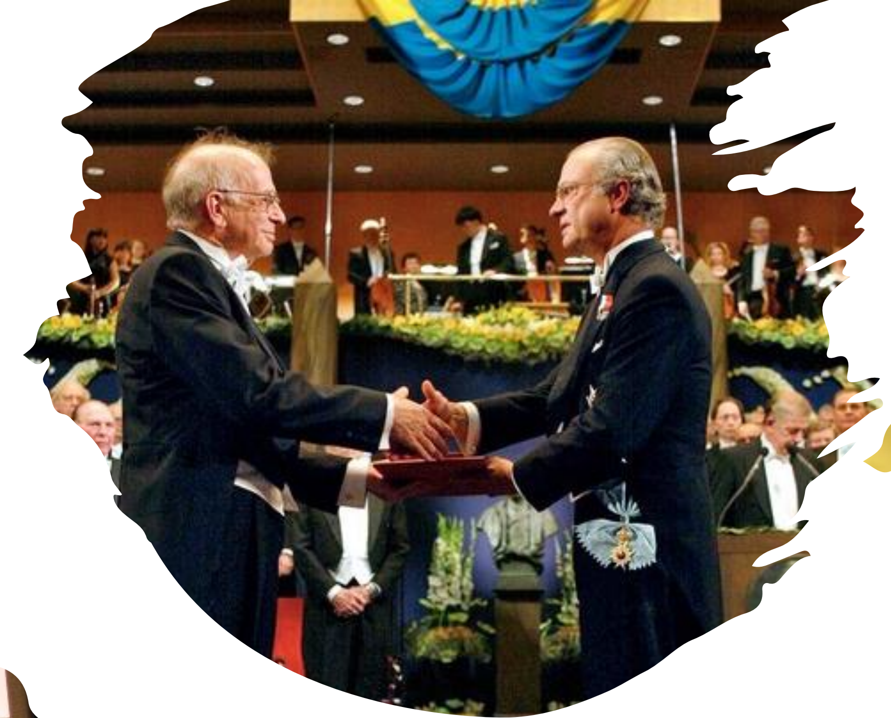


Daniel Kahneman

Nobel Memorial Prize in Economic Sciences 2002

For having integrated insights from psychological research into economic science, especially concerning human judgment and decision-making under uncertainty

Presidential Medal of Freedom 2013

AI101

Lecture 7: Uncertainty

Judea Pearl

ACM A.M. Turing Award 2011

For fundamental contributions to artificial intelligence through the development of a calculus for probabilistic and causal reasoning.

