MA5233 Computational Mathematics

Lecture 21: Finite Element Method

Simon Etter

National University of Singapore

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Finite Element Method

Finite element method

- Galerkin's method with special basis functions (hat functions).
- By far most common approach for solving PDEs.
- Finite element software packages are among the biggest and most complex software projects in scientific computing.
- Finite elements is among the biggest consumers of computing power.
  See e.g. `http://www.archer.ac.uk/status/codes/` for usage on British national supercomputer.

Outline of lecture

- Detailed discussion of finite elements in $d = 1$ dimension.
- Overview of finite elements in $d > 1$ dimensions

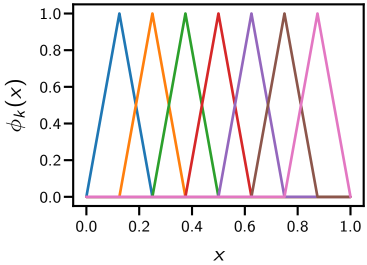
Finite Element Method

Def: Mesh on $[a, b]$

Sequence of points $a =: x_0 < x_1 < \ldots < x_n < x_{n+1} := b$.

Def: Hat functions for mesh $(x_k)_{k=0}^{n+1}$

$$\phi_k(x) := \begin{cases} \frac{x - x_{k-1}}{x_k - x_{k-1}} & \text{if } x_{k-1} \leq x \leq x_k, \\ \frac{x_{k+1} - x}{x_{k+1} - x_k} & \text{if } x_k \leq x \leq x_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \qquad k \in \{1, \ldots, n\}.$$



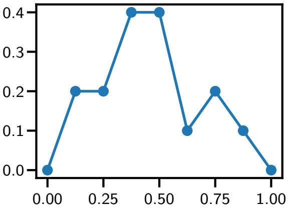
Finite Element Method

Def: Finite element method

Galerkin's method using the subspace $V_n := \mathrm{span}\{\phi_k\}$ spanned by the hat functions ϕ_k.

Remarks

- V_n is the space of continuous and piecewise linear functions with zero boundary conditions. A typical element $f \in V_n$ is given by



- Note that $V_n \not\subset C_0^1([0,1])$ but $V_n \subset H_0^1([0,1])$ as required by Galerkin's method.
- The hat functions $\phi_k(x)$ are similar to Lagrange polynomials in the sense that $\phi_k(x_\ell) = \delta_{k\ell}$.

Finite Element Method

Implementation

Galerkin matrix $A_{k\ell} := \int_0^1 \phi_k'(x)\,\phi_\ell'(x)\,dx$ can be computed explicitly:

- $A_{k\ell}$ is symmetric; hence it is enough to consider $k \geq \ell$.
- $A_{k\ell} = 0$ for $k > \ell + 1$ since then supports of ϕ_k and ϕ_ℓ don't overlap.
- Auxiliary computation:

$$\phi_k'(x) := \begin{cases} \frac{1}{x_k - x_{k-1}} & \text{if } x_{k-1} \leq x \leq x_k, \\ -\frac{1}{x_{k+1} - x_k} & \text{if } x_k \leq x \leq x_{k+1}, \\ 0 & \text{otherwise,} \end{cases}$$

- Diagonal:

$$\begin{aligned} A_{kk} &= \int_{x_{k-1}}^{x_k} \frac{1}{(x_k - x_{k-1})^2}\,dx + \int_{x_k}^{x_{k+1}} \frac{1}{(x_{k+1} - x_k)^2}\,dx \\ &= \frac{1}{x_k - x_{k-1}} + \frac{1}{x_{k+1} - x_k} \end{aligned}$$

- First off-diagonal:

$$A_{k+1,k} = -\int_{x_k}^{x_{k+1}} \frac{1}{(x_{k+1} - x_k)^2}\,dx = -\frac{1}{x_k - x_{k-1}}.$$

Finite Element Method

Implementation (continued)

Right-hand side: use composite trapezoidal rule with partition given by $(x_k)_{k=0}^{n+1}$, since then we can use $\phi_k(x_\ell) = \delta_{k\ell}$ to simplify the formula:

$$\begin{aligned} b_k &:= \int_0^1 f(x)\,\phi_k(x)\,dx \\ &\approx \sum_{i=0}^{n} \frac{x_{i+1} - x_i}{2} \Big(f(x_{i+1})\,\phi_k(x_{i+1}) + f(x_i)\,\phi_k(x_i) \Big) \\ &= \left(\frac{x_k - x_{k-1}}{2} + \frac{x_{k+1} - x_k}{2} \right) f(x_k) \end{aligned}$$

Evaluating the solution: compute $c = A^{-1} b$ and evaluate

$$u(x) = \sum_{k=1}^{n} c_k \phi_k(x).$$

Due to special structure of ϕ_k, this $u(x)$ corresponds to piecewise linear interpolation of data points (x_k, c_k) with $c_0 = c_{n+1} = 0$.

