MA3227 Numerical Analysis II

Lecture 3: Finite Differences in Two Dimensions

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Finite Differences in Two Dimensions

Introduction

Consider Poisson's equation on $\Omega = (0,1)^2$, i.e. the problem of finding $u : [0,1]^2 \to \mathbb{R}$ such that for all $(x_1, x_2) \in (0,1)^2$ we have

$$-\Delta u(x_1, x_2) = f(x_1, x_2), \qquad u(0, x_2) = u(1, x_2) = u(x_1, 0) = u(x_1, 1) = 0.$$

Finite difference discretisation works just like in 1D:

- Functions: $u(x_1, x_2) \to u[i_1, i_2] = u\big(\frac{i_1}{n+1}, \frac{i_2}{n+1}\big)$
- Derivatives:

$$\frac{\partial^2 u}{\partial x_1^2}\big(\tfrac{i_1}{n+1}, \tfrac{i_2}{n+1}\big) \approx (n+1)^2 \left(u[i_1+1, i_2] - 2u[i_1, i_2] + u[i_1-1, i_2]\right),$$
$$\frac{\partial^2 u}{\partial x_2^2}\big(\tfrac{i_1}{n+1}, \tfrac{i_2}{n+1}\big) \approx (n+1)^2 \left(u[i_1, i_2+1] - 2u[i_1, i_2] + u[i_1, i_2-1]\right)$$

  and thus

$$\Delta u\big(\tfrac{i_1}{n+1}, \tfrac{i_2}{n+1}\big) \approx (n+1)^2 \left(u[i_1+1,i_2] + u[i_1,i_2+1] - 4u[i_1,i_2] + u[i_1-1,i_2] + u[i_1,i_2-1]\right).$$

New issue: natural "layout" for point values $u[i_1, i_2]$ is a matrix, but for linear-algebra-purposes we would like $u[i_1, i_2]$ to be a vector.

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Vectorisation of a matrix

Given $u \in \mathbb{R}^{n \times n}$, we define $\text{vec}(u) \in \mathbb{R}^{n^2}$ through

$$\text{vec}(u)[i_1 + n\,(i_2 - 1)] := u[i_1, i_2].$$

Example

$\text{vec}(u)$ enumerates the entries of a 4×4 matrix u as follows:

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

.

Remarks

- We write $\Delta_n^{(1)}$ and $\Delta_n^{(2)}$ to indicate the dimension of Δ_n.
- Vectorisation of u leads to a more complicated sparsity pattern for the finite-difference-discretised Laplacian matrix $\Delta_n^{(2)}$. See next slide.

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Two-dimensional Laplacian matrix

$$
\Delta_n^{(2)} = (n+1)^2 \begin{pmatrix}
\begin{bmatrix} -4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -4 \end{bmatrix} & \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} & & \\
\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} & \ddots & \ddots & \\
& \ddots & \ddots & \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \\
& & \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} & \begin{bmatrix} -4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -4 \end{bmatrix}
\end{pmatrix}
$$

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Discussion

Solving Poisson's equation in 2D in principle works just like in 1D: assemble the linear system and solve.

However, there are a few technical complications:

- $\Delta_n^{(2)}$ has n^4 entries, but only $\mathcal{O}\left(n^2\right)$ of these entries are nonzero. Exploiting this special sparsity property is crucial to stay within the memory limits of a typical laptop.
  Julia's `SparseArrays` package provides a data type which allows us to represent sparse matrices efficiently.
- The structure of $\Delta_n^{(2)}$ is quite complicated. Assembling this matrix is tricky unless we use the Kronecker product trick introduced on the next slide.

