MA5233 Computational Mathematics

Lecture 12: Krylov Subspace Methods: Experiments

Simon Etter

National University of Singapore

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Krylov Subspace Methods: Experiments

Recap: Krylov subspace methods

Approximate solution to $Ax = b$ with $A \in \mathbb{K}^{N \times N}$ by

$$\tilde{x} := p_{n-1}(A)\, b \qquad \text{where} \qquad p_{n-1} := \underset{p_{n-1} \in \mathcal{P}_{n-1}}{\arg\min} \left\| \left(A p_{n-1}(A) - I \right) b \right\|.$$

Recap: convergence theory of Krylov subspace methods

$$\|A\,\tilde{x} - b\| \leq \kappa(V)\,\|b\| \min_{q_n \in \mathcal{P}_n} \max_{\lambda_k} \frac{|q_n(\lambda_k)|}{|q(0)|}.$$

If $\lambda_k \in [a, b]$ with $0 < a < b$, we have for $\kappa := \frac{b}{a}$ that

$$\min_{q_n \in \mathcal{P}_n} \max_{\lambda_k} \frac{|q_n(\lambda_k)|}{|q_n(0)|} \leq 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^n.$$

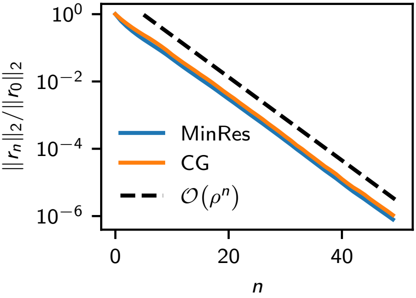
Aim of this lecture

Illustrate convergence theory with a few examples.

Krylov Subspace Methods: Experiments

Example: well-conditioned matrix

```
A = Diagonal(LinRange(1,50,1000))
b = rand(1000)
```



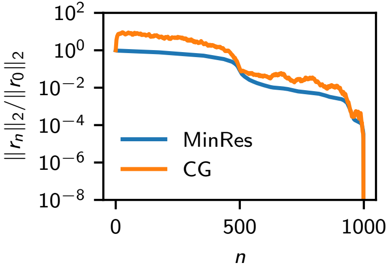
Observations

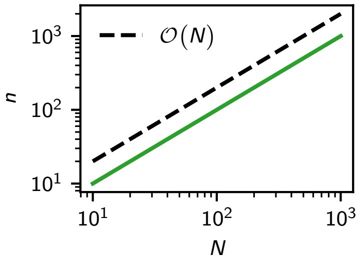
- As predicted, both MinRes and conjugate gradients converge exponentially with rate $\rho = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$ where $\kappa = 50$.

Krylov Subspace Methods: Experiments

Example: one-dimensional Poisson equation

```
A = -laplacian_1d(1000); b = rand(1000)
```





Right: number of CG iterations to achieve $\|A\tilde{x} - b\|_2 \leq 10^{-8}$.

Observations

- Virtually no convergence for $n < N$.
- Explanation: $\lambda_k \in [\pi^2, 4\,(N+1)^2]$. Hence (see next slide for last step)

$$\kappa = \mathcal{O}\big(N^2\big) \implies \rho = \tfrac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1} = \tfrac{1-\kappa^{-1/2}}{1+\kappa^{-1/2}} = 1-\mathcal{O}\big(N^{-1}\big) \implies n = \mathcal{O}(N).$$

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Theorem

Assume $\rho < 1$ but $\rho \approx 1$. Let $\varepsilon < 1$. Then,

$$\rho^n \approx \varepsilon \qquad \Longleftrightarrow \qquad n \approx \frac{|\log(\varepsilon)|}{1-\rho}.$$

Proof. Take logarithm of approximate equality on the left:

$$n \log(\rho) \approx \log(\varepsilon) \qquad \Longleftrightarrow \qquad n \approx \frac{\log(\varepsilon)}{\log(\rho)}.$$

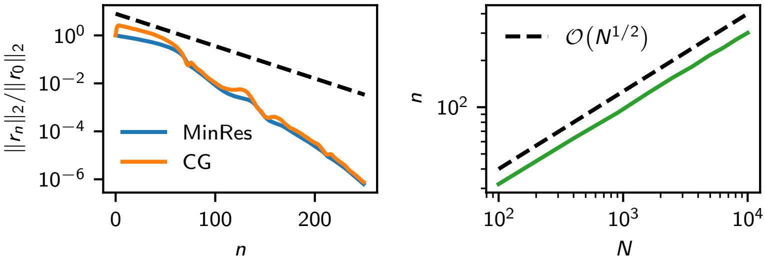
Use Taylor expansion $\log(1+x) = x + \mathcal{O}(x^2)$ for $x \to 0$ and $\log(\varepsilon) < 0$ to obtain

$$n \approx \frac{\log(\varepsilon)}{\log(\rho)} \approx \frac{\log(\varepsilon)}{\rho - 1} = \frac{|\log(\varepsilon)|}{1-\rho}.$$

Krylov Subspace Methods: Experiments

Example: two-dimensional Poisson equation

```
A = -laplacian_2d(100) # Laplacian on 100 x 100 grid
b = rand(100^2)
```



Right: number of CG iterations to achieve $\|A\tilde{x} - b\|_2 \leq 10^{-8}$.

Observations

- Number of steps $n = \mathcal{O}\big(N^{1/2}\big)$ to achieve fixed error reduction (right plot) is consistent with theory.
- Convergence history (left plot) is not consistent with theory. Theory provides only upper bound.

Krylov Subspace Methods: Experiments

ILU preconditioning

Recall: the main problem with LU factorisation was excessive fill-in.

Idea: simply discard any fill-in entries during factorisation.

The resulting factors $\tilde{L}, \tilde{U}$ will be wrong, i.e. $\tilde{L}\tilde{U} \neq A$, but maybe they are accurate enough that $P = \tilde{L}\tilde{U}$ is a good preconditioner?

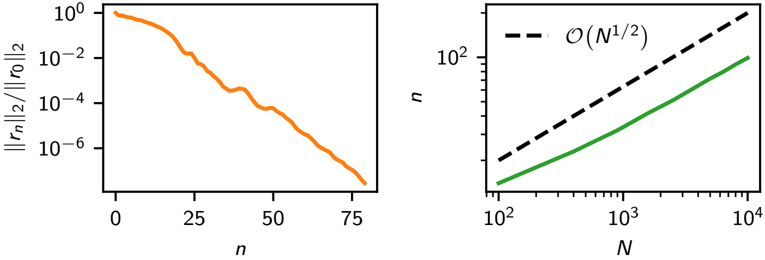
Remarks

- Allowing no fill-in at all is referred to as ILU(0).
- There are various versions of ILU preconditioners which allow some fill-in to improve the preconditioner quality.
- General idea: use LU for as much as we can afford, let Krylov method do the rest.

Krylov Subspace Methods: Experiments

Example: 2d Poisson equation with ILU(0) preconditioning

```
A = -laplacian_2d(100) # Laplacian on 100 x 100 grid
b = rand(100^2)
```



Right: number of CG iterations to achieve $\|A\tilde{x} - b\|_2 \leq 10^{-8}$.

Observations

- Convergence is faster, but $n = \mathcal{O}(N^{1/2})$ behaviour remains.