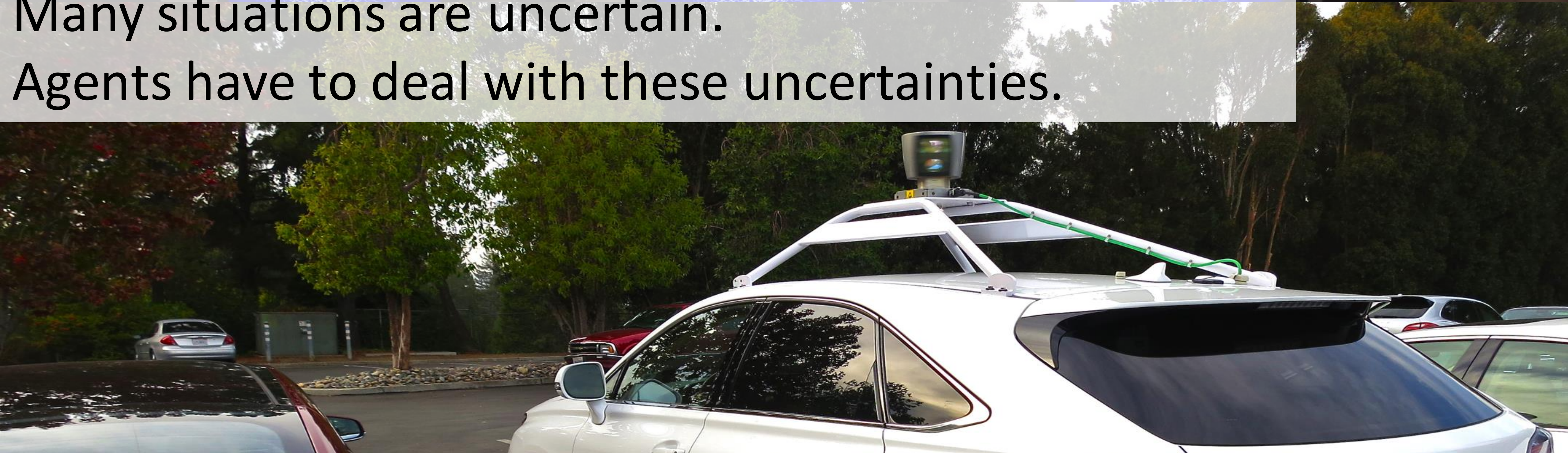
Recap
First Order Logic

Would like our AI to have knowledge about the world, and logically draw conclusions from it

- Notion of an object and use variables (placeholders) to specify abstract knowledge to specify true regularities
- Syntax & Semantics
- Unification
- Skolem
- Resolution
- Gödel’s Incompleteness Theorem

Many situations are uncertain.

Agents have to deal with these uncertainties.



So far…

So far, agents believed that:

- (logical) statements are true or false (maybe unknown)
- actions will always do what we think they do

Unfortunately, the real world is not like that:

- Agents almos never have access to the whole truth, i.e. the complete/perfect information

Agents must deal with **uncertainty**

Uncertainty
Outline

How can agents deal with uncertainties?

Today

- Uncertainty
- Probability
- Syntax and Semantics
- Inference
- Independence and Bayes' Rule
- Bayesian Networks
- Inference in Bayesian Networks

So far…

Example: Getting to the Airport

“We want to get to the airport to take a flight. When to we leave?”

- We have different actions for getting to the airport

  → action A_t = leave for the airport t minutes before departure

- Typical problems
  - Will a given action A_t get me to the airport in time?
  - Which action is the best choice for getting me to the airport

So far…

Example: Getting to the Airport

Risks involved in the plan A_{90} will get me to the airport (leaving 90min before departure)

- partial observability (road state, other drivers' plans, etc.)
- noisy sensors (traffic reports may be wrong)
- uncertainty in action outcomes (flat tire, accident, etc.)
- immense complexity of modeling and predicting traffic

A logically correct plan: A_{90} *will get me to the airport as long as my car doesn't break down, I don't run out of gas, no accident, the bridge doesn't fall down,* **etc**

Unfortunately, it is impossible to model all things that can go wrong (**qualification problem**)

A more cautious plan: A_{1440} *will get me to the airport*

What would you do?

Probabilities

Probabilities are **one way** of handling uncertainty

- E.g. A_{90} will get me to the airport with probability 0.5

They may **summarize the effects** that are due to

- **Laziness**
  - I don't want to list all things that must not go wrong
- **Theoretical Ignorance**
  - Some things just can't be known
  - e.g.: We cannot completely model the weather
- **Practical Ignorance**
  - Some things might not be known about the particular situation
  - e.g. Is there a traffic jam at A5?

Probabilities

How to Understand Probabilities

Probabilities can also be related to one's (subjective) beliefs

- A probability p means that I believe that the statement will be true in $p \cdot 100\%$ of the cases.
- E.g. there is a traffic jam in *10%* of the cases
- It does not mean that the street is jammed by a degree of *10%*

So, **probability theory** is about the **degree of belief** not the **degree of truth**

Probabilities of propositions change with new evidence:

- P(A_{45} gets me there in time | no reported accidents) = 0.06
- P(A_{45} gets me there in time | no reported accidents, 5 a.m.) = 0.15

Probabilities

The **degree of belief** view resolves tricky issues

Consider the probability that the sun will still exist tomorrow.

- Difficult to observe by an experiment

What is the chance that a patient has a particular disease?

