MA5233 Computational Mathematics

Lecture 19: Theory of PDEs

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Theory of PDEs

Differential equation

Equation in terms of an unknown function $u : \Omega \to \mathbb{R}^n$ and its derivatives which is to hold at every point in a connected domain $\Omega \subset \mathbb{R}^m$.

Solution is typically only unique if we also impose values of u and its derivatives on $\partial\Omega$.

- Differential equation is called *ordinary* or *initial value problem* if $\Omega \subset \mathbb{R}$ and we impose values of u and its derivatives at a single point.

  Example: $\dot{y}(t) = f\big(y(t)\big)$ for all $t \in [0, T]$, $\quad y(0) = y_0$.

- Differential equation is called *partial* or *boundary value problem* in all other cases.

  Example: $$\begin{aligned} -\Delta u(x) &= f(x) \text{ for all } x \in \Omega, \\ u(x) &= 0 \text{ for all } x \in \partial\Omega. \end{aligned}$$

Focus for the next few lectures: partial differential equations.
Focus for today: developing a mathematically sound theory of PDEs.

Motivating examples:

`https://youtu.be/ureGelZPi3o`, `https://youtu.be/O0kyDKu8K-k`

Theory of PDEs

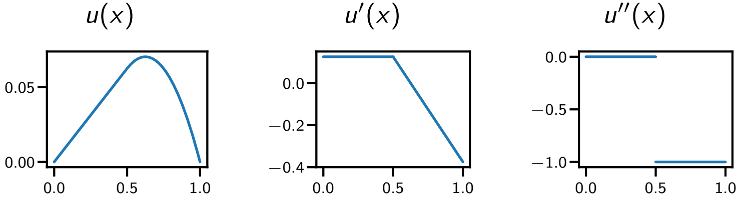
Introductory example

Consider the Poisson equation with Dirichlet boundary conditions,

$$-u'' = f \text{ on } [0,1], \qquad u(0) = u(1) = 0.$$

A pair f, u which "solves" this equation is given by

$$f(x) = \begin{cases} 0 & x < 0.5, \\ 1 & x > 0.5, \end{cases} \qquad u(x) = \begin{cases} \frac{x}{8} & x < 0.5, \\ \frac{(x-1/4)\,(1-x)}{2} & x > 0.5, \end{cases}$$



Observation: $u''(x)$ does not exist at $x = \frac{1}{2}$!

In what sense is $-u''(\frac{1}{2}) = f(\frac{1}{2})$ satisfied?

Introductory example (continued)

In example, we have $-u''(x) = f(x)$ for all $x \in [0, 1]$ except possibly at a single point $x = \frac{1}{2}$ depending on how we define $u''(x)$.

Since $\int f(x)\,dx = \int g(x)\,dx$ if $f(x) = g(x)$ except at a single point x, we have for any continuous function $v(x)$ and independently of how we define $u''(\frac{1}{2})$ that

$$-\int_0^1 u''(x)\,v(x)\,dx = \int_0^1 f(x)\,v(x)\,dx. \tag{1}$$

Conversely, if u satisfies (1) for all continuous functions $v(x)$ and $f(x), u''(x)$ are continuous, then we can set $v(x) = u''(x) + f(x)$ and obtain

$$0 = \int_0^1 \Big(f(x) + u''(x)\Big)\,v(x)\,dx = \int_0^1 \Big(f(x) + u''(x)\Big)^2\,dx$$

which shows that $-u''(x) = f(x)$ for all $x \in [0, 1]$.

Thus, equation (1) is in some sense equivalent to $-u''(x) = f(x)$.

More precise statements on next slide.

Introductory example (continued)

Repeated from previous slide for convenience:

$$-\int_0^1 u''(x)\, v(x)\, dx = \int_0^1 f(x)\, v(x)\, dx. \tag{1}$$

Summary:

- If $-u''(x) = f(x)$ holds for all $x \in [0, 1]$, then (1) holds for all continuous $v(x)$.
- If (1) holds for all continuous $v(x)$ and $u''(x), f(x)$ are continuous, then $-u''(x) = f(x)$ holds for all $x \in [0, 1]$.
- For the concrete pair f, u given on slide 3, (1) holds for all continuous $v(x)$ while $-u''(x) = f(x)$ does not make sense.

Conclusion: mathematical issues regarding how to interpret $-u''(x) = f(x)$ for discontinuous $f(x)$ disappear if we *define* that $u(x)$ solves $-u''(x) = f(x)$ if and only if (1) is satisfied for all continuous $v(x)$.

