MA5233 Computational Mathematics

Lecture 14: Polynomial Approximation

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Polynomial Approximation

Polynomial approximation

Given $f : [-1, 1] \to \mathbb{R}$, find polynomial $p \in \mathcal{P}_n$ minimising

$$\|f - p\|_{[-1,1]} := \sup_{x \in [-1,1]} |f(x) - p(x)|.$$

Why polynomial approximation?

Applications:

- Practical algorithm for evaluating "complicated" functions.
  Example: Krylov methods replace $A^{-1}b$ with $p(A)\,b$.
- Numerical integration. Hard: $\int f(x)\,dx$. Easy: $\int p(x)\,dx$.
- Basis in which to represent unknown functions.
  Example: finite element method for partial differential equations.

Key features of polynomials which make the above possible:

- Simple: polynomials require only addition and multiplication.
- Complete (Weierstrass approximation theorem): every continuous function can be uniformly approximated by polynomials.

Polynomial Approximation

Remarks on polynomial approximation problem

- Approximation on $[-1,1]$ is equivalent to approximation in any interval $[a,b]$:

$$\begin{aligned}\min_{\tilde{p}\in\mathcal{P}_n} \|f(\tilde{x}) - \tilde{p}(\tilde{x})\|_{[a,b]} &= \min_{\tilde{p}\in\mathcal{P}_n} \big\|f\big(\phi(x)\big) - \tilde{p}\big(\phi(x)\big)\big\|_{[-1,1]} \\ &= \min_{p\in\mathcal{P}_n} \big\|f\big(\phi(x)\big) - p(x)\big\|_{[-1,1]}\end{aligned}$$

  with $\qquad \phi : [-1,1] \to [a,b], \quad x \mapsto \frac{a+b}{2} + \frac{b-a}{2}\,x.$

- Supremum norm is often not exactly the error that you want to minimise, but it provides an upper bound on the desired error.

  Example: Consider $\|f\|_{p,[-1,1]} := \left(\int_{-1}^{1} |f(x)|^p \, dx\right)^{1/p}$. We have

$$\|f\|_{p,[-1,1]} \leq 2\,\|f\|_{[-1,1]}.$$

Polynomial Approximation

Methods of approximation

- Best approximation: $p = \arg\min \|f - p\|_{[-1,1]}$
  Rarely used in practice because hard to compute.
- Interpolation: $p(x_k) = f(x_k)$ for some $x_0, \ldots, x_n \in [-1, 1]$.
  Very easy to compute and "almost optimal" (precise statement will follow).
- L^2-projection: $\int_{-1}^{1} \big(f(x) - p(x)\big)\, x^k \, dx = 0$ for $k \in \{0, \ldots, n\}$.
  Useful for theory. Sometimes useful in practice.

Polynomial Approximation

Existence and uniqueness of best approximation [Tre13, Thm 10.1]

Given $f : [-1,1] \to \mathbb{R}$ and $n \in \mathbb{N}$, the minimiser

$$p^\star := \operatorname*{arg\,min}_{p \in \mathcal{P}_n} \|f - p\|_{[-1,1]}$$

exists and is unique.

Equioscillation theorem [Tre13, Thm 10.1]

A polynomial $p \in \mathcal{P}_n$ is equal to $p^\star$ if and only if there are $n+2$ points

$$-1 \le x_0 < x_1 < \ldots < x_n < x_{n+1} \le 1$$

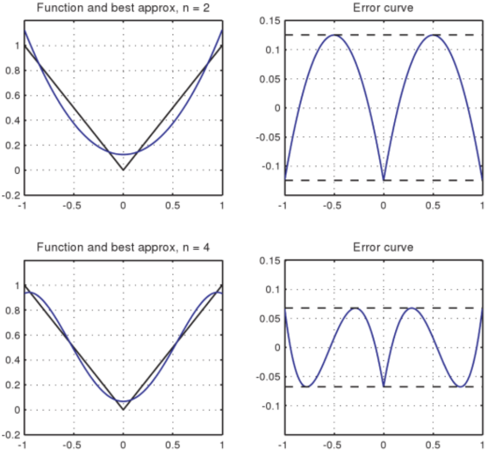
and $s \in \{-1, +1\}$ such that

$$f(x_k) - p(x_k) = s\,(-1)^k \, \|f - p^\star\|_{[-1,1]}.$$

See next slide for illustration.