Finite Element Method

Finite elements vs. finite differences

If we choose equispaced mesh $(x_k := \frac{k}{n+1})_{k=0}^{n+1}$, then $x_{k+1} - x_k = \frac{1}{n+1}$ and we obtain

$$A_{k\ell} = \begin{cases} 2\,(n+1) & \text{if } k = \ell, \\ -(n+1) & \text{if } |k-\ell| = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad b_k = \frac{1}{n+1} f(x_k).$$

These are the equations of finite differences!

More precisely, we have $A_{\mathrm{FD}} = (n+1)\,A_{\mathrm{FEM}}$ and $b_{\mathrm{FD}} = (n+1)\,b_{\mathrm{FEM}}$.

Conclusions:

- In simple cases, FEM is just a more complicated derivation of FD.
  This holds both in one and multiple dimensions.
- Advantages of FEM will be most apparent in $d > 1$ dimensions; hence we postpone discussion for now.

Finite Element Method

Thm: Finite element error estimates

Let V_n be the space of piecewise linear functions on an equispaced mesh with n vertices.

Assume the exact solution to Poisson's equation satisfies $u \in C^2([0,1])$.

Denote by u_n the Galerkin solution to Poisson's equation for V_n. Then,

$$\|u - u_n\|_{H^1([0,1])} = \mathcal{O}\big(n^{-1}\big), \qquad \|u - u_n\|_{L^2([0,1])} = \mathcal{O}\big(n^{-2}\big).$$

The following slides will provide a rough sketch of how this result can be derived from Céa's lemma and Aubin-Nitsche lemma.

Finite Element Method

Céa's lemma (recap)

$$\|u - u_n\|_{H^1([0,1])} \le C \inf_{v_n \in V_n} \|u - v_n\|_{H^1([0,1])}.$$

Approximation theorem

Assume V_n is the space of piecewise linear functions on an equispaced mesh with n vertices and $u \in C^0([0,1])$. Then,

$$\inf_{v_n \in V_n} \|u - v_n\|_{H^1([0,1])} = \mathcal{O}\big(n^{-1} \|u''\|_{[0,1]}\big).$$

Proof sketch. By Poincaré's inequality, we have

$$\|v\|_{H^1}^2 = \|v\|_{L^2}^2 + \|v'\|_{L^2}^2 \le C \|v'\|_{L^2}^2$$

for some $C > 0$. Hence it is enough to find some $v_n \in V_n$ such that

$$\|u' - v_n'\|_{L^2} \le \|u' - v_n'\|_{[0,1]} = \mathcal{O}\big(n^{-1}\big).$$

Since v_n is piecewise linear, v_n' is piecewise constant. Result follows by choosing these constants such that first term in local Taylor series expansion of $u' - v_n'$ cancels.

Finite Element Method

Aubin-Nitsche lemma (recap)

$$\|u-u_n\|^2_{L^2([0,1])} \le A\,\|u-u_n\|_{H^1([0,1])}\,\inf_{v\in V_n}\|g-v_n\|_{H^1([0,1])}. \qquad (1)$$

L^2 estimate

According to approximation theorem on previous slide, we have

$$\inf_{v_n\in V_n}\|g-v_n\|_{H^1([0,1])} = \mathcal{O}\big(n^{-1}\,\|g''\|_{[0,1]}\big) = \mathcal{O}\big(n^{-1}\,\|u-u_n\|_{[0,1]}\big).$$

Insert this estimate in (1) and cancel one factor of

$$\|u-u_n\|_{L^2} \le \|u-u_n\|_{[0,1]}$$

to arrive at

$$\|u-u_n\|_{L^2([0,1])} = \mathcal{O}\big(n^{-2}\big).$$

Finite Element Method

Finite elements in dimensions $d > 1$

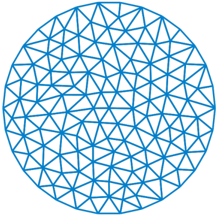
Much of the theory (weak derivatives, L^2 and H^k spaces, Lax-Milgram, Céa, Aubin-Nitsche, Strang) remains the same.

