Problem A1

This problem deals with combinatorial mathematics, illustrated by card distributions. This type of mathematics is essential for the evaluation of entropy in many physical systems. A deck of cards consists of $N = 52$ different cards (distinguishable). Before a game of cards, the cards are shuffled to obtain a random sequence (distribution).

a) How many distributions are there with N cards? Calculate this number for $N = 52$.

b) In statistical physics, the quantity $N!$ plays an important role. As sketched in the note "Stirling.pdf", a rather precise value is obtained with Stirling's formula for large N:

$$N! \simeq N^N e^{-N} \sqrt{2\pi N}$$

How accurate is this formula for $N = 52$?

c) In statistical physics, we are often interested in the logarithm of $N!$. Show that Stirling's formula is equivalent to

$$\ln(N!) \simeq N \ln(N) - N,$$

where we ignore terms of order $\ln(N)$. What is the accuracy of this estimate for $N = 52$?

d) A hand in bridge consists of $m = 13$ cards and their sequence is unimportant. The number of different hands is given by the binomial formula

$$\binom{N}{m} = \frac{N!}{m!(N-m)!}$$

Repeat the argument for this formula.

e) How many different hands can be dealt in bridge where $N = 52$ and $m = 13$?

f) Now we wish to divide N cards into three piles, one with m cards, one with l cards, and one with $N - m - l$ cards. Piles with 0 cards are at first not considered. In how many ways can this be done? We now include the possibility that some of the three numbers could be zero. Does this change the result?

g) Generalize this result to an arbitrary number of piles, M, with n_i cards in the i'th pile and $\sum_{i=1}^{M} n_i = N$.

h) A game of bridge depends on all four hands. How many different card distributions are there?

Problem A2

Compute the limits $x \to 0^+$ and $x \to \infty$ of the following expressions. Note that $x \to 0^+$ means that x approaches zero through positive values. Note also that in some cases it may be easier to rewrite the expression before taking the limit.

Hint: Remind yourself of L'Hôpital's rule.

a)

$$e^{-\frac{1}{x}}$$

b)

$$xe^{-x}$$

c)

$$\frac{e^{-\frac{1}{x}}}{x^2}$$

d)

$$\frac{e^x - e^{-x}}{e^x + e^{-x}}$$

e)

$$\frac{\epsilon_1 e^{-\frac{\epsilon_1}{x}} + \epsilon_2 e^{-\frac{\epsilon_2}{x}}}{e^{-\frac{\epsilon_1}{x}} + e^{-\frac{\epsilon_2}{x}}}, \qquad \text{where } \epsilon_2 > \epsilon_1,$$

f)

$$\ln(e^x + e^{-x}) - x$$

g)

$$\ln(e^x + e^{-x}) + \frac{-xe^x + xe^{-x}}{e^x + e^{-x}}$$

h) Challenge yourself by inventing and solving more advanced limit problems.