

MA5233 Computational Mathematics

Homework Sheet 4

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Recommended deadline: 19 September 2019

Final deadline: 19 September 2019, 7pm

Consider the following definitions.

- A matrix $A \in \mathbb{R}^{n \times n}$ is called *column-wise diagonally dominant* if for all $j \in \{1, \dots, n\}$ we have

$$A_{jj} \geq \sum_{i \neq j} |A_{ij}|.$$

The notation $i \neq j$ below the sum indicates that the sum is taken over $i \in \{1, \dots, n\} \setminus \{j\}$.

- A matrix $A \in \mathbb{R}^{n \times n}$ is called *symmetric positive semi-definite* (SPSD) if A is symmetric ($A^T = A$) and for all $v \in \mathbb{R}^n$ we have $v^T A v \geq 0$.

This homework sheet will

- examine the relationships between these two classes of matrices, and
- study the consequences of the above properties regarding pivoting in the LU factorisation.

You may have noticed that this homework sheet is somewhat longer than usual. To compensate for this, there will be no homework sheet released on the 12 September (next week), and this sheet will carry twice the weight compared to the sheets so far.

1 Alternative definition of symmetric positive semi-definite

Show that a symmetric matrix A is positive semi-definite if and only if all of its eigenvalues λ_k are nonnegative.

2 Diagonally dominant vs. symmetric positive semi-definite

1. Show Gershgorin's circle theorem, which states that the eigenvalues of a matrix $A \in \mathbb{C}^{n \times n}$ are contained in the union of all disks with centres $c = A_{jj}$ and radii $r = \sum_{i \neq j} |A_{ij}|$.

Hints.

- An alternative formulation of Gershgorin's circle theorem is

$$Ax = \lambda x \quad \implies \quad \lambda \in \bigcup_{j=1}^n D\left(A_{jj}, \sum_{i \neq j} |A_{ij}|\right)$$

where $D(c, r) := \{z \in \mathbb{C} \mid |z - c| \leq r\}$ denotes the disk in the complex plane with centre $c \in \mathbb{C}$ and radius $r > 0$.

- Consider a left eigenvector y , i.e. a vector y such that $y^T A = \lambda y^T$ for some $y \in \mathbb{R}^n$, and assume y is scaled such that $\|y\|_\infty = 1$. The claim then follows from the equation $y^T A = \lambda y^T$.
2. Show that a symmetric column-wise diagonally dominant matrix A is symmetric positive semi-definite.
 3. Provide a symmetric positive semi-definite 2×2 matrix which is not column-wise diagonally dominant.

3 Pivoting for diagonally dominant matrices

Assume the matrix

$$A = \begin{pmatrix} \alpha & b^T \\ c & D \end{pmatrix}$$

with $\alpha \in \mathbb{R}$, $c, b \in \mathbb{R}^{n-1}$ and $D \in \mathbb{R}^{(n-1) \times (n-1)}$ is column-wise diagonally dominant, and consider a single step of LU factorisation applied to this matrix,

$$\begin{pmatrix} \alpha & b^T \\ c & D \end{pmatrix} = \begin{pmatrix} 1 & \\ \frac{c}{\alpha} & I \end{pmatrix} \begin{pmatrix} \alpha & b^T \\ D - \frac{cb^T}{\alpha} \end{pmatrix}.$$

1. Show that the lower-right block $\tilde{A} := D - \frac{cb^T}{\alpha}$ of the resulting matrix is column-wise diagonally dominant.

Hint. You will need the inequalities

$$D_{jj} - |b_j| \geq \sum_{i \neq j} |D_{ij}| \quad \text{and} \quad \alpha - |c_j| \geq \sum_{i \neq j} |c_i|$$

which hold for all $j \in \{1, \dots, n-1\}$ and follow from the diagonal dominance of A . The notation $i \neq j$ below the sums indicates that the sum is taken over $i \in \{1, \dots, n-1\} \setminus \{j\}$.

2. Show that column-pivoting does not swap any rows when selecting the pivot for the first column of A .
3. Conclude that LU factorisation with column pivoting applied to A does not swap any rows.

4 Further remarks (unmarked)

One can show that if A is column-wise diagonally dominant, then its LU factors satisfy

$$\max_{ij} |L_{ij}| \leq 1, \quad \max_{ij} |U_{ij}| \leq 2 \max_{ij} |A_{ij}| \quad (1)$$

(the statement regarding L is easy to show, the statement regarding U requires quite a bit of work). If we assume e.g. the norm $\|M\|_\infty = \max_i \sum_j |M_{ij}|$, then this implies

$$\|L\|_\infty \leq n, \quad \|U\|_\infty \leq 2n \|A\|_\infty$$

and hence the exponential growth of the entries of the U -factor that we observed for the Wilkinson matrix on sheet 2 is not possible for column-wise diagonally dominant matrices.

If A is symmetric positive semi-definite, then column pivoting may swap rows. However, it can be shown that even without column pivoting, similar entry-wise bounds as in (1) hold; hence exponential blow-up of rounding errors as we have observed in the LU factorisation of the Wilkinson matrix is again not possible if A is symmetric positive semi-definite.

The following books contain more details regarding the above statements.

- B. Wendroff. *Theoretical Numerical Analysis*. DOI:10.1016/C2013-0-12559-7
- N. Higham. *Accuracy and Stability of Numerical Algorithms* DOI:10.1137/1.9780898718027