MA5233 Computational Mathematics

Lecture 20: Galerkin's Method

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Galerkin's Method

Recap: Weak formulation of Poisson's equation

We have seen in Lecture 19 that Poisson's equation with homogeneous Dirichlet boundary conditions,

$$-u''(x) = f(x), \qquad u(0) = u(1) = 0,$$

is more reasonably interpreted as the following problem.
Given $f \in L^2([0,1])$, find $u \in H_0^1([0,1])$ such that for all $v \in H_0^1([0,1])$ we have

$$\int_0^1 u'(x)\, v'(x)\, dx = \int_0^1 f(x)\, v(x)\, dx. \tag{1}$$

An obvious way to determine approximate numerical solutions $u_n \approx u$ to this problem is as follows.

Galerkin's method

Given n-dimensional subspace $V_n \subset H_0^1([0,1])$, find $u_n \in V_n$ such that (1) holds for all $v \in V_n$.

Galerkin's Method

Reducing Galerkin's problem to a linear system

Assume we have a basis $\phi_1(x), \ldots, \phi_n(x)$ for $V_n \subset H_0^1([0,1])$.

Then, we have

$$u_n(x) = \sum_{k=1}^{n} c_k \, \phi_k(x)$$

and the coefficients $c \in \mathbb{R}^n$ can be determined as the solution to the linear system $Ac = b$ where

$$A_{k\ell} := \int_0^1 \phi_k'(x) \, \phi_\ell'(x) \, dx, \qquad b_k := \int_0^1 f(x) \, \phi_k(x) \, dx.$$

Proof. It follows from the linearity of $a(u,v)$ and $b(v)$ that if $a(u,v) = b(v)$ holds for $v \in \{\phi_1, \ldots, \phi_n\}$, then it holds for all $v \in V_n$.

Furthermore, we have

$$a(u, \phi_k) = \textstyle\sum_{\ell=1}^{n} c_\ell \, a(\phi_k, \phi_\ell)$$

and thus $a(u, \phi_k) = b(\phi_k)$ with $k \in \{1, \ldots, n\}$ yields n linear equations for the c_ℓ which have a unique solution according to Lax-Milgram theorem.

Galerkin's Method

Example: Galerkin's method with sine functions

Consider the space V_n spanned by the trigonometric polynomials

$$\phi_k(x) := \sqrt{2}\,\sin(\pi k x).$$

We have $\sin(\pi k x) \in C_0^\infty([0,1])$; hence $V_n \subset H_0^1([0,1])$.

The Galerkin matrix A is given by

$$\begin{aligned} A_{k\ell} &= 2\int_0^1 \Big(\tfrac{\partial}{\partial x}\sin(\pi k x)\Big)\Big(\tfrac{\partial}{\partial x}\sin(\pi \ell x)\Big)\,dx \\ &= 2\pi^2 k\ell \int_0^1 \cos(\pi k x)\cos(\pi \ell x)\,dx = \pi^2 k\ell\,\delta_{k\ell}, \end{aligned}$$

i.e. $A_{k\ell}$ is a diagonal matrix with diagonal $A_{kk} = (\pi k)^2$.

The formula for A can also be derived from $-\phi_k''(x) = (\pi k)^2\,\phi_k(x)$.

$\int_0^1 \cos(\pi k x)\cos(\pi \ell x)\,dx = \frac{1}{2}\,\delta_{k\ell}$ can be shown by expanding $\cos(x) = \frac{1}{x}\big(e^{\iota x} + e^{-\iota x}\big)$.

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Example: Galerkin's method with sine functions (continued)

We compute b using a composite trapezoidal rule approximation,

$$b_k = \sqrt{2}\int_0^1 f(x)\,\sin(\pi k x) \approx \tfrac{\sqrt{2}}{n+1}\sum_{i=1}^{n} f(\tfrac{i}{n+1})\,\sin(\pi k \tfrac{i}{n+1}),$$

which can be evaluated using a `RODFT00`-type sine transform, see `http://www.fftw.org/fftw3_doc/1d-Real_002dodd-DFTs-_0028DSTs_0029.html`.

Once $c := A^{-1}b$ has been computed, we can evaluate the approximate solution $u_n(x)$ at arbitrary points x using

$$u_n(x) := \sqrt{2}\sum_{k=1}^{n} c_k \sin(\pi k x).$$

Observation

Finite difference solution was given as vector of point values $\tilde{u}_k \approx u(\frac{k}{n+1})$.
Galerkin solution is given as a function $u_n(x) \in V_n$.