Theory of PDEs

Topics to discuss next

- Introduce a generalised notion of derivatives for functions which are not differentiable in the classical sense.
- Continue "reinterpreting" $-u''(x) = f(x)$ until we arrive at a form for which we can prove existence and uniqueness of solutions.

Theory of PDEs

Def: C^k and L^2 function spaces

Consider $f : [a, b] \to \mathbb{R}$. We define:

- $f \in C^k([a, b])$ if f has k continuous derivatives.
- $f \in C_0^k([a, b])$ if $f \in C^k([a, b])$ and $f(a) = f(b) = 0$.
- $f \in L^2([a, b])$ if $\int_a^b f(x)^2\, dx$ is well-defined and finite.

Def: L^2 inner product and norm

$$\langle f, g\rangle_{L^2([a,b])} := \int_a^b f(x)\, g(x)\, dx,$$

$$\|f\|_{L^2([a,b])} := \sqrt{\langle f, f\rangle_{L^2([a,b])}} = \sqrt{\int_a^b f(x)^2\, dx}.$$

By Cauchy-Schwarz inequality, we have

$$\langle f, g\rangle_{L^2([a,b])} \leq \|f\|_{L^2([a,b])}\, \|g\|_{L^2([a,b])};$$

hence $\langle f, g\rangle_{L^2([a,b])}$ is bounded for all $f, g \in L^2([a, b])$.

Remark on L^2 norm

Strictly speaking, $\|f\|_{L^2([a,b])}$ is not a norm since for e.g.

$$f(x) = \begin{cases} 1 & \text{if } x = 0.5, \\ 0 & \text{otherwise} \end{cases}$$

we get $\|f\|_{L^2([0,1])} = 0$ but $f \neq 0$.

We fix this by interpreting a function $f \in L^2([a,b])$ as a *representative* of the set of all functions $g : [a,b] \to \mathbb{R}$ such that $\|f - g\|_{L^2([a,b])} = 0$.

Simply put, we *define* that $f = g$ for $f, g \in L^2([a,b])$ if $\|f - g\|_{L^2([a,b])} = 0$.

Important consequence: the value $f(x)$ of $f \in L^2([a,b])$ at a single point x is not well defined.

Theory of PDEs

Def: Weak derivative

Let $f, g \in L^2([a,b])$. We say g is a *weak derivative of f* if for all $v \in C_0^1([a,b])$ we have

$$\int_a^b f(x)\, v'(x)\, dx = -\int_a^b g(x)\, v(x)\, dx.$$

Such weak derivatives g may not exist for a given $f \in L^2([a,b])$, but if they do they are unique and we write $f' := g$.

If $f \in C^1$, then the weak derivatives exists and it agrees with the classical derivative.

Rationale. We obtain using integration by parts and $v \in C_0^1([a,b]) \implies v(a) = v(b) = 0$ that

$$\int_a^b \underbrace{f(x)}_{\downarrow}\, \underbrace{v'(x)}_{\uparrow}\, dx = f(b)\, \underbrace{v(b)}_{0} - f(a)\, \underbrace{v(a)}_{0} - \int_a^b f'(x)\, v(x)\, dx$$

$$= -\int_a^b f'(x)\, v(x)\, dx.$$

Theory of PDEs

Thm: Weak derivatives of piecewise C^1 functions

Assume $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and continuously differentiable except on a countable set $X \subset [a, b]$. Then, the weak derivative $f'(x)$ exists and it agrees with the classical derivative on $[a, b] \setminus X$.

Proof. Assume $X = \{\hat{x}\}$ for simplicity (result for more points in X can be shown analogously). Then, we have for every $v \in C_0^1([a, b])$ that

$$\begin{aligned}
\int_a^b f(x)\, v'(x)\, dx &= \int_a^{\hat{x}} f(x)\, v'(x)\, dx + \int_{\hat{x}}^b f(x)\, v'(x)\, dx \\
&= f(\hat{x})\, v(\hat{x}) - f(a)\, v(a) - \int_a^{\hat{x}} f'(x)\, v(x)\, dx + \dots \\
&\qquad f(b)\, v(b) - f(\hat{x})\, v(\hat{x}) - \int_{\hat{x}}^b f'(x)\, v(x)\, dx \\
&= -\int_a^b f'(x)\, v(x)\, dx.
\end{aligned}$$

Theory of PDEs

Introductory example (continued)

With the notation developed on the last few slides, we may formulate our reinterpretation of $-u''(x) = f(x)$ from slide 5 in a precise way as follows.