- A medical doctor wants to consider other patients who are similar. But if you gather too much information to compare patients, there are no similar patients left!

Probabilities
Basics

The state or sample space can be seen as a set of all samples:

- E.g. the sample space of a (single) die roll is 1,2,3,4,5,6 (sides)

$$\Omega$$

$$\omega \in \Omega$$

A probability space or probability model is a sample space with an assignment of probabilities per possible sample

E.g. $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$

$$P(\omega) \text{ for every } \omega \in \Omega$$

$$\sum_{\omega} P(\omega) = 1$$

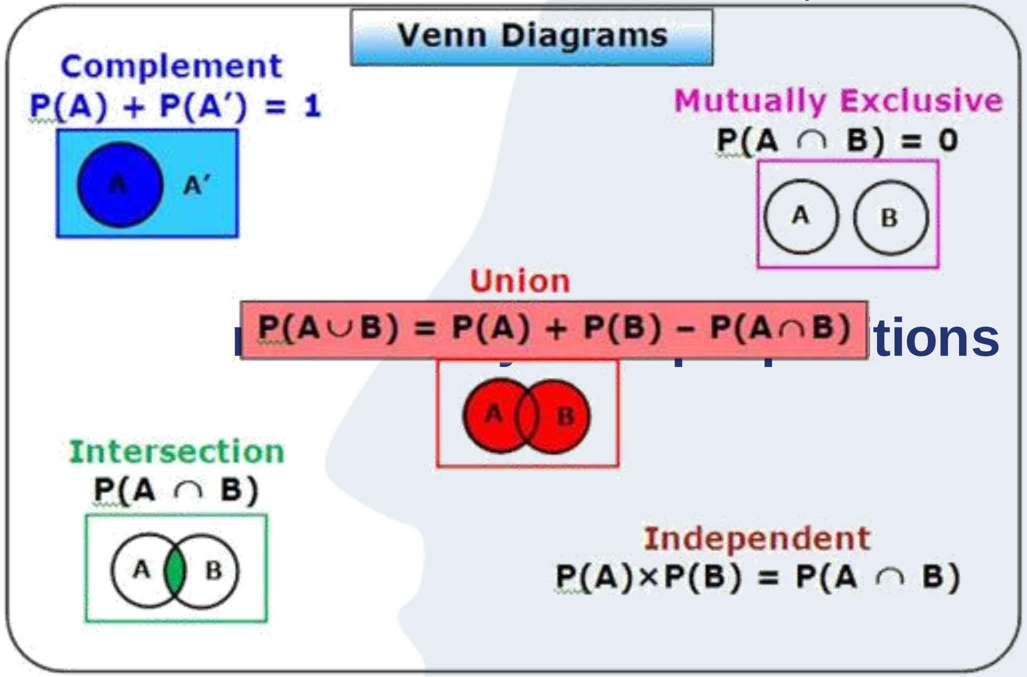
An event A is any subset of the sample space:

E.g. $P(\text{roll greater } 4) = P(5) + P(6) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$$P(A) = \sum_{\omega \in A} P(\omega)$$

Probabilities
Kolmogorov's Axioms of Probability

Due to Benjamin Owens



1. **All probabilities are between 0 and 1**

$$0 \leq P(a) \leq 1$$

2. **Necessarily true propositions have probability 1, tions have probability 0**

$$P(false) = 0, P(tue) = 1$$

3. **The probability of a disjunction is**

$$P(a \vee b) = P(a) + P(b) - P(a \wedge b)$$

4. **These axioms restrict the set of probabilistic beliefs that an agent can (reasonably) hold.**
   - similar to logical constraints like A and $\neg A$ can't both be true

*[Wikipedia] In gambling, a Dutch book is a set of odds and bets, established by the bookmaker, that ensures that the bookmaker will profit, at the expense of the gamblers, regardless of the outcome of the event (a horse race, for example) on which the gamblers bet.

