

MA5233 Computational Mathematics

Lecture 22: Time-dependent PDEs

Simon Etter



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Time-dependent PDEs

Introduction

Last two topics:

- ▶ ODEs \approx differential equations in time.
- ▶ PDEs \approx differential equations in space.

New topic: differential equations in both space and time.

Heat equation

Given $u_0 : [0, 1] \rightarrow \mathbb{R}$, find $u : [0, 1] \times [0, T] \rightarrow \mathbb{R}$ such that for all $x \in [0, 1]$ and $t \in [0, T]$ it holds

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t), & \text{(PDE)} \\ u(x, 0) = u_0(x), & \text{(initial conditions)} \\ u(0, t) = u(1, t) = 0. & \text{(boundary conditions)} \end{cases}$$

Heat equation is the simplest time-dependent PDE and serves as a role model for more complicated equations.

See Lecture 5 for physical interpretation of this equation.

Time-dependent PDEs

Discretisation of time-dependent PDEs

Most numerical methods for time-dependent PDEs are derived according to the following scheme.

0. Original PDE:

$$\text{Find } u : [0, 1] \times [0, T] \rightarrow \mathbb{R} \text{ such that } \frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t). \quad (1)$$

1. Discretise in space using finite differences / Galerkin method.

This replaces (1) with

$$\text{Find } u : [0, T] \rightarrow \mathbb{R}^n \text{ such that } \frac{\partial u}{\partial t}(t) = Au(t). \quad (2)$$

where $u(t) \in \mathbb{R}^n$ is the vector of point values for finite differences or vector of expansion coefficients for Galerkin's method.

2. Discretise in time using Runge-Kutta / multistep method.

This replaces (2) with

$$\text{Find } (u_k \in \mathbb{R}^n)_{k=0}^m \text{ such that } u_{k+1} = u_k + B u_k \frac{T}{m}.$$

where $u_k \approx u(T \frac{k}{m})$.

The next slides demonstrate this procedure using FD and explicit Euler.

Time-dependent PDEs

Finite difference and explicit Euler discretisation of heat equation

0. Original PDE: $\frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t)$

1. Finite difference discretisation (assuming $(x_k := \frac{k}{n+1})_{k=0}^{n+1}$):

$$\frac{\partial u}{\partial t}(x_k, t) \approx (n+1)^2 \left(u(x_{k-1}, t) - 2u(x_k, t) + u(x_{k+1}, t) \right).$$

Let us write this as $\frac{\partial u}{\partial t}(t) = Au(t)$.

2. Explicit Euler time-stepping (assuming $(t_\ell := T \frac{\ell}{m})_{\ell=0}^m$):

$$u(t_{\ell+1}) \approx u(t_\ell) + Au(t_\ell) \frac{T}{m}.$$

See `example()` in `22_time_dependent_pdes.jl`.

Time-dependent PDEs

Error analysis

Same setup as for Runge-Kutta methods:

- ▶ Exact time propagator $\Phi_t : u(x, 0) \mapsto u(x, t)$.
- ▶ Numerical time propagator $\tilde{\Phi}_t : u(x_k, 0) \mapsto \tilde{u}(x_k, t)$.

Note that Φ_t maps functions to functions, while $\tilde{\Phi}_t$ maps point-values to point-values. As before, we assume that functions are implicitly converted to point-values if this is required by context.

Assumptions on numerical propagator:

- ▶ Consistency: $\|\tilde{\Phi}_t(u) - \Phi_t(u)\| = \mathcal{O}(t(t^p + n^{-q}))$ for some $p, q > 0$.
- ▶ Stability: $\|\tilde{\Phi}_t(u_2) - \tilde{\Phi}_t(u_1)\| \leq (1 + \tilde{L}t) \|u_2 - u_1\|$ for some $\tilde{L} > 0$.

Main new ingredient:

- ▶ space-discretised ODE $\frac{\partial u}{\partial t} = Au(t)$ is only an approximation to the exact equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}(t)$.
- ▶ Hence, consistency error involves a spatial component $\mathcal{O}(t n^{-q})$ and a temporal component $\mathcal{O}(t^{p+1})$.

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Theorem

Under the assumptions on previous slide, we have

$$\|\tilde{u}(T) - u(T)\| = \mathcal{O}(m^{-p} + n^{-q})$$

where $u(T)$ denotes the vector of point-values of the exact solution and $\tilde{u}(T)$ is the numerically computed approximation using m equidistant steps in time and n equispaced grid points in space.

Proof. The proof is a minor modification of what we have done for the Runge-Kutta schemes in Lecture 16. As before, let us introduce the shorthand notation

$$\Phi(u) := \Phi_{T/m}(u), \quad \tilde{\Phi}(u) := \tilde{\Phi}_{T/m}(u), \quad u_k := u\left(\frac{Tk}{m}\right), \quad \tilde{u}_k := \tilde{u}\left(\frac{Tk}{m}\right).$$

Both u_k and \tilde{u}_k are vectors of point values.