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Kronecker product

Let a, b be vectors and A, B be matrices. We define

$$a \otimes b := \begin{pmatrix} a[1]\, b \\ \vdots \\ a[n]\, b \end{pmatrix}, \qquad A \otimes B := \begin{pmatrix} A[1,1]\, B & \cdots & A[1,n]\, B \\ \vdots & \ddots & \vdots \\ A[n,1]\, B & \cdots & A[n,n]\, B \end{pmatrix}.$$

Remarks

- $h = f \otimes g$ corresponds to the product $h(x_1, x_2) = g(x_1)\, f(x_2)$.
- Let I be the identity matrix. We have the correspondences

$$\tfrac{\partial^2}{\partial x_1^2} \quad \longrightarrow \quad I \otimes \Delta_n^{(1)}, \qquad \tfrac{\partial^2}{\partial x_2^2} \quad \longrightarrow \quad \Delta_n^{(1)} \otimes I.$$

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Two-dimensional Laplacian

It follows from previous slide that the finite-difference discretisation of

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

is given by

$$\Delta_n^{(2)} = \Delta_n^{(1)} \otimes I + I \otimes \Delta_n^{(1)}$$

Implementing $\Delta_n^{(2)}$ is straightforward with this formula.
See `laplacian_2d()`.

Convergence analysis for two-dimensional finite differences

The "stability & consistency $\Rightarrow$ convergence" trick also works in 2D.

- Consistency can be shown using Taylor expansions just like in 1D.
- Stability of $\Delta_n^{(2)}$ requires us to study the eigenvalues of $\Delta_n^{(2)}$.
  These eigenvalues can be derived from the results on the next slide.

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Lemma

$$(A \otimes B)\,(a \otimes b) = (Aa) \otimes (Bb)$$

Proof. Straightforward but tedious computations.

Eigenvalues and -vectors of 2D Laplacian

Let λ_k, u_k be eigenpairs of $\Delta_n^{(1)}$. Then, eigenpairs of $\Delta_n^{(2)}$ are

$$\lambda_{k_1,k_2} = \lambda_{k_1} + \lambda_{k_2}, \qquad u_{k_1,k_2} = u_{k_1} \otimes u_{k_2}.$$

Proof.

$$\begin{aligned}
\Delta_n^{(2)} u_{k_1,k_2} &= \left(\Delta_n^{(1)} \otimes I + I \otimes \Delta_n^{(1)}\right)\left(u_{k_1} \otimes u_{k_2}\right) \\
&= \left(\Delta_n^{(1)} u_{k_1}\right) \otimes u_{k_2} + u_{k_1} \otimes \left(\Delta_n^{(1)} u_{k_2}\right) \\
&= \lambda_{k_1}\, u_{k_1} \otimes u_{k_2} + \lambda_{k_2}\, u_{k_1} \otimes u_{k_2} \\
&= \left(\lambda_{k_1} + \lambda_{k_2}\right) u_{k_1} \otimes u_{k_2} \\
&= \left(\lambda_{k_1} + \lambda_{k_2}\right) u_{k_1,k_2}.
\end{aligned}$$

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Stability of 2D Laplacian

It follows from results on previous slide that

$$|\lambda_{\mathrm{min}}| = |\lambda_{1,1}| = 2\,|\lambda_1| = 2\pi^2 + \mathcal{O}\big(n^{-2}\big).$$

Hence, we have $\|(\Delta_n^{(2)})^{-1}\|_{2,n} = \mathcal{O}(1)$, i.e. 2D Laplacian is stable.

Convergence of 2D finite differences

Combining the consistency and stability result, we obtain

$$\|u - u_n\|_{n,2} = \mathcal{O}\big(n^{-2}\big)$$

as in 1D.

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Summary

- The structure of $\Delta_n^{(2)}$ is "tridiagonal + two far off-diagonals".
- Eigenpairs of $\Delta_n^{(2)}$ are given by

$$\lambda_{k_1,k_2} = \lambda_{k_1} + \lambda_{k_2}, \qquad u_{k_1,k_2} = u_{k_1} \otimes u_{k_2}$$

  with λ_k, u_k the eigenpairs of $\Delta_n^{(1)}$.