Probabilities

You better do not violate the axioms of probability

Dutch Book* Theorem, Bruno de Finetti (1931)

- an agent (in the example it is Agent 1) who bets according to probabilities that violate the axioms of probability can be forced to bet so as to lose money *regardless of outcome!*

Bruno de Finetti

Example:

- suppose Agent 1 believes the following: $P(a) = 0.4, P(b) = 0.3, P(a \vee b) = 0.8$
- Agent 2 can now select a set of events and bet on them according to these probabilities so that she cannot loose (Agent 2 offers e.g. \$6 against Agent 1's \$4 for proposition a)

Agent 1		Agent 2		Outcome for Agent 1			
proposition	belief	bet	stakes	$a \wedge b$	$a \wedge \neg b$	$\neg a \wedge b$	$\neg a \wedge \neg b$
a	0.4	a	4:6	-6	-6	4	4
b	0.3	b	3:7	-7	3	-7	3
$a \vee b$	0.8	$\neg(a \vee b)$	2:8	2	2	2	-8
				-11	**-1**	**-1**	**-1**

Axioms of Probability are violated because $P(a \vee b) > P(a) + P(b)$

Agent 2 now designs a game, choosing to bet \$4 on a, \$3 on b and \$2 on not(avb). E.g., if a holds, Agent 1 would have to pay \$6, i.e. -6

So summing up, Agent 1 always loses money

Probabilities

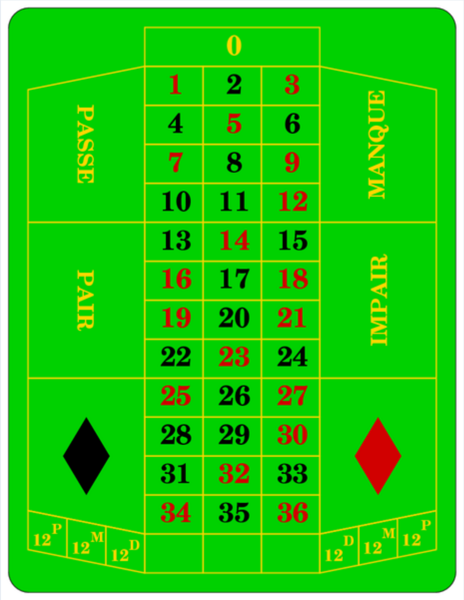
Let's make our life easier: Instead of events, let us use random variables

A random variable is a function from atomic events to some range of values

Example: Roulette

- Atomic events: numbers 0-36
- Random variables with outcomes true or false
  - Rouge | Noir, Pair | Impair, Passe | Manque
  - Transversale, Carre, Cheval
  - Douzaines premier | Milieu | Dernier
  - …

  E.g. rouge(36) = true



The probability function P over atomic events induces a probability distribution over all random variables X

$$P(X = x_i) = \sum_{\omega : X(\omega) = x_i} P(\omega)$$

Probabilities

Propositions, or towards uncertain knowledge

Think of a proposition as the event where the proposition is true:

Often in AI applications, the sample points are defined by the values of a set of random variables

- i.e. the sample space is the Cartesian product of the ranges of the variables

With Boolean variables, sample points = propositional logic model

- E.g. A=*true*, B=*false*, or $a \wedge \neg b$

Proposition = disjunction of atomic events in which it is true

- E.g. $(a \vee b) \equiv (\neg a \wedge b) \vee (a \wedge \neg b) \vee (a \wedge b) \quad \rightarrow P(a \vee b) = P(\neg a \wedge b) + P(a \wedge \neg b) + P(a \wedge b)$

Probabilities

Syntax of Propositions

Propositional or Boolean random can be true or false

- E.g. hasUmbrella,
- $hasUmbrella = true$ is a proposition, and can be simply written as $hasUmbrella$

Discrete random variables (finite or infinite)

- E.g. $Weather$ is one of $\langle sunny, rain, cloudy, snow \rangle$
- $Weather = rain$ is a proposition
- Values must be exhaustive and mutually exclusive

Continuous random variables (bounded or unbounded)

- E.g $Temp$ is an unbound variable
- $Temp = 25.5$, $Temp > 23$ are propositions

Joint Distributions

Uncertain/Quantified Truth Table

A joint distribution gives the probability of combined events

- E.g. The probability that $X=x$ and $Y=y$ is true $P(x, y) \equiv P(X = x \wedge Y = y)$

Cancer

Smoking

	no	benigne	maligne
no	0.768	0.024	0.008
few	0.132	0.012	0.006
many	0.035	0.010	0.005

The joint distribution allows us to answer any question! But how?

