MA5233 Computational Mathematics

Lecture 19: Theory of PDEs

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Theory of PDEs

Differential equation

Equation in terms of an unknown function $u : \Omega \to \mathbb{R}^n$ and its derivatives which is to hold at every point in a connected domain $\Omega \subset \mathbb{R}^m$.

Solution is typically only unique if we also impose values of u and its derivatives on $\partial\Omega$.

- Differential equation is called *ordinary* or *initial value problem* if $\Omega \subset \mathbb{R}$ and we impose values of u and its derivatives at a single point.

  Example: $\dot{y}(t) = f\big(y(t)\big)$ for all $t \in [0, T]$, $\quad y(0) = y_0$.

- Differential equation is called *partial* or *boundary value problem* in all other cases.

  Example:
  $$-\Delta u(x) = f(x) \text{ for all } x \in \Omega,$$
  $$u(x) = 0 \text{ for all } x \in \partial\Omega.$$

Focus for the next few lectures: partial differential equations.
Focus for today: developing a mathematically sound theory of PDEs.

Motivating examples:

`https://youtu.be/ureGelZPi3o`, `https://youtu.be/O0kyDKu8K-k`

Theory of PDEs

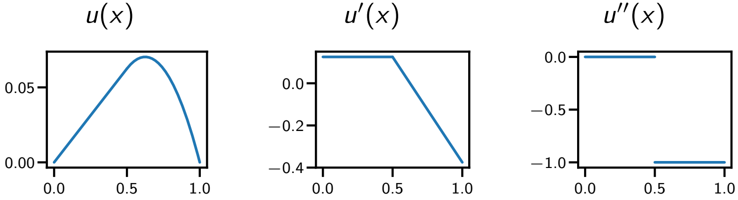
Introductory example

Consider the Poisson equation with Dirichlet boundary conditions,

$$-u'' = f \text{ on } [0,1], \qquad u(0) = u(1) = 0.$$

A pair f, u which "solves" this equation is given by

$$f(x) = \begin{cases} 0 & x < 0.5, \\ 1 & x > 0.5, \end{cases} \qquad u(x) = \begin{cases} \frac{x}{8} & x < 0.5, \\ \frac{(x-1/4)\,(1-x)}{2} & x > 0.5, \end{cases}$$



Observation: $u''(x)$ solution is discontinuous at $x = \frac{1}{2}$!

In what sense is $-u''(\frac{1}{2}) = f(\frac{1}{2})$ satisfied?

Theory of PDEs

Introductory example (continued)

In example, we have $-u''(x) = f(x)$ for all $x \in [0,1]$ except at a single point $x = \frac{1}{2}$.

In particular, for any continuous function $\phi(x)$ we have

$$-\int_0^1 u''(x)\,\phi(x)\,dx = \int_0^1 f(x)\,\phi(x)\,dx. \tag{1}$$

Conversely, if (1) is satisfied for all continuous functions $\phi(x)$ and $f(x), u''(x)$ are continuous, then we can set $\phi(x) = u''(x) + f(x)$ and obtain

$$0 = \int_0^1 \Big(f(x) + u''(x)\Big)\,\phi(x)\,dx = \int_0^1 \Big(f(x) + u''(x)\Big)^2\,dx$$

which shows that $-u''(x) = f(x)$ for all $x \in [0,1]$.

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Introductory example (continued)

Repeated from previous slide for convenience:

$$-\int_0^1 u''(x)\,\phi(x)\,dx = \int_0^1 f(x)\,\phi(x)\,dx. \tag{1}$$

Summary:

- If $-u''(x) = f(x)$ holds for all $x \in [0,1]$, then (1) holds for all continuous $\phi(x)$.
- If (1) holds for all continuous $\phi(x)$ and $u''(x), f(x)$ are continuous, then $-u''(x) = f(x)$ holds for all $x \in [0,1]$.
- For the concrete pair f, u given above, (1) holds for all continuous $\phi(x)$ while $-u''(x) = f(x)$ does not make sense.