Galerkin's Method

Error analysis for Galerkin's method

As usual, we are interested in an estimate $\|u - u_n\| = \mathcal{O}\big(f(n)\big)$.

Such an estimate will follow easily from the following observation.

Galerkin orthogonality

Assume V is a vector space, $V_n \subset V$, and $a : V \times V \to \mathbb{R}$, $b : V \to \mathbb{R}$ are linear in all arguments.

Assume u and u_n satisfy, respecively,

$$\begin{aligned} a(u, v) &= b(v) \qquad \text{for all } v \in V, \\ a(u_n, v_n) &= b(v_n) \qquad \text{for all } v_n \in V_n. \end{aligned}$$

Then, we have

$$a(u - u_n, v_n) = 0 \qquad \text{for all } v_n \in V_n.$$

Proof. Note that $v_n \in V_n$ and $V_n \subset V$ implies $v_n \in V$. Therefore,

$$a(u - u_n, v_n) = a(u, v_n) - a(u_n, v_n) = b(v_n) - b(v_n) = 0.$$

Galerkin's Method

Céa's lemma

Assume V, $a(u,v)$ and $b(v)$ are as in Lax-Milgram theorem, i.e.

- $a(u,v)$ and $b(v)$ are linear in all arguments.
- $a(u,v)$ is bounded: $\exists A > 0$ such that $|a(u,v)| \leq A \, \|u\|_V \, \|v\|_V$.
- $a(u,v)$ is coercive: $\exists c > 0$ such that $a(v,v) \geq c\|v\|_V^2$.
- $b(v)$ is bounded: $\exists B > 0$ such that $|b(v)| \leq B \, \|v\|_V$.

Assume $V_n \subset V$ is a linear subspace, and u and u_n satisfy, respectively,

$$\begin{aligned} a(u,v) &= b(v) \qquad &\text{for all } v \in V, \\ a(u_n, v_n) &= b(v_n) \qquad &\text{for all } v_n \in V_n. \end{aligned}$$

Then,

$$\|u - u_n\|_V \leq \tfrac{A}{c} \inf_{v_n \in V_n} \|u - v_n\|_V.$$

Proof.

$$\begin{aligned} c \, \|u - u_n\|_V^2 &\leq a(u - u_n, u - u_n) \\ &= a(u - u_n, u - v_n) + \underbrace{a(u - u_n, v_n - u_n)}_{=0 \text{ (Galerkin orth.)}} \\ &\leq A \, \|u - u_n\|_V \, \|u - v_n\|_V. \end{aligned}$$

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Consequences of Céa's lemma

- Galerkin solution u_n is quasi-optimal, i.e. approximation error is within a constant factor of best approximation error for u in V_n.
- Asymptotics of $\|u - u_n\|_V$ can be derived from asymptotics of $\inf_{v_n \in V_n} \|u - v_n\|_V$.

Céa's lemma applied to Galerkin with sine functions

It can be shown that any $f \in H_0^1([0,1)$ can be expanded into a sine series

$$f(x) = \sqrt{2} \sum_{k=1}^{\infty} \hat{f}_k \sin\bigl(\pi k x\bigr) \quad \text{where} \quad \hat{f}_k := \sqrt{2} \int_0^1 f(x) \sin\bigl(\pi k x\bigr)\, dx,$$

and

$$\|f\|_{L^2([0,1])}^2 = \sum_{k=1}^{\infty} \hat{f}_k^2, \qquad \|f'\|_{L^2([0,1])}^2 = \sum_{k=1}^{\infty} (\pi k\, \hat{f}_k)^2.$$

Hence, $\|u - u_n\|_{H^1([0,1])}$ with $u_n \in V_n := \operatorname{span}\bigl\{\sin(\pi k x) \mid k \in \{1, \dots, n\}\bigr\}$ is minimised if we choose $u_n(x) = \sqrt{2} \sum_{k=1}^{n} \hat{u}_k \sin(\pi k x)$ since then

$$\|u - u_n\|_{H^1([0,1])}^2 = \sum_{k=n+1}^{\infty} (1 + \pi^2 k^2)\, \hat{u}_k^2.$$

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Céa's lemma applied to Galerkin with sine functions (continued)

Comparing coefficients in

$$-u''(x) = \sqrt{2}\sum_{k=1}^{\infty} \hat{u}_k\,(\pi k)^2 \sin(\pi k x) = \sqrt{2}\sum_{k=1}^{\infty} \hat{f}_k \sin(\pi k x) = f(x),$$

we conclude that $\hat{u}_k = \frac{\hat{f}_k}{(\pi k)^2}$. Furthermore, it follows from formulae for A and b from slides 4 and 5 that $(\hat{u}_n)_k = \frac{\hat{f}_k}{(\pi k)^2}$.