Marginalization (or Summing Out)

We do not want to talk always about all variable!

For any set of variables X and Y we can compute the probability

$$P(Y) = \sum_{i=1}^{n} P(x_i, Y)$$

The resulting distribution is called marginal distribution and its probabilities are the marginal probabilities

Cancer

Smoking	no	benigne	maligne
no	0.768	0.024	0.008
few	0.132	0.012	0.006
many	0.035	0.010	0.005

$$P(Y = few) = P(no, few) + P(benigne, few) + P(maligne, few) = 0.15$$

Conditional Probabilities
are kind of “probability distribution for a sub-population”

A **conditional probability** can be described as the probability of X=x under the assumption that Y=y is true:

$$P(x|y) = \frac{P(x \wedge y)}{P(y)}$$

The **Product rule** gives us an alternative formulation

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

The **Chain rule** can be derived by successive application of the Product rule:

$$P(X_1, X_2, ..., X_n) = P(X_1, ..., X_{n-1})P(X_n|X_1, ..., X_{n-1})$$
$$= P(X_1, ..., X_{n-2})P(X_{n-1}|X_1, ..., X_{n-2})P(X_n|X_1, ..., X_{n-1})$$
$$= \prod_{i=1}^{n} P(X_i|X_1, ..., X_{i-1})$$

Conditional Probabilities
Example

Cancer

Smoking	no	benigne	maligne
no	0.768	0.024	0.008
few	0.132	0.012	0.006
many	0.035	0.010	0.005

Lets assume $P(Y = few)$

We already now that $P(Y = few) = 0.15$

Now we want to calculate $P(X = maligne|Y = few) = \frac{P(maligne, few)}{P(Y = few)} = \frac{0.006}{0.15} = 0.04$

Lets assume $P(X = maligne)$

We calculate $P(maligne) = P(maligne, no) + P(maligne, few) + P(maligne, many) = 0.019$

Now we want to calculate $P(Y = few|X = maligne) = \frac{P(maligne, few)}{P(maligne)} = \frac{0.006}{0.019} = 0.316$

Joint Distributions

Can we reduce the complexity of the joint distribution?

Yes, if we can make use of independencies

X and Y are independent from another if one of the following is true;

$$P(X|Y) = P(X)$$
$$P(Y|X) = P(Y)$$
$$P(X, Y) = P(X)P(Y)$$

Independent variables are not effected by the other variable

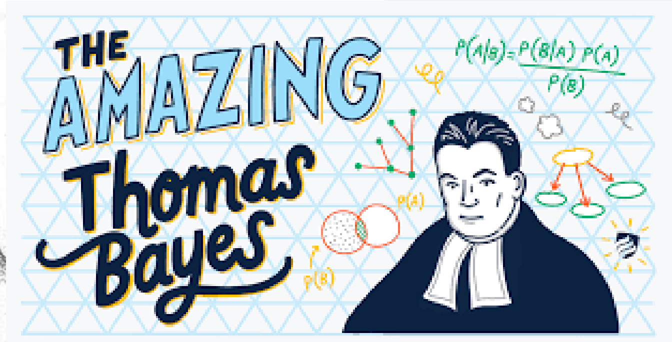
- This reduces the amount of possible values

But…

- **Absolut independent variables are rare**
- **i.e. in cancer research there are a lot of variables, none of which are independent**

Bayes Rule

In contrast to logic, no input & output (if P>0)!