Polynomial Approximation

Equioscillation theorem, illustrated



Observation: $f(x) - p^{\star}(x)$ equioscillates in $n + 3$ points in both examples.

Figure copied from Trefethen's ATAP. See reference at the end.

Polynomial Approximation

Review of best approximation

Good:

- There are iterative algorithms for computing best approximations. Search for "Remez algorithm" if you want to know more.

Bad:

- These algorithms are expensive and may fail to converge.
- Theory presented so far does not provide convergence rates.

Conclusion:

- We need other approximation algorithms to overcome these issues.

Polynomial Approximation

Existence and uniqueness of interpolant

Given $f : [-1, 1] \to \mathbb{R}$ and $n+1$ distinct points $x_0, \ldots, x_n \in [-1, 1]$, there exists a unique $p \in \mathcal{P}_n$ such that

$$p(x_k) = f(x_k) \qquad \text{for } k \in \{0, \ldots, n\}.$$

Proof: existence. The interpolant $p(x)$ is given by

$$p(x) = \sum_{j=0}^{n} f(x_j)\, \ell_j(x)$$

with $\ell_j(x)$ the Lagrange polynomials introduced on the next slide.

Proof: uniqueness. Assume $p, q \in \mathcal{P}_n$ are two interpolants to f. It follows from

$$p(x) - q(x) \in \mathcal{P}_n \qquad \text{and} \qquad p(x_k) - q(x_k) = 0 \quad \text{for } k \in \{0, \ldots, n\}$$

that $p(x) - q(x) = 0$.

Polynomial Approximation

Lagrange polynomials

Consider $n+1$ distinct points $x_0, \dots, x_n \in [-1, 1]$.
The Lagrange polynomials $\ell_j(x)$ with $j \in \{0, \dots, n\}$ are given by

$$\ell_j(x) := \prod_{i \neq j} \frac{x - x_i}{x_j - x_i}.$$

These polynomials satisfy

$$\ell_j(x_k) = \prod_{i \neq j} \frac{x_k - x_i}{x_j - x_i} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases}$$

Polynomial Approximation

Example

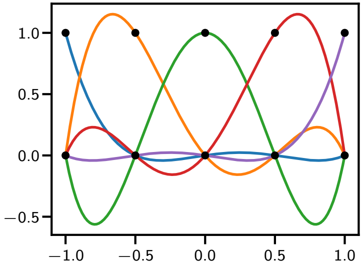
Consider the five points

$$x_0 = -1, \quad x_1 = -0.5, \quad x_2 = 0, \quad x_3 = 0.5, \quad x_4 = 1.$$

The Lagrange polynomial $\ell_2(x)$ is given by

$$\ell_2(x) = \frac{(x - x_0)}{(x_2 - x_0)} \frac{(x - x_1)}{(x_2 - x_1)} \frac{(x - x_3)}{(x_2 - x_3)} \frac{(x - x_4)}{(x_2 - x_4)}.$$

It is shown in green in the plot below (the other lines show other ℓ_j).



Polynomial Approximation

Interpolation error estimate [SM03, Thm 6.2]

Assume $f : [-1, 1] \to \mathbb{R}$ has $n + 1$ continuous derivatives.
Let p be the interpolant to f in the $n + 1$ points $x_0, \ldots, x_n \in [-1, 1]$.
For every $x \in [-1, 1]$, there exists a $\xi \in [-1, 1]$ such that

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i).$$

Interpretation

Polynomial interpolant $p(x)$ approximates $f(x)$ well if both of the following conditions are satisfied:

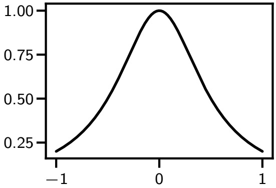
- $f(x)$ has many derivatives and these derivatives are small.
  Such functions are called *smooth*.
- The *node polyomial* $\ell(x) := \prod_{i=0}^{n} (x - x_i)$ is small throughout $[-1, 1]$.