Main new features:

- Imposing boundary conditions becomes more complicated since $H^1(\Omega) \notin C^0(\Omega)$ for $\Omega \subset \mathbb{R}^d$ with $d > 1$.
  Solution: trace operator (not discussed since not core topic of this module).
- Error estimation requires results for polynomial approximation in $d > 1$ dimensions. Theory gets more complicated, but result is the same: for linear finite elements, we get $\mathcal{O}(h^{-1})$ convergence in H^1 and $\mathcal{O}(h^{-2})$ convergence in L^2 with h the meshwidth (will be introduced later).
- Data structures and algorithms become much more involved.

Finite Element Method

Mesh in two dimensions



Computer representation:

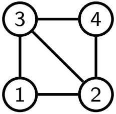
- $V \subset \mathbb{R}^2$: (finite) set of vertices.
- $T \subset \binom{V}{3}$: set of triangles.

T is a set of triplets $(v_1, v_2, v_3) \in V^3$ with v_1, v_2, v_3 pairwise distinct. Such a triplet represents the triangle

$$\left\{ t_1\, v_1 + t_2\, v_2 + t_3\, v_3 \mid t_1, t_2, t_3 \geq 0, t_1 + t_2 + t_3 = 1 \right\}.$$

Picture source: `https://en.wikipedia.org/wiki/File:Finite_element_triangulation.svg`.

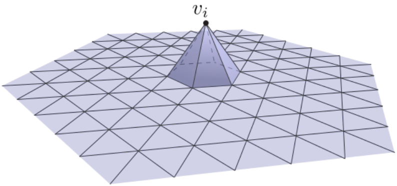
Example



$$\longrightarrow \qquad T = \{(1, 2, 3), (2, 3, 4)\}.$$

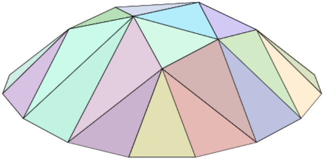
Finite Element Method

Hat functions in dimensions $d > 1$



Picture source: `http://brickisland.net/cs177/?p=309`.

Hat functions span the space of continuous and piecewise linear functions.
A typical function $f \in V_n$ is given by

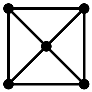
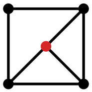


Picture source: `https://en.wikipedia.org/wiki/File:Piecewise_linear_function2D.svg`.

Finite Element Method

Conforming meshes

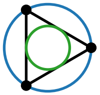
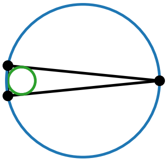
A mesh is called *conforming* if all vertices are in the corners of their adjacent triangles.

Conforming:  Non-conforming: 

Conformity is necessary for the hat functions to be in $H^1(\Omega)$.

Mesh parameters

- Mesh width $h := \max r_{\text{circ}}$: largest radius of circumscribed circle of all triangles.
- Shape regularity measure $\rho := \max \frac{r_{\text{circ}}}{r_{\text{in}}}$: largest ratio of radii of circumscribed and inscribed circles of all triangles.

ρ small:  ρ large: 

There are other definitions of mesh width and shape regularity, but they are all equivalent in the sense that h measures the size and ρ the deformity of the triangles.

Finite Element Method

Error estimate

Given a sequence of meshes with width $h \to 0$ and regularity ρ bounded and assuming $u \in H^2$, we have

$$\|u - u_n\|_{L^2} = \mathcal{O}\big(h^2\big), \qquad \|u - u_n\|_{H^1} = \mathcal{O}\big(h^1\big).$$

Galerkin matrix A

- Every row / column pair of A corresponds to a vertex in the mesh.
- We have $A(k, \ell) \neq 0$ if and only if vertices k and ℓ are connected by an edge.
- Condition number $\kappa(A)$ increases for $n \to \infty$.

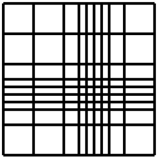
Finite Element Method

Finite elements vs. finite differences

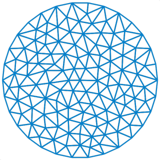
- Finite differences requires a cartesian mesh.
- Finite elements requires a triangular mesh.

Triangular meshes are more convenient for approximating complicated domains, and refining in just a particular region.

cartesian



triangular



However, both methods are just a collection of tricks to turn a PDE into a linear system, and for practical purposes it is irrelevant whether this linear system has been derived using the FD or FE strategy.