Rev. Thomas Bayes (c. 1702 – 1761)
English theologian and mathematician

Product rule: $P(x,y) = P(x|y)P(y) = P(y|x)P(x)$

Bayer’ rule:

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)}$$

Probability of the evidence, given the belief is true. This is called **Likelihood**

Probability of the hypothesis before considering the evidence. This is called **Prior**

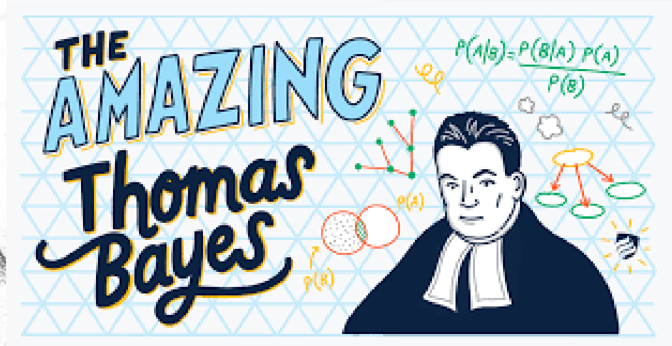
Probability of the hypothesis X after the evidence Y. This is called **Posterior**

Probability of the evidence Y under any circumstance. This is called **Marginalization**

Bayes Rule

Example: AIDS-Test

Bayer' rule: $P(x|y) = \dfrac{P(y|x)P(x)}{P(y)}$



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Events

- Aids = a person is infected or not
- Positive = a person has a positive test result

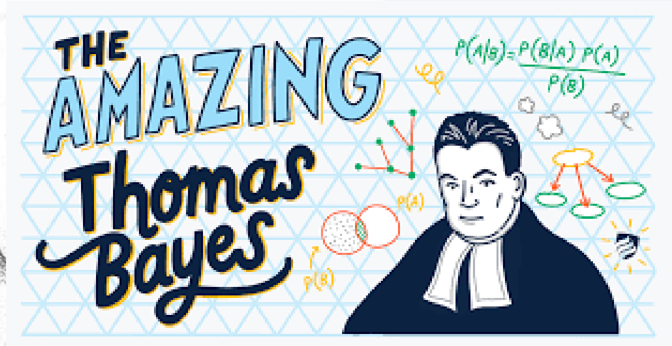
Probabilities

$$P(positive, aids) = 0.99,$$
$$P(negative, aids) = 0.01,$$
$$P(positive, \neg aids) = 0.005,$$
$$P(negative, \neg aids) = 0.995$$

Is this test reliable?