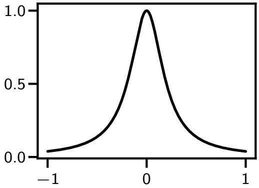
Polynomial Approximation

Example

Consider equispaced points $\left(x_i = \frac{2i-n}{n}\right)_{i=0}^{n}$ and the two functions

$$f_1(x) = \frac{1}{1+4x^2}, \qquad f_2(x) = \frac{1}{1+25x^2}.$$

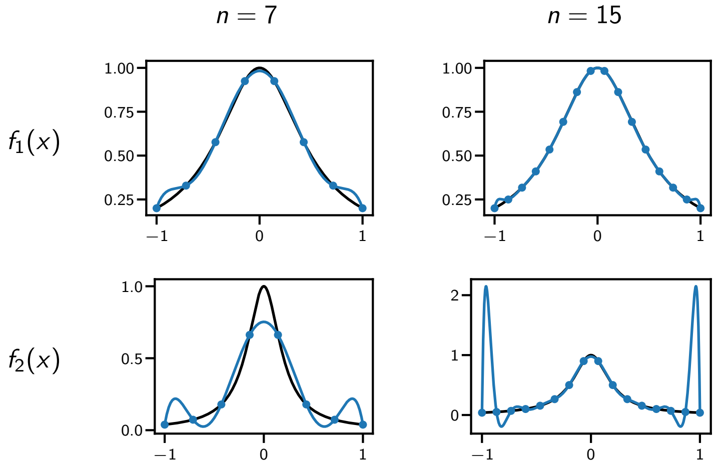




Note that $f_1(x)$ is smoother than $f_2(x)$.

Polynomial Approximation

Interpolants in $n+1$ equispaced points $\left(x_i = \frac{2i-n}{n}\right)_{i=0}^{n}$

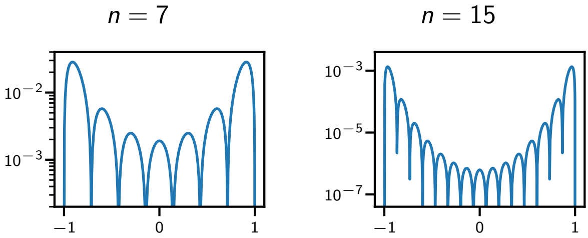


Observation: interpolant to f_1 converges while interpolant to f_2 diverges!

Polynomial Approximation

Explanation

Consider node polynomial $\ell(x) := \prod_{i=0}^{n}(x - x_i)$:



Observation: $\ell(x)$ is much larger for $x = \pm 1$ than for $x \approx 0$.

Previous slide suggests that the limits of $\left|f^{(n+1)}(\xi)\right| \left|\ell(x)\right|$ for $n \to \infty$ are:

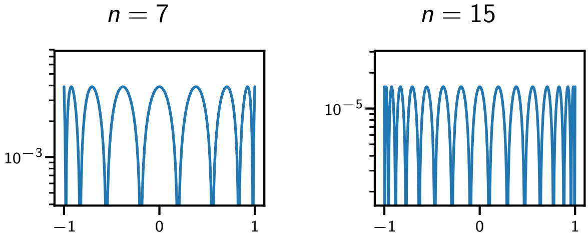
	$x \approx 0$	$x \approx \pm 1$
$f_1(x)$	0	0
$f_2(x)$	0	∞

Polynomial Approximation

Conclusion from interpolation error estimate

Uniform accuracy is achieved if we choose the interpolation points x_i such that $\ell(x) = \prod_{i=0}^{n}(x - x_i)$ equioscillates on $[-1, 1]$.

Put differently, $\ell(x)$ should look like this:



We determine such points by reversing the problem:

- Find equioscillating polynomial $T_{n+1} \in \mathcal{P}_{n+1}$.
- Choose x_i as the $n + 1$ roots of $T_{n+1}(x)$.

The polynomials $T_k(x)$ are known as *Chebyshev polynomials*.

Introducing them requires some preparation.

