MA5233 Computational Mathematics

Homework Sheet 3

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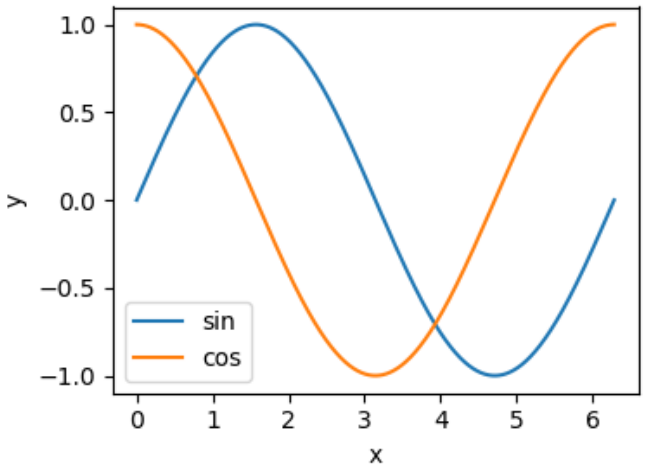
Recommended deadline: 5 September 2019
Final deadline: 12 September 2019, 7pm

1 Getting started with PyPlot (unmarked)

The second exercise of this sheet will ask you to create plots, which you can do using the `PyPlot` package. To install this package, open the Julia REPL, type `]` followed by `add PyPlot` and press enter. After this, you can load the package by typing `using PyPlot` like any other package. Please contact me if you have any problems installing `PyPlot`.

The following example illustrates most of the commands that you will need for this assignment:

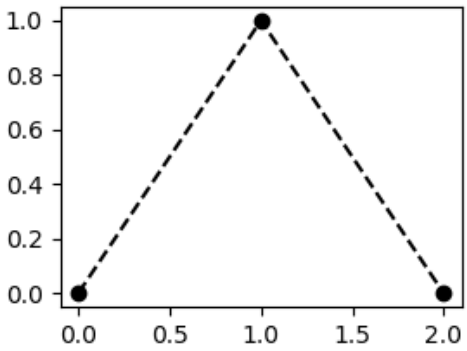
```
fig = figure(figsize=(4,3))
x = LinRange(0,2pi,1000)
plot(x, sin.(x), label="sin")
plot(x, cos.(x), label="cos")
legend(loc="best")
xlabel("x")
ylabel("y")
tight_layout()  # resizes the plot so x and y label are not cut off
savefig("test.png")
close(fig)
```



Other commands which may prove useful are:

- `semilogx()`: use logarithmic scale for x axis
- `semilogy()`: use logarithmic scale for y axis.
- `loglog()`: use logarithmic scale for both axes.
- `plot(x,y,format,...)` with `format` being a combination of the following allows you to change the line format.
  - Line styles: `-` (solid), `--` (dashed), `-.` (dash-dotted), `:` (dotted).
  - Markers: see https://matplotlib.org/3.1.1/api/markers_api.html.
  - Colours: `C0` to `C9` (default colour palette), `k` (black)

  Example: `plot([0,1,2], [0,1,0], "k--o")` yields the following.



2 Poisson equation with Dirichlet and Neumann boundary conditions

The aim of this exercise is to write code which solves the Poisson equation

$$\begin{cases} -u''(x) = f(x) & \text{for } x \in [0,1] \\ u(0) = 0, \quad u'(1) = 0 \end{cases} \tag{1}$$

with Dirichlet boundary conditions at $x = 0$ and Neumann boundary conditions at $x = 1$.

As in class, we discretise this equation by replacing the continuous function $u : [0,1] \to \mathbb{R}$ with the vector of point-values

$$u = \begin{pmatrix} u\left(\frac{1}{n}\right) & u\left(\frac{2}{n}\right) & \dots & u\left(\frac{n-1}{n}\right) & u\left(\frac{n}{n}\right) \end{pmatrix}^T. \tag{2}$$