Bayes Rule

Example: AIDS-Test



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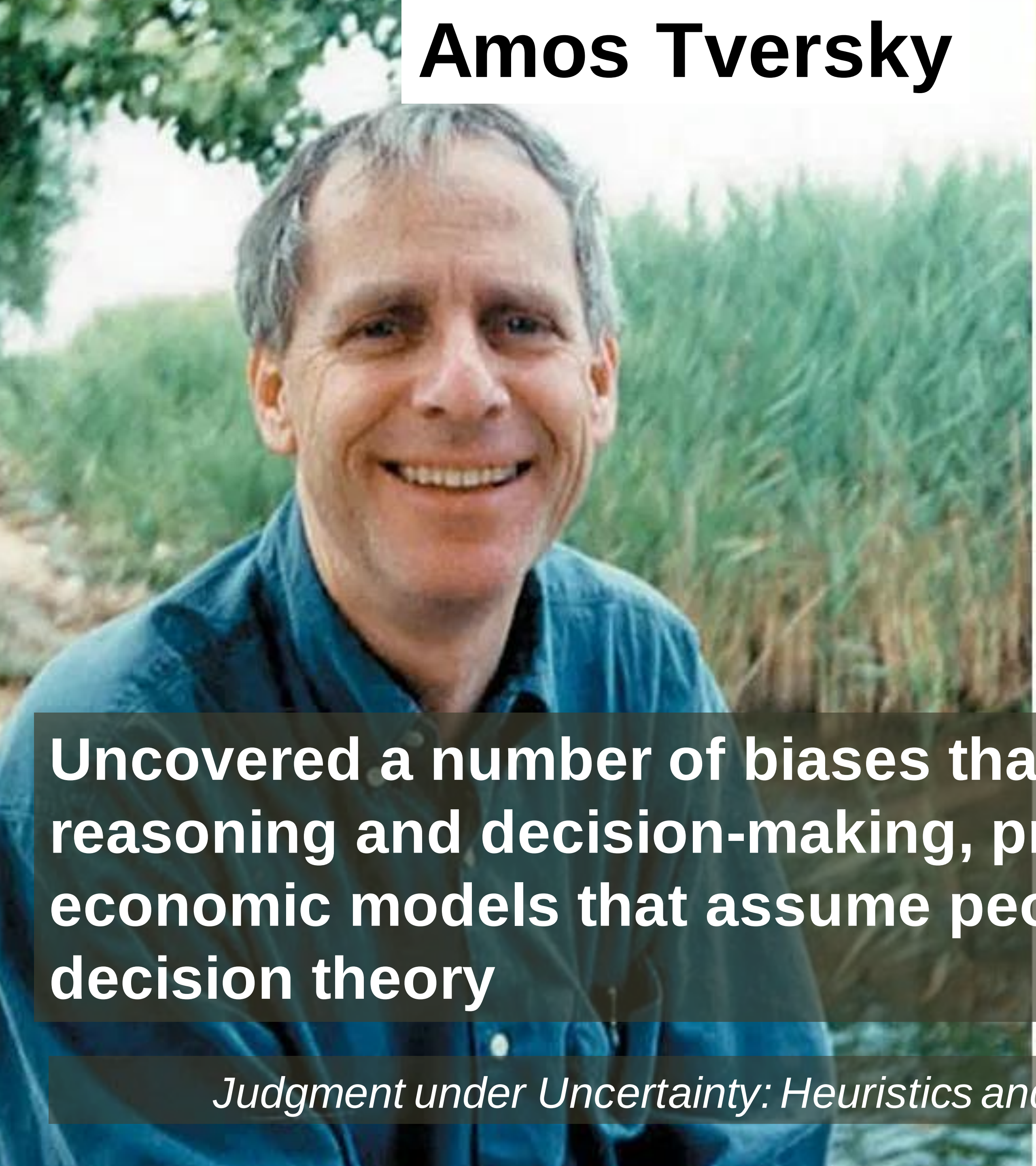
Bayer' rule: $P(x|y) = \frac{P(y|x)P(x)}{P(y)}$