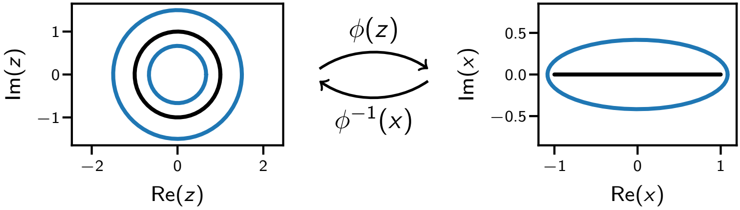
Polynomial Approximation

Joukowsky map

$$\phi(z) := \frac{z + z^{-1}}{2}, \qquad \phi_{\pm}^{-1}(x) := x \pm \sqrt{x^2 - 1}.$$

Properties:

1. $\phi(z)$ maps the unit circle $\{|z| = 1\}$ to $[-1, 1]$.
2. $\phi_+^{-1}(z) = \left(\phi_-^{-1}(z)\right)^{-1}$



Polynomial Approximation

Proof of inverse.

$$\frac{z+z^{-1}}{2}=x \iff z^2-2zx+1=0 \iff z=x\pm\sqrt{x^2-1}.$$

Proof of Property 1. We have $z^{-1}=\frac{\bar{z}}{|z|^2}$; hence for $|z|=1$ we obtain

$$\phi(z)=\frac{z+\bar{z}}{2}=\mathrm{Re}(z)\in[-1,1].$$

Proof of Property 2. Immediate consequence of $\phi(z)=\phi(z^{-1})$.

Polynomial Approximation

Chebyshev polynomials

$$T_n(x) := \frac{\phi^{-1}(x)^n + \left(\phi^{-1}(x)\right)^{-n}}{2}$$

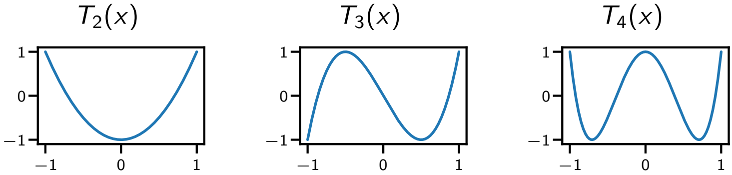
Note: definition is independent of choice of branch of $\phi_\pm^{-1}(x)$.

Properties:

1. $T_n(x)$ is indeed a polynomial and satisfies the recurrence relation

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

2. $\left|T_n(x_i)\right| = \|T_n\| = 1$ for $x_i = \cos\left(\pi \frac{i}{n}\right)$ and $i \in \{0, \dots, n\}$.
3. $T_n(x_i) = 0$ for $x_i = \cos\left(\pi \frac{2i-1}{2n}\right)$ and $i \in \{1, \dots n\}$.



Polynomial Approximation

Proof of Property 1.

- $T_k(x) \in \mathcal{P}_k$ follows from recurrence relation.
- Formulae for $T_0(x)$ and $T_1(x)$ are obvious.
- To show recurrence formula, set $x = \phi(z)$ and compute

$$\begin{aligned} 2xT_k(x) &= \frac{1}{2}\left(z + z^{-1}\right)\left(z^k + z^{-k}\right) \\ &= \frac{z^{k+1} + z^{-k-1}}{2} + \frac{z^{k-1} + z^{-k+1}}{2} \\ &= T_{k+1}(x) + T_{k-1}(x). \end{aligned}$$

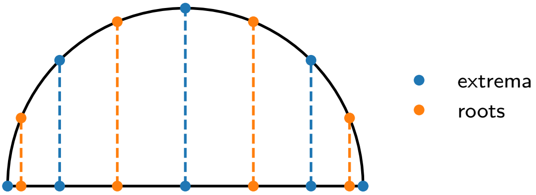
Polynomial Approximation

Proof of Property 2. We know that $|\phi^{-1}(x)| = 1$ for $x \in [-1, 1]$; hence $\|T_k(x)\| \leq 1$. Since $T_n(x) = \mathrm{Re}\big(\phi^{-1}(x)\big)$ for $x \in [-1, 1]$, this upper bound is attained for

$$\phi^{-1}(x_i)^n = \pm 1 \iff \phi^{-1}(x_i) = \exp\big(\pi\iota\tfrac{i}{n}\big) \iff x_i = \cos\big(\pi\tfrac{i}{n}\big).$$