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Proof (continued). We compute

$$\begin{aligned}\|\tilde{u}(T) - u(T)\| &= \|\tilde{\Phi}(\tilde{u}_{m-1}) - \Phi(u_{m-1})\| \\ &\leq \|\tilde{\Phi}(\tilde{u}_{m-1}) - \tilde{\Phi}(u_{m-1})\| + \|\tilde{\Phi}(u_{m-1}) - \Phi(u_{m-1})\| \\ &\leq \left(1 + \frac{\tilde{L}T}{m}\right) \|\tilde{u}_{m-1} - u_{m-1}\| + \mathcal{O}(m^{-1}(m^{-p} + n^{-q})) \\ &\leq \dots \\ &\leq 0 + \left(\sum_{k=0}^{m-1} \left(1 + \frac{\tilde{L}T}{m}\right)^k\right) \mathcal{O}(m^{-1}(m^{-p} + n^{-q})) \\ &\leq \left(1 + \frac{\tilde{L}T}{m}\right)^{m-1} \mathcal{O}(m^{-p} + n^{-q})\end{aligned}$$

Claim follows after observing that since $1 + x \leq \exp(x)$, we have

$$\left(1 + \frac{\tilde{L}T}{m}\right)^{m-1} \leq \exp\left(\tilde{L}T \frac{m-1}{m}\right) \leq \exp(\tilde{L}T).$$

Time-dependent PDEs

Consistency of FD + explicit Euler

$$\tilde{u}(t) = u(0) + Au(0) t$$

$$u(t) = u(0) + \frac{\partial u}{\partial t}(0) t + \mathcal{O}(t^2)$$

Since

$$\frac{\partial u}{\partial t}(x_k, 0) = \frac{\partial^2 u}{\partial x^2}(x_k, 0) = (Au)_k + \mathcal{O}(n^{-2}),$$

we have $\tilde{u}(t) - u(t) = \mathcal{O}(t(t + n^{-2}))$.

Stability of FD + explicit Euler

$$\begin{aligned}\|\tilde{\Phi}_t(u_2) - \tilde{\Phi}_t(u_1)\| &\leq \|u_2 - u_1\| + t \|Au_2 - Au_1\| \\ &\leq (1 + \|A\|t) \|u_2 - u_1\|.\end{aligned}$$

Conclusion

Error for finite difference and explicit Euler discretisation with m equidistant time-steps and n mesh points is given by

$$\text{error} = \mathcal{O}(m^{-1} + n^{-2}).$$

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Discussion

- ▶ Solving time-dependent PDEs involves two limits $m, n \rightarrow \infty$.
- ▶ Analysis shows that error can effectively be decomposed into temporal and spatial components.
- ▶ Optimal efficiency is achieved if these two components are of equal magnitude, i.e. if $m^{-p} \approx n^{-q}$.

Concrete examples (assuming finite difference discretisation in space):

- ▶ For explicit or implicit Euler, choose $m \propto n^2$.
- ▶ For explicit or implicit midpoint, choose $m \propto n$.

See `convergence()`.

Observation

For explicit methods, solution blows up if m is too small compared to n .

Time-dependent PDEs

Asymptotic behaviour of heat equation

Physical intuition: $u(x, t) \rightarrow 0$ for $t \rightarrow \infty$ due to boundary conditions.

Mathematical analysis:

- ▶ If $u_0(x) = \sin(\pi kx)$, then $u(x, t) = \sin(\pi kx) \exp(-(\pi t)^2 x)$.
- ▶ Heat equation is linear: if $u_k(x, t)$ solves heat equation with initial conditions $u_{0,k}(x)$ for $k \in \{1, 2\}$, then $u_1(x, t) + u_2(x, t)$ solves heat equation for initial conditions $u_{0,1}(x) + u_{0,2}(x)$.
- ▶ Fourier theory: any (reasonable) $u_0(x)$ can be written as

$$u_0(x) = \sum_{k=1}^{\infty} \hat{u}_k \sin(\pi kx).$$

Conclusion:

$$u(x, t) = \sum_{k=1}^{\infty} \hat{u}_k \sin(\pi kx) \exp(-(\pi k)^2 t),$$

i.e. solutions decay exponentially in time.

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Asymptotic behaviour of Runge-Kutta methods

Runge-Kutta with time-step Δt applied to the ODE $\dot{u} = Au$ produces solutions \tilde{u}_k which go to zero for $k \rightarrow \infty$ if and only if the stability function $R(z)$ satisfies $|R(\lambda \Delta t)| < 1$ for all eigenvalues λ of A .

Time step constraint for FD + explicit Runge-Kutta method

- ▶ We showed in Lecture 5 that the eigenvalues of the finite difference matrix A satisfy $\lambda_k \in (-4(n+1)^2, 0)$.
- ▶ We showed in Lecture 17 that stability functions $R(z)$ for explicit Euler and midpoint rule satisfy $|R(z)| \geq 1$ for $z \in \mathbb{R} \setminus (-2, 0)$.

Conclusion: stability constraint $|R(\lambda \Delta t)| < 1$ is satisfied if and only if

$$4(n+1)^2 \Delta t \leq 2 \quad \Longleftrightarrow \quad \Delta t \leq \frac{1}{2} (n+1)^{-2}.$$

Discussion

- ▶ Time step constraint is acceptable for explicit Euler method since convergence anyway requires $\Delta t \propto n^{-2}$.
- ▶ Time step constraint means explicit midpoint offers no advantage over explicit Euler: for convergence, $\Delta t \propto n^{-1}$ would be enough, but stability imposes $\Delta t \propto n^{-2}$.