Conclusion: we may *define* that $u(x)$ solves $-u''(x) = f(x)$ if (1) holds for all continuous $\phi(x)$.

Unlike $-\Delta u = f$, which we derived from a flux law and conservation of mass (see Lecture 5), the above reinterpretation of this PDE has no physical meaning.
It is just a mathematical trick to arrive at a sound theory.
Also, note that this is not the final definition of what we mean by $-u''(x) = f(x)$.

Theory of PDEs

Def: Function spaces

Consider function $f : [a,b] \to \mathbb{R}$. We define:

- $f \in C^k([a,b])$ if f has k continuous derivatives.
- $f \in C_0^k([a,b])$ if $f \in C^k([a,b])$ and $f(a) = f(b) = 0$.
- $f \in L^2([a,b])$ if $\int_a^b f(x)^2 \, dx$ is well-defined and finite.

Def: L^2 inner product and norm

$$\langle f, g \rangle_{L^2([a,b])} := \int_a^b f(x)\, g(x) \, dx,$$

$$\|f\|_{L^2([a,b])} := \sqrt{\langle f, f \rangle_{L^2([a,b])}} = \sqrt{\int_a^b f(x)^2 \, dx}.$$

By Cauchy-Schwarz inequality, we have

$$\langle f, g \rangle_{L^2([a,b])} \leq \|f\|_{L^2([a,b])} \, \|g\|_{L^2([a,b])};$$

hence $\langle f, g \rangle_{L^2([a,b])}$ is bounded for all $f, g \in L^2([a,b])$.

Theory of PDEs

Remark on L^2 norm

Strictly speaking, $\|f\|_{L^2([a,b])}$ is not a norm since for e.g.

$$f(x) = \begin{cases} 1 & \text{if } x = 0.5, \\ 0 & \text{otherwise} \end{cases}$$

we get $\|f\|_{L^2([0,1])} = 0$ but $f \neq 0$.

We fix this by interpreting a function $f \in L^2([a,b])$ as a *representative* of the set of all functions $g : [a,b] \to \mathbb{R}$ such that $\|f - g\|_{L^2([a,b])} = 0$.

Simply put, we define that $f = g$ for $f, g \in L^2([a,b])$ if $\|f - g\|_{L^2([a,b])} = 0$.

Important consequence: the value of function $f \in L^2([a,b])$ at a single point x is not well defined.

Theory of PDEs

Def: Weak derivative

Let $f, g \in L^2([a, b])$. We say g is a *weak derivative of f* if for all $\phi \in C_0^\infty([a, b])$ we have

$$\int_a^b f(x)\, \tfrac{\partial \phi}{\partial x_i}(x)\, dx = -\int_a^b g(x)\, \phi(x)\, dx.$$

Such weak derivatives g may not exist for a given $f \in L^2([a, b])$, but if they do they are unique and we write $\frac{\partial f}{\partial x} := g$.
If $f \in C^1$, then the weak derivatives exists and it agrees with the classical derivative.

Rationale. We obtain using integration by parts and $\phi \in C_0^\infty([a, b]) \iff \phi(a) = \phi(b) = 0$ that

$$\int_a^b \underbrace{f(x)}_{\downarrow}\, \underbrace{\tfrac{\partial \phi}{\partial x}(x)}_{\uparrow}\, dx = f(b)\, \underbrace{\phi(b)}_{0} - f(a)\, \underbrace{\phi(a)}_{0} - \int_a^b \tfrac{\partial f}{\partial x}(x)\, \phi(x)\, dx$$

$$= -\int_a^b \tfrac{\partial f}{\partial x}(x)\, \phi(x)\, dx.$$

Theory of PDEs

Thm: Weak derivatives of piecewise C^1 functions

Assume $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and continuously differentiable except on a countable set $X \subset [a,b]$. Then, the weak derivative $\frac{\partial f}{\partial x}$ exists and it agrees with the classical derivative on $[a,b] \setminus X$.

