beta\mu}} = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} + \frac{1}{e^{-\beta(\epsilon+\mu)} + 1},$$

and the limits are

$$\lim_{\mu\to-\infty} N = 0 \qquad \text{and} \qquad \lim_{\mu\to\infty} N = 2.$$

Plot of $N$ as a function of $\beta\mu$ for $\beta\epsilon = 1$ (red curve) and $\beta\epsilon = 4$ (blue curve)




c) The grand partition function for the case of bosons is

$$Z = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{-\beta((m-n)\epsilon-\mu(m+n))} = \sum_{n=0}^{\infty} e^{n\beta(\epsilon+\mu)} \sum_{m=0}^{\infty} e^{-m\beta(\epsilon-\mu)} = \frac{1}{1-e^{\beta(\epsilon+\mu)}} \cdot \frac{1}{1-e^{-\beta(\epsilon-\mu)}}.$$

d) According to the Bose-Einstein distribution, the number of bosons in a single particle state with energy $x$ is

$$\bar{n}_{\text{BE}} = \frac{1}{e^{\beta(x-\mu)} - 1}.$$

The number of bosons in the system is the number of bosons in the single particle state with energy $-\epsilon$ plus the number of bosons in the single particle state with energy $\epsilon$, that is

$$N = \frac{1}{e^{-\beta(\epsilon+\mu)} - 1} + \frac{1}{e^{\beta(\epsilon-\mu)} - 1}.$$

The limits are

$$\lim_{\mu \to -\infty} N = 0 \qquad \text{and} \qquad \lim_{\mu \to -\epsilon} N = \infty.$$

There can be infinitely many particles in the system, and this is achieved for $\mu$ approaching $-\epsilon$ from the left.

Plot of $N$ as a function of $\beta\mu$ for $\beta\epsilon = 1$