MA3227 Numerical Analysis II

Lab Session 2

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1 Fill path theorem

The file `lab02_fill_paths.jl` defines a function `example_sparse_matrix()` which assembles some sparse matrix A. Predict the fill-in in the LU factorisation of A using the fill-path theorem and check your result numerically.

Hints.

- Use `L,U = lu(A, Val(false))` to compute the LU factorisation of A. The extra argument `Val(false)` turns off pivoting.
- You can see the sparsity pattern of a matrix `A` more easily if you look at `A .!= 0` rather than the numerical values of A.
- You can generate further practice exercises by modifying the code of `example_sparse_matrix()`.

2 GMRES polynomials

We have seen in class that the GMRES solutions x_k satisfy the bound

$$\|Ax_k - b\|_2 \le C \min_{q_k \in \mathcal{P}_k} \max_{\lambda_\ell} \frac{|q_k(\lambda_\ell)|}{|q_k(0)|}$$

where $C > 0$ is some constant independent of k, λ_ℓ are the eigenvalues of A, and $\mathcal{P}_k = \{p(x) \mid p(x) = \sum_{\ell=0}^{k} c_\ell \, x^\ell\}$ is the space of polynomials of degree $\le k$.

1. Run the function `repeated_eigenvalues()` in `lab02_gmres_polynomials.jl` and try to understand the plot that it generates. Can you explain why the second plot goes to 0 already for $k = 5$ while the first plot reaches 0 only for $k = 6$?
2. Run the function `alternating_eigenvalues()` in `lab02_gmres_polynomials.jl`. Can you explain the staircase pattern of the plot?

3 Arnoldi iteration (if time permits)

In Lecture 6, we wrote a function `arnoldi(A,b,k) -> Q,H` which implements the Arnoldi iteration given $A \in \mathbb{R}^{N \times k}$, $b \in \mathbb{R}^N$ and $k \in \mathbb{N}$. Try to write that function yourself, referring to the lecture notes and codes as little as possible. Also, write tests to verify that your code is correct.

Hints.

- The defining properties of the Arnoldi iteration are:
  - $Q \in \mathbb{R}^{N \times (k+1)}$ is orthogonal.
  - $H \in \mathbb{R}^{(k+1) \times k}$ is Hessenberg ($H[i,j] = 0$ if $i > j+1$).
  - We have $Q[:,1] = \frac{b}{\|b\|_2}$ and $AQ[:,1:k] \approx QH$.
- Recall the basic orthogonalisation operation: given $a, b \in \mathbb{R}^N$, set

  $$\hat{b} := b - \frac{a^T b}{a^T a} a.$$

  Then, $\hat{b} \perp a$ and $\operatorname{span}\{a, b\} = \operatorname{span}\{a, \hat{b}\}$.