Hence, Galerkin solution u_n is *exactly* the best approximation of u in V_n.

Further observation

If $\hat{u}_k = \mathcal{O}(k^{-p})$ with $p > \frac{1}{2}$, then

$$\|u - u_n\|_{L^2([0,1])} = \sqrt{\textstyle\sum_{k=n+1}^{\infty} \hat{u}_k^2} = \mathcal{O}(n^{-p})$$

but

$$\|u' - u_n'\|_{L^2([0,1])} = \sqrt{\textstyle\sum_{k=n+1}^{\infty} (\pi k)^2\, \hat{u}_k^2} = \mathcal{O}(n^{-p+1}),$$

i.e. L^2-norm of error converges faster than H^1-norm by one factor of n.

This observation is typical and can be explained by Aubin-Nitsche lemma.

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Aubin-Nitsche lemma

Introduction:

- Céa's lemma applied to Poisson's equation yields error estimates in the H^1-norm.
- Many applications require error estimates in L^2-norm.
- Aubin-Nitsche lemma allows us to derive L^2 from H^1 estimates.
- We follow notation from Céa's lemma in the following.

Definitions:

- L^2 functional: $b^\star(v) := \int_0^1 \big(u(x) - u_n(x)\big)\, v(x)\, dx$.
  Note that $b^\star(u - u_n) = \|u - u_n\|^2_{L^2([0,1])}$.
- Dual problem: find $g \in H_0^1([0,1])$ such that

$$a(v, g) = b^\star(v) \qquad \text{for all } v \in H_0^1([0,1]).$$

Result:

$$\|u - u_n\|^2_{L^2([0,1])} \leq A\, \|u - u_n\|_{H^1([0,1])} \inf_{v \in V_n} \|g - v_n\|_{H^1([0,1])}.$$

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Proof of Aubin-Nitsche. We have for any $v_n \in V_n$ that

$$\begin{aligned} \|u - u_n\|^2_{L^2([0,1])} &= b^\star(u - u_n) = a(u - u_n, g) = a(u - u_n, g - v_n) \\ &\leq A \, \|u - u_n\|_{H^1([0,1])} \, \|g - v_n\|_{H^1([0,1])}. \end{aligned}$$

where in the third step we used Galerkin orthogonality $a(u - u_n, v_n) = 0$.

Aubin-Nitsche lemma applied to sine functions example

Analogously as on slide 9, we conclude

$$\hat{g}_k = \begin{cases} 0 & \text{if } k \leq n \\ \frac{\hat{u}_k}{(\pi k)^2} & \text{otherwise.} \end{cases}$$

If $\hat{u}_k = \mathcal{O}(k^{-p})$, Aubin-Nitsche estimate thus boils down to

$$\|u - u_n\|^2_{L^2} = \mathcal{O}\left(n^{-2p}\right) \leq \mathcal{O}\left(n^{-p+1}\right) \mathcal{O}\left(n^{-p-1}\right) = \|u - u_n\|_{H^1} \, \|g - v_n\|_{H^1}.$$

Discussion of Aubin-Nitsche lemma

For sine functions example, Aubin-Nitsche is just a complicated way of rederiving the observation from slide 9.

The value of Aubin-Nitsche lemma is that it also works in other settings.
(We will see an example in the next lecture.)

Aubin-Nitsche lemma in words:
L^2 error converges faster than H^1 error if dual solution g can be well approximated in V_n.

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Variational crimes

Computing $A_{k\ell} := \int_0^1 \phi'_k(x)\,\phi'_\ell(x)\,dx$ and $b_k := \int_0^1 f(x)\,\phi_k(x)\,dx$ will almost always require quadrature; hence the numerically computed u_n will involve further approximations than just replacing $V \to V_n$.

These extra approximations are known as *variational crimes*.

Rule of thumb: variational crimes do not affect speed of convergence if variational crime errors converge at least as fast as Galerkin error.

A more precise version of this rule is known as *Strang's lemma*.

Example

$$\text{Galerkin error is } \|u - u_n\| = \mathcal{O}(n^{-p})$$

$$\downarrow$$

When increasing n, increase number of quadrature points such that quadrature error is also $\mathcal{O}(n^{-p})$ (or better).