However, unlike what we have seen before this vector now includes the right end-point $u\left(\frac{n}{n}\right) = u(1)$ since we only know the value $u'(1)$ but not $u(1)$. It is because of this

extra unknown that we chose the denominators in (2) as n rather than $n+1$ in order to maintain the convention that n denotes the number of unknowns. Furthermore, we use the usual approximation

$$u''\big(\tfrac{k}{n}\big) \approx n^2 \Big(u\big(\tfrac{k-1}{n}\big) - 2u\big(\tfrac{k}{n}\big) + u\big(\tfrac{k+1}{n}\big)\Big)$$

for the second derivative, and in a first try we use the approximation

$$u'(1) \approx n \Big(u\big(\tfrac{n}{n}\big) - u\big(\tfrac{n-1}{n}\big)\Big) = 0$$

to discretise the Neumann boundary condition. The linear system for the unknown point-values u_n then becomes $A_n u_n = b_n$ where

$$A_n := \begin{pmatrix} 2n^2 & -n^2 & & & \\ -n^2 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & -n^2 & 2n^2 & -n^2 \\ & & & -n & n \end{pmatrix}, \quad u_n \approx \begin{pmatrix} u\big(\frac{1}{n}\big) \\ u\big(\frac{2}{n}\big) \\ \dots \\ u\big(\frac{n-1}{n}\big) \\ u\big(\frac{n}{n}\big) \end{pmatrix}, \quad b_n := \begin{pmatrix} f\big(\frac{1}{n}\big) \\ f\big(\frac{2}{n}\big) \\ \dots \\ f\big(\frac{n-1}{n}\big) \\ 0 \end{pmatrix}. \tag{3}$$

1. Assume u is the solution to the continuous Poisson equation (1). Show that the consistency error decays as

$$\|A_n u - b_n\|_{2,n} := \frac{1}{\sqrt{n}} \sqrt{\sum_{k=1}^{n} \Big(u(\tfrac{k}{n}) - u_n[k]\Big)^2} = \mathcal{O}\big(n^{-1}\big).$$

2. Derive a three-point approximation

$$u'(1) \approx D_n u := c_0\, u\big(\tfrac{n}{n}\big) + c_1\, u\big(\tfrac{n-1}{n}\big) + c_2\, u\big(\tfrac{n-2}{n}\big)$$

which is accurate to second order, i.e. $D_n u - u'(1) = \mathcal{O}\big(n^{-2}\big)$.

Hint. Taylor-expand $u\big(\frac{n-k}{n}\big)$ around $x = 1$ and determine the c_k by matching the coefficients in $D_n u - u'(1) = \mathcal{O}\big(n^{-2}\big)$.

3. Propose modifications $\hat{A}_n$ and $\hat{b}_n$ to the A_n and b_n from (3) such that after modification, the consistency error satisfies

$$\|\hat{A}_n u - \hat{b}_n\|_{2,n} = \mathcal{O}\big(n^{-2}\big).$$

4. Let us assume without proof that both A_n and $\hat{A}_n$ are stable, i.e. $\|A_n\|_{2,n}$, $\|\hat{A}_n\|_{2,n}$ remain bounded for $n \to \infty$. Then, the findings from Tasks 1 and 3 imply that the solutions u_n and $\hat{u}_n$ to $A_n u_n = b_n$ and $\hat{A}_n \hat{u}_n = \hat{b}_n$ converge as

$$\|u - u_n\|_{2,n} = \mathcal{O}\big(n^{-1}\big) \qquad \text{and} \qquad \|u - \hat{u}_n\|_{2,n} = \mathcal{O}\big(n^{-2}\big),$$

respectively. Verify this numerically by plotting $\|u - u_n\|_{2,n}$ and $\|u - \hat{u}_n\|_{2,n}$ for $n \in \{2^2, \ldots, 2^{12}\}$ and some function $u(x)$ for which you can explicitly compute the correct right-hand side $f(x)$.

Your submission for this task should consist of the aforementioned plot in either PDF or PNG format and a Julia file containing all the code needed to reproduce your plot.

Hint. The matrix $\tilde{A}_n$ from Task 3 is no longer tridiagonal. I recommend that you assemble this matrix by first assembling the tridiagonal part using the `Tridiagonal()` function, then convert to a sparse matrix using `sparse()`, and finally add extra and/or modify entries using `A[i,j] = v`.

Example:

```
julia> A = sparse(Tridiagonal(
           fill(1,3),
           fill(2,4),
           fill(3,3)
       ))
       A[4,2] = 4
       A[4,3] = 5
       Matrix(A)
4×4 Array{Int64,2}:
 2  3  0  0
 1  2  3  0
 0  1  2  3
 0  4  5  2
```