MA3227 Numerical Analysis II

Lecture 1: Poisson Equation

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Poisson Equation

Introduction

- Partial differential equations (PDEs) are equations in terms of an unknown function $u(x)$ and its derivatives.
- Many different phenomena can be described in terms of PDEs, e.g.
  - fluid dynamics (Navier-Stokes),
  - electromagnetism (Maxwell),
  - quantum mechanics (Schrödinger),
  - option pricing (Black-Scholes).
- Poisson's equation is the simplest and most fundamental PDE.
- This module will present all numerical techniques for PDEs at the example of Poisson's equation for ease of exposition.

Poisson Equation

Poisson equation

Given $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$, find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u(x) = f(x) & \forall x \in \Omega, \\ \quad u(x) = 0 & \forall x \in \partial\Omega. \end{cases}$$

$\Delta := \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ is called the Laplacian operator.

Poisson equation describes diffusion (heat, particles, etc.).

- $f(x)$: distribution of heat / particle sources.
- $u(x)$: steady-state distr. of temperature / particle concentration.

Poisson Equation

Derivation of the Poisson equation

- Fick's law of diffusion: net flux is given by

$$J = -D\,\nabla u = -D \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{pmatrix}.$$

- Conservation of mass: for all $\Omega' \subset \Omega$, we must have that

$$\frac{\partial}{\partial t} \int_{\Omega'} u\,dx = -\int_{\partial\Omega'} n \cdot J\,dx + \int_{\Omega'} f\,dx.$$

- Divergence / Gauss's theorem:

$$\int_{\partial\Omega'} n \cdot J\,dx = \int_{\Omega'} \nabla \cdot J\,dx = \int_{\Omega'} \left(\frac{\partial J_1}{\partial x_1} + \ldots + \frac{\partial J_n}{\partial x_n} \right) dx$$

Poisson Equation

Derivation of the Poisson equation

- Combining conservation of mass with divergence theorem yields

$$\int_{\Omega'} \left(\frac{\partial u}{\partial t} + \nabla \cdot J - f \right) dx = 0 \qquad \forall \Omega' \subset \Omega.$$

  Hence

$$\frac{\partial u}{\partial t} + \nabla \cdot J - f = 0.$$

- Inserting steady-state condition $\frac{\partial u}{\partial t} = 0$ and Fick's law $J = -D \nabla u$ with $D = 1$ yields the Poisson equation

$$-\nabla \cdot (\nabla u) = -\Delta u = f.$$

- Boundary condition $u|_{\partial\Omega} = 0$: temperature at boundary is zero / particles reaching the boundary never reenter.
  Other boundary conditions are possible. We focus on $u|_{\partial\Omega} = 0$ for simplicity.

Poisson Equation

Exercise 1

Find $u : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u(x) = 2 & \forall x \in (0,1), \\ u(x) = 0 & \forall x \in \{0,1\}. \end{cases}$$

Exercise 2

Find $u : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u(x) = \sin(\pi x) & \forall x \in (0,1), \\ u(x) = 0 & \forall x \in \{0,1\}. \end{cases}$$

Discussion

- Unlike in the above examples, it is in general not possible to write down an explicit solution for Poisson's equation.
- Our aim in the next lecture is to *discretise* Poisson's equation, i.e. to transform it such that it can be solved numerically.

Poisson Equation

Summary

- Poisson's equation $-\Delta u = f$ is a simple PDE describing diffusion.
- Poisson's equation can be derived from conservation of mass and Fick's flux law $J = -\nabla u$.