

Example for finite difference discretisation

Given $f, a: [0, 1] \rightarrow \mathbb{R}$, find $u: [0, 1] \rightarrow \mathbb{R}$ such that

$$-\frac{\partial}{\partial x} \left(a(x) \frac{\partial u}{\partial x}(x) \right) = f(x) \quad \forall x \in (0, 1)$$

and $u(0) = u(1) = 0$.

This equation describes heat flow in a medium with spatially varying heat conductivity constant.

Finite difference discretisation:

$$\begin{aligned} \frac{\partial}{\partial x} (a u') \left(\frac{i}{n+1} \right) &\approx \dots \\ &\approx \frac{a\left(\frac{i+1/2}{n+1}\right) u\left(\frac{i+1/2}{n+1}\right) - a\left(\frac{i-1/2}{n+1}\right) u\left(\frac{i-1/2}{n+1}\right)}{1/(n+1)} \\ &\approx (n+1) \left(a\left(\frac{i+1/2}{n+1}\right) \frac{u\left(\frac{i+1}{n+1}\right) - u\left(\frac{i}{n+1}\right)}{1/(n+1)} \right. \\ &\quad \left. - a\left(\frac{i-1/2}{n+1}\right) \frac{u\left(\frac{i}{n+1}\right) - u\left(\frac{i-1}{n+1}\right)}{1/(n+1)} \right) \\ &= (n+1)^2 \left[a\left(\frac{i+1/2}{n+1}\right) u\left(\frac{i+1}{n+1}\right) \right. \\ &\quad \left. - \left(a\left(\frac{i+1/2}{n+1}\right) + a\left(\frac{i-1/2}{n+1}\right) \right) u\left(\frac{i}{n+1}\right) \right. \\ &\quad \left. + a\left(\frac{i-1/2}{n+1}\right) u\left(\frac{i-1}{n+1}\right) \right] \end{aligned}$$

Associated matrix:

$$\Delta_{n,a} = (n+1)^2 \begin{pmatrix} -a\left(\frac{1-1/2}{n+1}\right) - a\left(\frac{1+1/2}{n+1}\right) & & & & \\ & a\left(\frac{2-1/2}{n+1}\right) & & & \\ & & \ddots & & \\ & & & -a\left(\frac{2-1/2}{n+1}\right) - a\left(\frac{2+1/2}{n+1}\right) & \\ & & & & \ddots & \\ & & & & & a\left(\frac{2+1/2}{n+1}\right) \end{pmatrix}$$

See $\text{laplacian}(n, a)$.