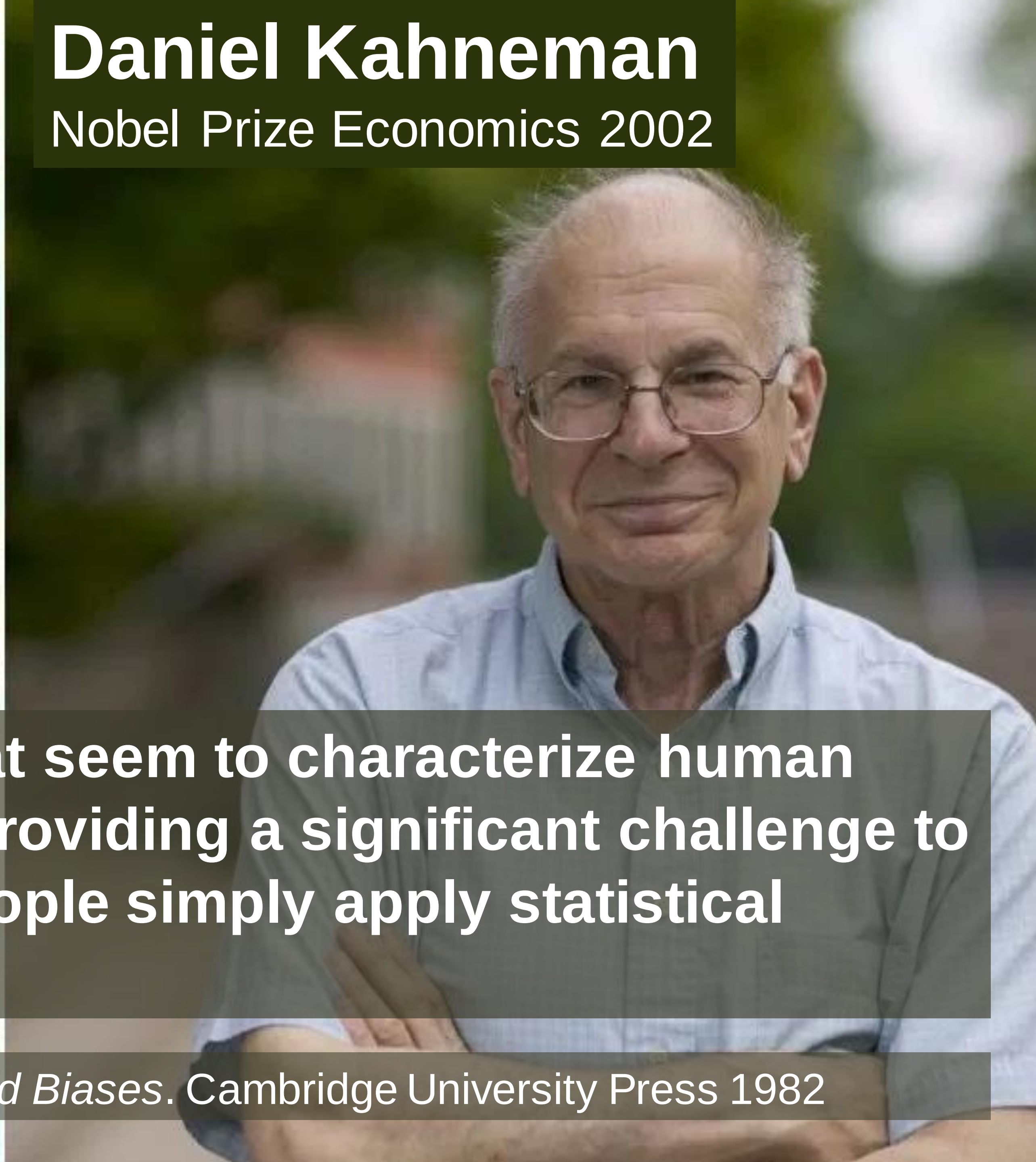
Probabilities

$$P(positive, aids) = 0.99,$$
$$P(negative, aids) = 0.01,$$
$$P(positive, \neg aids) = 0.005,$$
$$P(negative, \neg aids) = 0.995$$

Lets assume the risk for you to having aids is $P(aids) = 0.0001$ *(Prior)* and now you have a positive test result. Should you panic?

$$P(a|p) = \frac{P(p|a)P(a)}{P(p)} = \frac{P(p|a)P(a)}{P(p|a)P(a) + P(p|\neg a)P(\neg a)}$$
$$= \frac{0.99 \cdot 0.0001}{0.99 \cdot 0.0001 + 0.005 \cdot 0.9999} = \mathbf{0.0194}$$





Amos Tversky

Daniel Kahneman
Nobel Prize Economics 2002

Uncovered a number of biases that seem to characterize human reasoning and decision-making, providing a significant challenge to economic models that assume people simply apply statistical decision theory

Judgment under Uncertainty: Heuristics and Biases. Cambridge University Press 1982

Uncertainty in AI

How can we deal with uncertainty on a computer?

Recall Joint distribution is enumerating everything

- Worst-case run time: $O(2^n)$
  - n = # of RVs
- Space is $O(2^n)$ too
  - Size of the table of the joint distribution

Mission over? No!! Our mission has just started

Main idea: make use of independencies to compress the representation

Summary

- Uncertainty is omnipresent
- Uncertainty can be captured using probability distributions
- Not following the axioms of probability theory makes you lose
- Maringalization, Bayes rule, chain rule, …

You should be able to:

- Argue why not following the axoims of probabilties is bad
- Compute marginals from joint distriubtions