Proof of Property 3. Since $|\phi^{-1}(x)| = 1$ and $T_k(x) = \mathrm{Re}\big(\phi^{-1}(x)\big)$ for $x \in [-1, 1]$, the roots are given by

$$\phi(x_i)^n = \pm\iota \iff \phi^{-1}(x_i) = \exp\big(\pi\iota\tfrac{2i-1}{2n}\big) \iff x_i = \cos\big(\pi\tfrac{2i-1}{2n}\big).$$



Polynomial Approximation

Chebyshev points

Choosing the roots $x_i = \cos\left(\pi \frac{2i-1}{2n}\right)$ of $T_n(x)$ as interpolation points leads to equioscillating node polynomial $\ell(x) \propto T_n(x)$.

While not exactly equioscillating, the extrema $x_i = \cos\left(\pi \frac{i}{n}\right)$ are also good interpolation points and are frequently used in practice.

Both sets of points are called *Chebyshev points*, and all of the following statements hold for either choice of Chebyshev points.

Convergence theory for approximation in Chebyshev points

Our starting point for deriving the Chebyshev points was the estimate

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i).$$

This estimate is somewhat unsatisfying as it only applies if $k \geq n+1$ with k the number of derivatives of f.

For interpolation in Chebyshev points, the estimate can be extended to the regime $k < n+1$ as shown on next slide.

Polynomial Approximation

Chebyshev interpolation error [Tre13, Thm 7.2]

Assume $f(x)$ is $k-1$ times continuously differentiable and $f^{(k)}$ is bounded and continuous except at a finite number of discontinuities.

Let $p \in \mathcal{P}_n$ be the interpolant to f in $n+1$ Chebyshev points.

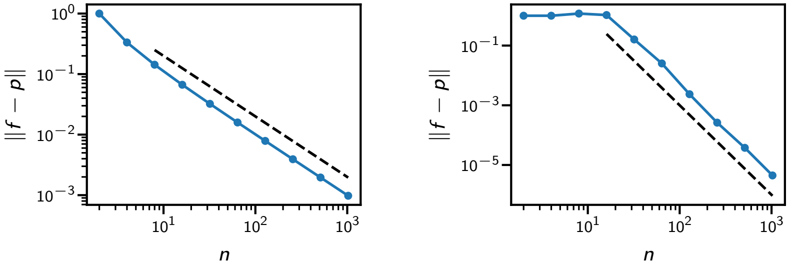
Then, there exists a $C > 0$ independent of n and f such that

$$e_n = \|f - p\|_{[-1,1]} \leq C \, \|f^{(k)}\|_{[-1,1]} \, n^{-k}.$$

Polynomial Approximation

Examples

$f_1(x) = |x| \to e_n = \mathcal{O}(n^{-1})$ $\qquad$ $f_2(x) = |\sin(4\pi x)|^3 \to e_n = \mathcal{O}(n^{-3})$



Additional observation for f_2:

We first need to resolve oscillations of $\sin(x)$ before we see convergence.

Polynomial Approximation

Infinitely differentiable functions

Previous theorem: f has k derivatives $\rightarrow e_n = \mathcal{O}(n^{-k})$. What if $k = \infty$?

We distinguish two cases depending on whether for every point $x_0 \in [-1, 1]$, there exists an $\epsilon > 0$ such that the Taylor series

$$\text{Taylor}[f, x_0](x) := \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

converges to $f(x)$ for all x such that $|x - x_0| \leq \varepsilon$.

- If yes: $f(x)$ is called *analytic*. Polynomial approximation converges exponentially (theorem will follow).
- If no: polynomial approximation converges superalgebraically but subexponentially.

Terminology

- Algebraic convergence: $e_n = \mathcal{O}\left(n^{-k}\right)$ for some $k \in \mathbb{N}$.
- Exponential convergence: $e_n = \mathcal{O}\left(a^n\right)$ for some $a < 1$.

