

Answers A

Problem 1.12

The volume per molecule at room temperature and atmospheric pressure is

$$\frac{V}{N} = \frac{kT}{P} = \frac{1.381 \cdot 10^{-23} \text{ J/K} \cdot 300 \text{ K}}{1.013 \cdot 10^5 \text{ N/m}^2} = 4.09 \cdot 10^{-26} \text{ m}^3.$$

Taking the cube root gives

$$\left(\frac{V}{N}\right)^{1/3} = 3.45 \cdot 10^{-9} \text{ m}.$$

In comparison, the bond length of an N_2 molecule is $1.098 \cdot 10^{-10} \text{ m}$, and the bond length of an H_2O molecule is $0.958 \cdot 10^{-10} \text{ m}$, so the average distance between the molecules is roughly ten times larger than the size of a small molecule.

Problem 1.22(a-c)

a) First consider a single molecule, whose x -component of the velocity vector is v_x . Let us assume that v_x is positive. This molecule will only hit the area A within the time Δt if its distance from the wall in the x -direction is at most $v_x \Delta t$. A molecule with x -component of the velocity equal to $-v_x$ will not hit the wall. Let the number of particles per volume with x -component of the velocity equal to $\pm v_x$ be N_{v_x}/V . Half of those are flying towards the wall and the other half away from the wall. The number of molecules with x -component of the velocity equal to $\pm v_x$ hitting the wall within the time interval Δt is hence

$$\Delta N_{v_x} = \frac{1}{2} \frac{N_{v_x}}{V} A v_x \Delta t$$

Assume that m is the same for all molecules. From equation (1.12) in Schroeder, we know that the contribution to the pressure from one of these molecules is $P_1 = m v_x^2 / V$. Hence

$$\Delta N_{v_x} = \frac{1}{2} \frac{P_1 N_{v_x}}{m v_x} A \Delta t = \frac{1}{2} \frac{P_{v_x}}{m v_x} A \Delta t,$$

where P_{v_x} is the contribution to the pressure coming from the molecules with x -component of the velocity equal to $\pm v_x$. We rearrange this equation into

$$P_{v_x} A \Delta t = 2 m v_x \Delta N_{v_x}.$$

Now sum this equation over all possible values of v_x to get (the contribution from $v_x = 0$ is zero anyway, so it can be ignored)

$$P A \Delta t = 2 m \sum_{v_x > 0} v_x \Delta N_{v_x}.$$

Now define $\overline{v_x} = \frac{1}{\Delta N} \sum_{v_x > 0} v_x \Delta N_{v_x}$, where $\Delta N = \sum_{v_x > 0} \Delta N_{v_x}$ is the total number of molecules hitting the wall within the time Δt . Then

$$P A \Delta t = 2 m \overline{v_x} \Delta N$$

and hence

$$\Delta N = \frac{PA\Delta t}{2m\bar{v}_x}.$$

b) From (1.15) in Schroeder we have $kT = m\bar{v}_x^2$, so

$$\sqrt{v_x^2} = \sqrt{\frac{kT}{m}}.$$

c) From the first and second questions and the ideal gas law we get

$$\frac{dN}{dt} = -\frac{PA}{2m\bar{v}_x} = -\frac{NkTA}{2Vm\bar{v}_x} \approx -\frac{NkTA}{2Vm\sqrt{kT/m}} = -\frac{A}{2V}\sqrt{\frac{kT}{m}}N.$$

We have added a minus compared to above, since above we were interested in how many molecules hit the wall, and now we are interested in how fast N is reduced because molecules escape. Solving this equation gives $N(t) = N(0)e^{-t/\tau}$ with

$$\tau = \frac{2V}{A}\sqrt{\frac{m}{kT}}.$$

Problem 1.33

The sign of the work W is determined from $W = -\int_{V_i}^{V_f} PdV$.

The change in energy ΔU is positive if PV increases, negative if PV decreases, and zero if PV does not change. This is so, because when the number of particles N is fixed, then U is only a function of the temperature $T = PV/(Nk)$, and U increases with T . Note also that $\Delta U = 0$ for a cyclic process.

The sign of the heat Q is determined from $Q = \Delta U - W$.

	A	B	C	Total
W	$-$	0	$+$	$+$
ΔU	$+$	$+$	$-$	0
Q	$+$	$+$	$-$	$-$

First, the system expands while it is being heated. It is then heated further at constant volume. Afterwards it is cooled and compressed back to the initial volume. The net effect is that we do work on the system, and the system produces heat.