We say $u : [0,1] \to \mathbb{R}$ "solves" $-u''(x) = f(x)$ if u has two weak derivatives and for all $v \in C^0([0,1])$ we have that

$$-\int_0^1 u''(x)\, v(x)\, dx = \int_0^1 f(x)\, v(x)\, dx. \tag{1}$$

However, it turns out that this is still not the most mathematically convenient formulation of the problem. In order to get to the final formulation, we need some more notation.

Theory of PDEs

Def: Sobolev spaces

Consider $f \in L^2([a,b])$. We define:

- $f \in H^k([a,b])$ if f has k weak derivatives.
- $f \in H_0^k([a,b])$ if $f \in H^k([a,b])$ and $f^{(\ell)}(a) = f^{(\ell)}(b) = 0$ for all $\ell \in \{0, \ldots k-1\}$.

Remark: boundary conditions in Sobolev spaces

According to definition, we have $f \in H^1([a,b])$ if $f \in L^2(\Omega)$, f has one weak derivative and $f(a) = f(b) = 0$.

It is not obvious that the last requirement makes sense:
since f is in $L^2([a,b])$, it is only defined up to modification with $\delta f \in L^2([a,b])$ such that $\|\delta f\|_{L^2([a,b])} = 0$, see "Remark on L^2 norm".

This issue is resolved as follows:
It can be shown that if $f \in H^1([a,b])$, then there exists exactly one δf of the above form such that $f + \delta f \in C^0([a,b])$.
The intended meaning of $f(a) = f(b) = 0$ is that this condition is satisfied for the representative of $f \in L^2([a,b])$ which is also in $C^0([a,b])$.

Theory of PDEs

Def: Sobolev inner product and spaces

Given $f, g \in H^k([a,b])$, we define

$$\langle f, g \rangle_{H^k([a,b])} := \sum_{\ell=0}^{k} \langle f^{(\ell)}, g^{(\ell)} \rangle_{L^2([a,b])},$$

$$\|f\|_{H^k([a,b])} := \sqrt{\langle f, f \rangle_{H^k([a,b])}}.$$

Since the weak derivatives $f^{(\ell)}, g^{(\ell)}$ are in $L^2([a,b])$, it follows that $\langle f, g \rangle_{H^k([a,b])}$ and $\|f\|_{H^k([a,b])}$ are bounded for $f, g \in H^k([a,b])$.

Example

$$\langle f, g \rangle_{H^1([a,b])} = \int_a^b f(x)\, g(x)\, dx + \int_a^b f'(x)\, g'(x)\, dx.$$

Theory of PDEs

Hilbert space

A vector space V equipped with an inner product $\langle f, g \rangle_V$ is called a *Hilbert space* if V is complete under the norm $\|f\|_V := \sqrt{\langle f, f \rangle_V}$.

Examples

- $\mathbb{R}^n$ with inner product $\langle a, b \rangle := a^T b$.
- $L^2([a, b])$ with L^2 inner product.
- $H^1([a, b])$ and $H_0^1([a, b])$ with H^1 inner product.

Theory of PDEs

Def: Weak solution to Poisson's equation

We say $u : [0,1] \to \mathbb{R}$ is a weak solution to the Poisson equation

$$-u'' = f \text{ on } [0,1], \qquad u(0) = u(1) = 0, \tag{2}$$

if $u \in H_0^1([0,1])$ and for all $v \in H_0^1([0,1])$ it holds

$$\int_0^1 u'(x)\, v'(x)\, dx = \int_0^1 f(x)\, v(x)\, dx.$$

Remarks

- If $u \in H_0^1([0,1]) \cap C^2([0,1])$ is a weak solution, then u satisfies (2).
- If $u \in C^2([0,1])$ satisfies (2), then u is a weak solution.
- If $u \in C^1([a,b])$, $u \in C^2([0,1] \setminus X)$ for some countable set $X \subset [0,1]$ and u satisfies (2) except on X, then u is a weak solution.

It follows from the third point that the solution from the introductory example is a weak solution.