Polynomial Approximation

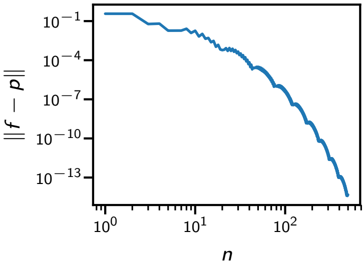
Example: convergence for infinitely differentiable functions

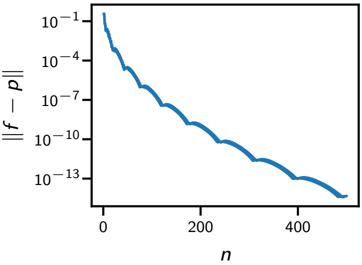
Consider $f(x) := \exp\left(-\frac{1}{|x|}\right)$.

$f(x)$ has infinitely many continuous derivatives, but it is not analytic:

$$\text{Taylor}[f, 0](x) = 0 \qquad \text{while} \qquad f(x) \neq 0 \text{ for } x \neq 0.$$

Convergence of Chebyshev interpolation is better than algebraic but worse than exponential:





Polynomial Approximation

Analytic functions

Definitions:

- An infinitely differentiable function $f(x)$ is called *analytic at a point* $x_0 \in \mathbb{C}$ if there exists $\varepsilon > 0$ such that $\text{Taylor}[f, x_0](x) = f(x)$ for all $x \in \mathbb{C}$ such that $|x - x_0| < \varepsilon$.
- $f(x)$ is called *analytic on a set* $\Omega \subset \mathbb{C}$ if it is analytic at every $x_0 \in \Omega$.

Properties of analytic functions:

- $f(x), g(x)$ analytic at $x_0 \implies f(x) + g(x)$ analytic at x_0.
- $f(x), g(x)$ analytic at $x_0 \implies f(x)g(x)$ analytic at x_0.
- $g(x)$ anal. at x_0, $f(x)$ anal. at $g(x_0) \implies f(g(x))$ analytic at x_0.

Examples of analytic functions:

- Analytic everywhere: $\exp(x)$, $\sin(x)$, $\cos(x)$.
- Analytic except at 0: $\frac{1}{x}$, $\sqrt{x}$, $\log(x)$.

Polynomial Approximation

Interpolation error for analytic functions [Tre13, Thm 8.2]

Assume $f(x)$ is analytic and bounded on the *Bernstein ellipse*

$$E(r) := \left\{ x \in \mathbb{C} \mid \tfrac{1}{r} < |\phi_{\pm}^{-1}(x)| < r \right\}, \qquad r \geq 1.$$

Let $p \in \mathcal{P}_n$ be the interpolant to f in $n+1$ Chebyshev points.

Then, there exists a $C > 0$ independent of f and n such that

$$\|f - p\|_{[-1,1]} \leq C \, \|f\|_{E(r)} \, r^{-n}.$$

Remark

Usually $f(x)$ is unbounded on $E(r^\star)$ with $r^\star := \sup\{r \text{ as above}\}$.

In this case $\|f\|_{E(r)} \to \infty$ as $r \to r^\star$, i.e. $\|f - p\|$ converges exponentially with any rate $r < r^\star$ but not with rate $r^\star$.

However, this technical subtlety is usually of no practical relevance.

Polynomial Approximation

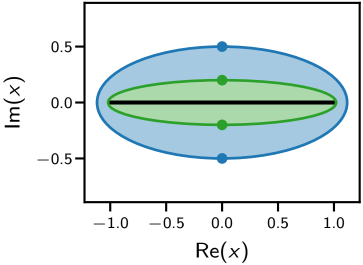
Example

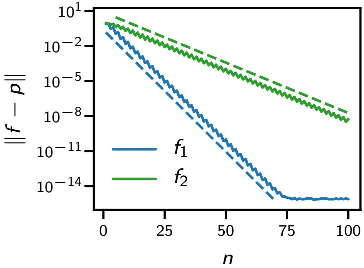
Recall the two functions $f_1(x) = \dfrac{1}{1+4x^2}, \quad f_2(x) = \dfrac{1}{1+25x^2}.$

$f_1(x)$ is analytic except at $x_{\pm}^{(1)} := \pm\frac{\iota}{2}$.

$f_2(x)$ is analytic except at $x_{\pm}^{(2)} := \pm\frac{\iota}{5}$.