Problem 1.34

a) From $W = -\int_{V_i}^{V_f} PdV$, $U = \frac{5}{2}NkT = \frac{5}{2}PV$, and $Q = \Delta U - W$, we obtain

	A	B	C	D
W	0	$-P_2(V_2 - V_1)$	0	$P_1(V_2 - V_1)$
ΔU	$\frac{5}{2}V_1(P_2 - P_1)$	$\frac{5}{2}P_2(V_2 - V_1)$	$-\frac{5}{2}(P_2 - P_1)V_2$	$-\frac{5}{2}P_1(V_2 - V_1)$
Q	$\frac{5}{2}V_1(P_2 - P_1)$	$\frac{5}{2}P_2(V_2 - V_1)$	$-\frac{5}{2}(P_2 - P_1)V_2$	$-\frac{5}{2}P_1(V_2 - V_1)$

b)

A: Heat is added to the gas while the piston is held fixed.

B: Heat is added to the gas while the piston moves outwards.

C: Heat is removed from the gas while the piston is held fixed.

D: Heat is removed from the gas while the piston moves inwards.

c)

$$W_{\text{net}} = -(P_2 - P_1)(V_2 - V_1)$$

$$\Delta U_{\text{net}} = 0$$

$$Q_{\text{net}} = (P_2 - P_1)(V_2 - V_1)$$

ΔU_{net} is zero because it is a cyclic process, W_{net} is minus the area encircled by the curve, and $Q_{\text{net}} = -W_{\text{net}}$.

Problem 2.12(b-d)

b) We know by definition that

$$e^{\ln(x)} = x.$$

Writing $y = e^x$, we obtain

$$e^{\ln(e^y)} = e^y.$$

Utilizing that exp is a monotonically increasing function, it follows that

$$\ln(e^y) = y.$$

Now

$$e^{\ln(ab)} = ab = e^{\ln(a)} e^{\ln(b)} = e^{\ln(a) + \ln(b)}$$

and taking \ln on both sides, we get

$$\ln(ab) = \ln(a) + \ln(b).$$

Also

$$\ln(a^b) = \ln(e^{\ln(a)b}) = b \ln(a).$$

c)

$$1 = \frac{dx}{dx} = \frac{d}{dx} (e^{\ln(x)}) = e^{\ln(x)} \frac{d \ln(x)}{dx} = x \frac{d \ln(x)}{dx}$$

so

$$\frac{d \ln(x)}{dx} = \frac{1}{x}$$

d)

$$\ln(1+x) \approx \ln(1) + \left(\frac{d \ln(1+x)}{dx} \right) \Big|_{x=0} x = \ln(e^0) + \left(\frac{1}{1+x} \right) \Big|_{x=0} x = x$$

and

$$\ln(1.1) = 0.0953, \quad \ln(1.01) = 0.009950$$

Problem A1

a) Number of distributions = $N!$

$$52! = 8.07 \cdot 10^{67}$$

b) $52! \approx \left(\frac{52}{e}\right)^{52} \sqrt{2\pi \cdot 52} = 8.05 \cdot 10^{67}$
 Deviation = $\frac{8.05-8.07}{8.07} = -0.0016 = -0.16\%$

c) $\ln(N!) = N \ln(N/e) + \frac{1}{2} \ln(2\pi N) = N \ln(N) - N + \frac{1}{2} \ln(2\pi N) \approx N \ln(N) - N$
 $\ln(52!) = 156.36$
 $52 \ln(52) - 52 = 153.46$
 Deviation = $\frac{153.46-156.36}{156.36} = -0.0185 = -1.85\%$

d) We want to pick m cards out of N cards. For the first card we pick, we have N possibilities, for the next $N - 1$, for the next $N - 2$, etc, down to $N - m + 1$. It does, however, not matter in which order, we select the m cards. We therefore need to divide by the number of ways in which the m cards can be permuted, which is $m!$. This gives

$$\frac{N \cdot (N - 1) \cdots (N - m + 1)}{m!} = \frac{N!}{m!(N - m)!}.$$

e)

$$\frac{52!}{13!(52 - 13)!} = 6.35 \cdot 10^{11}$$

f)

$$\frac{N!}{l!m!(N - l - m)!}$$

This is independent of whether some of the piles have zero cards.

g)

$$\frac{N!}{n_1!n_2! \cdots n_M!}$$

h)

$$\frac{52!}{(13!)^4} = 5.36 \cdot 10^{28}$$

Problem A2

a) 0 and 1

b) 0 and 0

c) 0 and 0

d) 0 and 1

e) ϵ_1 and $(\epsilon_1 + \epsilon_2)/2$

f) $\ln(2)$ and 0

g) $\ln(2)$ and 0

h) How can you test if your answer is correct?