Theory of PDEs

Abstract weak formulation of Poisson's equation

The key point of reinterpreting $-u'' = f$ according to the definition on previous slide is that the problem can now be formulated as follows:

> Given Hilbert space V, bilinear $a : V \times V \to \mathbb{R}$ and linear $b : V \to \mathbb{R}$, find $u \in V$ such that
>
> $$a(u, v) = b(v) \qquad \text{for all } v \in V.$$

For Poisson's equation, we have $V = H_0^1([0, 1])$,

$$a(u, v) := \int_0^1 u'(x)\, v'(x)\, dx, \qquad b(v) := \int_0^1 f(x)\, v(x)\, dx.$$

Terminology

- Bilinear $a : V \times V \to \mathbb{R}$ is called *bilinear form*.
- Linear $b : V \to \mathbb{R}$ is called *functional*.

Theory of PDEs

Lax-Milgram theorem

The abstract problem from previous slide has a unique solution and it holds $\|u\|_V \leq \frac{B}{A}$ if all of the following conditions are satisfied:

- $a(u,v)$ is bounded: $\exists A > 0$ such that $|a(u,v)| \leq A \|u\|_V \|v\|_V$.
- $a(u,v)$ is coercive: $\exists c > 0$ such that $a(v,v) \geq c\|v\|_V^2$.
- $b(v)$ is bounded: $\exists B > 0$ such that $|b(v)| \leq B \|v\|_V$.

If $a(u,v) = a(v,u)$ is symmetric, then LM is also called Riesz representation theorem.

Application of Lax-Milgram to Poisson equation

Boundedness of $a(u,v)$ and $b(v)$ is straightforward:

$$\begin{aligned} |a(u,v)| &= \left|\langle u', v'\rangle_{L^2([0,1])}\right| \\ &\leq \|u'\|_{L^2([0,1])} \|v'\|_{L^2([0,1])} \\ &\leq \|u\|_{H^1([0,1])} \|v\|_{H^1([0,1])} \end{aligned} \qquad \begin{aligned} |b(v)| &= \left|\langle f, v\rangle_{L^2([0,1])}\right| \\ &\leq \|f\|_{L^2([0,1])} \|v\|_{L^2([0,1])} \\ &\leq \|f\|_{L^2([0,1])} \|v\|_{H^1([0,1])} \end{aligned}$$

Hence $A = 1$ and $B = \|f\|_{L^2([0,1])}$.

Showing coercivity of $a(u,v)$ requires Poincaré's inequality, see next slide.

Coercivity of Poisson's equation

For $V = H_0^1([0,1])$, showing coercivity amounts to finding $c > 0$ such that

$$0a(v,v) \geq c\left(\|v\|^2_{L^2([0,1])} + \|v'\|^2_{L^2([0,1])}\right). \tag{3}$$

By definition, we have that $a(v,v) = \langle v', v' \rangle_{L^2([0,1])} = \|v'\|^2_{L^2([0,1])}$; hence if we can find $C > 0$ such that

$$\|v\|_{L^2([0,1])} \leq C\,\|v'\|_{L^2([0,1])}, \tag{4}$$

we get

$$\|v\|^2_{L^2([0,1])} + \|v'\|^2_{L^2([0,1])} \leq (C^2+1)\,\|v'\|^2_{L^2([0,1])} = (C^2+1)\,a(v,v)$$

which is (3) with $c = (C^2+1)^{-1}$.

The bound (4) indeed holds, and it is known as Poincaré inequality.

Poincaré inequality

We have $\|v\|_{L^2([0,1])} \leq C \|v'\|_{L^2([0,1])}$ for all $v \in H_0^1([0,1])$ and some $C > 0$ independent of v.

Discussion

- Loosely speaking, Poincaré's inequality says v' small $\implies$ v small.
- This is not true in general: $v(x) := C$ with $C \in \mathbb{R}$ has derivative $v'(x) = 0$ but $v(x)$ can be arbitrarily large.
- Poincaré's inequality can hold because we restrict $v \in H_0^1([0,1])$, which limits the above counterexample to $C = 0$.

Theory of PDEs

What you should remember from this lecture

- Idea of weak derivatives.
- Function spaces $C^k, C_0^k, L^2, H^1, H_0^1$, their inner products and norms.
- Weak formulation of Poisson's equation.
- Lax-Milgram theorem.
- Poincaré inequality.

Final remarks

This module (MA5233) is about solving PDEs numerically, and this lecture is intended to provide a minimal background for the discussion of such numerical methods.

If you want to know more about the theory of PDEs, take a module which is fully dedicated to this topic, e.g. MA4211 or MA5206.