According to theorem, interpolation in Chebyshev points converges and rate of convergence is given by $r_k = \min\left|\phi_{\pm}^{-1}(x_{\pm}^{(k)})\right|$.





Polynomial Approximation

Review of polynomial interpolation

- Interpolant exists, is unique and can be easily evaluated.
- Accuracy of interpolant depends on smoothness of f and distribution of interpolation points.

  Bad: equispaced points. Good: Chebyshev points (either type).
- Algebraic convergence for f with finitely many derivatives. Exponential convergence for analytic f.

To be discussed next

- Conditioning of interpolation problem.
- How much worse is interpolation compared to best approximation?

Polynomial Approximation

Notation

- $\|f\| := \|f\|_{[-1,1]}$, i.e. I drop the subscript $[-1,1]$ for brevity.
- $\mathcal{B}$: space of bounded functions $[-1,1] \to \mathbb{R}$.
- $p^\star$: best approximation to $f \in \mathcal{B}$.
- $P : \mathcal{B} \to \mathcal{P}_n$: interpolation operator for the points $x_0, \ldots, x_n$.

Observation

P is a linear operator: for all $\alpha \in \mathbb{R}$ and $f, g \in \mathcal{B}$ we have that

$$P(\alpha f) = \alpha Pf, \qquad P(f+g) = Pf + Pg.$$

Lebesgue constant (supremum norm of P)

$$\|P\| = \sup_{f \in \mathcal{B}} \frac{\|Pf\|}{\|f\|}$$

$\|P\|$ measures conditioning of polynomial interpolation:

$$\|P(f + \Delta f) - Pf\| = \|P \Delta f\| \leq \|P\| \|\Delta f\|$$

Polynomial Approximation

Application of Lebesgue constant

$$\|f - Pf\| \leq (1 + \|P\|) \|f - p^\star\|.$$

Proof. We obtain using $p^\star = Pp^\star$ that

$$\begin{aligned}
\|f - Pf\| &\leq \|f - p^\star\| + \|p^\star - Pf\| \\
&= \|f - p^\star\| + \|P(p^\star - f)\| \\
&\leq (1 + \|P\|) \|f - p^\star\|.
\end{aligned}$$

Conclusion

Interpolation problem is well-conditioned and interpolant close to optimal if and only if $\|P\|$ is small.

Polynomial Approximation

Formula for Lebesgue constant

$$\|P\| = \|\lambda\| = \sup_{x\in[-1,1]} \lambda(x) \qquad \text{where} \qquad \lambda(x) := \sum_{j=0}^{n} |\ell_j(x)|.$$

Proof. Using Lagrange's interpolation formula, we get

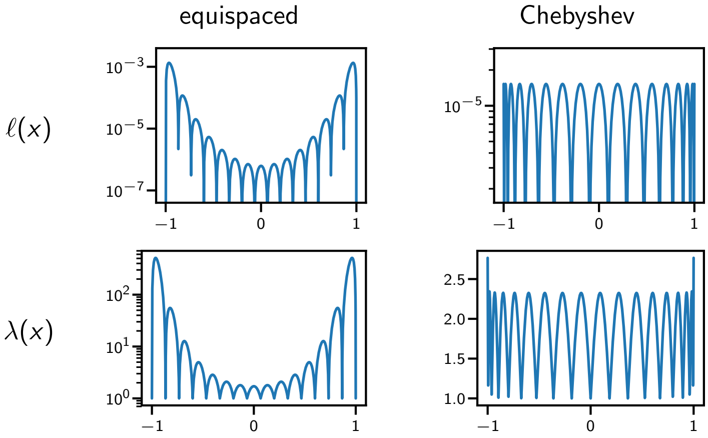
$$\begin{aligned}
\|Pf\| &= \sup_{x\in[-1,1]} \left| \sum_{j=0}^{n} f(x_j)\, \ell_j(x) \right| \\
&\le \sup_{x\in[-1,1]} \sum_{j=0}^{n} |f(x_j)|\, |\ell_j(x)| \\
&\le \|f\| \sup_{x\in[-1,1]} \lambda(x)
\end{aligned}$$

and hence

$$\|P\| = \sup_{f\in\mathcal{B}} \frac{\|Pf\|}{\|f\|} = \sup_{x\in[-1,1]} \lambda(x).$$

Polynomial Approximation

Lebesgue constant in practice



Observation:

- $\lambda(x)$ behaves similarly to $\ell(x)$.
- $\|P\|$ is small for Chebyshev points and large for equispaced points.