Proof. Assume $X = \{\hat{x}\}$ for simplicity (result for more points in X can be shown analogously). Then, we have for every $\phi \in C_0^\infty([a,b])$ that

$$\begin{aligned}
\int_a^b f(x)\,\tfrac{\partial \phi}{\partial x}(x)\,dx &= \int_a^{\hat{x}} f(x)\,\tfrac{\partial \phi}{\partial x}(x)\,dx + \int_{\hat{x}}^b f(x)\,\tfrac{\partial \phi}{\partial x}(x)\,dx \\
&= f(\hat{x})\,\phi(\hat{x}) - f(a)\,\phi(a) - \int_a^{\hat{x}} \tfrac{\partial f}{\partial x}(x)\,\phi(x)\,dx + \dots \\
&\qquad f(b)\,\phi(b) - f(\hat{x})\,\phi(\hat{x}) - \int_{\hat{x}}^b \tfrac{\partial f}{\partial x}(x)\,\phi(x)\,dx \\
&= -\int_a^b \tfrac{\partial f}{\partial x}(x)\,\phi(x)\,dx.
\end{aligned}$$

Theory of PDEs

Introductory example (continued)

We have seen that there are pairs f, u which "solve" $-u''(x) = f(x)$ in some sense even though $u \notin C^2$.

We have seen that a reasonable reformulation of the PDE is as follows:

> We say $u : [0,1] \to \mathbb{R}$ "solves" $-u''(x) = f(x)$ if u has two weak derivatives and for all $\phi \in C^0([0,1])$ we have that
>
> $$-\int_0^1 u''(x)\,\phi(x)\,dx = \int_0^1 f(x)\,\phi(x)\,dx.$$

However, it turns out that this still is not the most convenient formulation of the problem. If we restrict ϕ to $\phi \in C_0^1([0,1])$, then we obtain by integration by parts that

$$\int_0^1 u'(x)\,\phi'(x)\,dx = \int_0^1 f(x)\,\phi(x)\,dx.$$

This is closer to our final definition of what we mean by $-u''(x) = f(x)$, but it is not quite the final definition yet. In order to get there, we need some more notation.

Theory of PDEs

Def: Sobolev spaces

Consider $f \in L^2([a,b])$. We define:

- $f \in H^k([a,b])$ if f has k weak derivatives.
- $f \in H_0^k([a,b])$ if $f \in H^k([a,b])$ and $\frac{\partial^\ell f}{\partial x^\ell}(a) = \frac{\partial^\ell f}{\partial x^\ell}(b) = 0$ for all $\ell \in \{0, \dots k-1\}$.

Remark: boundary conditions in Sobolev spaces

According to definition, we have $f \in H^1([a,b])$ if $f \in L^2(\Omega)$, f has one weak derivative and $f(a) = f(b) = 0$.

It is not obvious that the last requirement makes sense: since f is only in $L^2([a,b])$, it is only defined up to modification with a function $\delta f \in L^2([a,b])$ such that $\|\delta f\|_{L^2([a,b])} = 0$, see "Remark on L^2 norm".

However, in one dimension it can be shown that if $f \in H^1([a,b])$, then there exists exactly one δf of the above form such that $f + \delta f \in C^0([a,b])$.

The intended meaning of $f(a) = f(b) = 0$ is that this condition is satisfied for the representative of $f \in L^2([a,b])$ which is also in $C^0([a,b])$.

Theory of PDEs

Def: Weak solution to Poisson's equation

We say $u : [a, b] \to \mathbb{R}$ is a weak solution to the Poisson equation

$$-u'' = f \text{ on } [a, b], \qquad u(a) = u(b) = 0, \tag{2}$$

if $u \in H_0^1([a, b])$ and for all $v \in H_0^1([a, b])$ it holds

$$\int_a^b u'(x)\, v'(x)\, dx = \int_a^b f(x)\, v(x)\, dx.$$

Remarks

- If $u \in C^2([a, b])$ satisfies (2), then u is a weak solution.
- If $u \in C^1([a, b])$, $u \in C^2([a, b] \setminus X)$ for some countable set $X \subset [a, b]$ and u satisfies (2) except on x, then u is a weak solution.

It follows from the second point that the solution from the introductory example is a weak solution.

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[to be continued]