Polynomial Approximation

Lebesgue constant in practice

One can show the following bounds [Tre13, Thm 15.2]:

- Equispaced points: $\|P\| = \|\lambda\| > \frac{2^{n-2}}{n^2}$.
- Chebyshev points: $\|P\| \leq 1 + \frac{2}{\pi}\log(n+1)$.

Conclusion

- Chebyshev interpolation is well-conditioned and close to optimal.
- Interpolation in equispaced points is neither.

Interpolation algorithms

When proving existence of interpolants, we used the formulae

$$p(x) = \sum_{j=0}^{n} f(x_j)\,\ell_j(x), \qquad \ell_j(x) = \prod_{i \neq j} \frac{x - x_i}{x_j - x_i}.$$

Good: one can show that this formula is backward-stable.
Bad: every evaluation of formula takes $\mathcal{O}(n^2)$ FLOP.

Each $\ell_j(x)$ requires $\mathcal{O}(n)$ FLOP. Need to evaluate n such functions.

See homework sheet for a more efficient algorithm.

Polynomial Approximation

Convergence of Krylov subspace methods

In lecture on Krylov subspace methods, we have seen the statement

$$\min_{q_n \in \mathcal{P}_n} \max_{x \in [1,\kappa]} \frac{|q_n(x)|}{|q_n(0)|} \le 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^n . \tag{1}$$

We now have the tools necessary to prove this claim.

Proof of (1). Choose q_n as the shifted Chebyshev polynomial

$$q_n(x) := T_n\big(L(x)\big), \qquad L(x) := 1 - \tfrac{2}{\kappa-1}\,(x-1).$$

Since $L\big([1,\kappa]\big) = [-1,1]$, we have

$$\max_{x \in [1,\kappa]} \tfrac{|q_n(x)|}{|q_n(0)|} = \tfrac{1}{|q_n(0)|}$$

and it remains to show

$$|q_n(0)| = T_n\big(1 + \tfrac{2}{\kappa-1}\big) = T_n\big(\tfrac{\kappa+1}{\kappa-1}\big) \ge \tfrac{1}{2}\left(\tfrac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1}\right)^n .$$

Polynomial Approximation

Proof of (1), *continued.* Recall the formulae

$$T_n(x) := \frac{\phi^{-1}(x)^n + \left(\phi^{-1}(x)\right)^{-n}}{2}, \qquad \phi^{-1}(x) := x + \sqrt{x^2 - 1}.$$

We compute

$$\begin{aligned}\phi^{-1}\left(\tfrac{\kappa+1}{\kappa-1}\right) &= \tfrac{\kappa+1}{\kappa-1} + \sqrt{\left(\tfrac{\kappa+1}{\kappa-1}\right)^2 - 1} = \tfrac{\kappa+1+\sqrt{(\kappa+1)^2-(\kappa-1)^2}}{\kappa-1} \\ &= \tfrac{\kappa+2\sqrt{\kappa}+1}{\kappa-1} = \tfrac{(\sqrt{\kappa}+1)^2}{(\sqrt{\kappa}+1)(\sqrt{\kappa}-1)} = \tfrac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1}.\end{aligned}$$

Hence,

$$T_n\left(\tfrac{\kappa+1}{\kappa-1}\right) = \tfrac{1}{2}\left(\left(\tfrac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1}\right)^n + \left(\tfrac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1}\right)^{-n}\right) \geq \tfrac{1}{2}\left(\tfrac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1}\right)^n.$$

Polynomial Approximation

References and further reading

- L. N. Trefethen. *Approximation Theory and Approximation Practice.* Society for Industrial and Applied Mathematics (2013),
  Extensive discussion of Chebyshev polynomials, and very readable.
- E. Suli and D. F. Mayers. *An Introduction to Numerical Analysis.* Cambridge University Press (2003), `doi:10.1017/CBO9780511801181`
  Can be accessed online for free via the library website!