Circuit Theory and Electronic Fundamentals
Lab Report - T2

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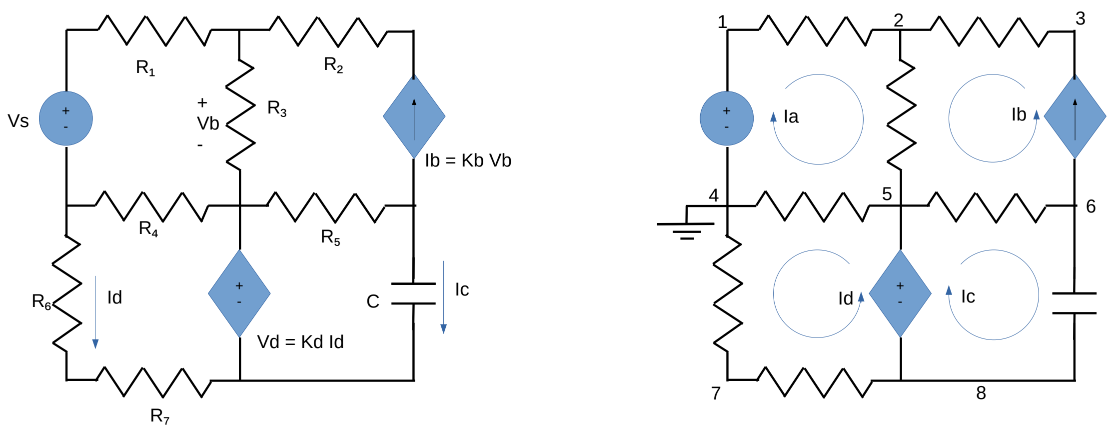
Contents

1 Introduction

In this laboratorial session we were exposed to an electrical circuit (Figure 1a) composed of four meshes, eight nodes and eleven total elements: 7 resistors (R_1 through R_7), 1 voltage source (v_s), 1 capacitor (C), 1 voltage controlled current source and, finally, 1 current controlled voltage source. The assumed node names and current directions are represented in Figure 1b.



(a) (b)

Figure 1: Circuit diagram (a) and assumed node names and current direction (b)

For the theoretical analysis (section 2), we used the nodal method for $t < 0$ (which results in the current through the capacitor being zero), in which, through Kirchoffs Current Law we determined the voltages for all the nodes and, consequently by applying Ohm's Law to resistors R_1, R_2 and R_6, the currents in the circuit meshes. Then, using Thévenins's theorem we calculated the Equivalent Resistance seen from the terminals of the capacitor, which we then used to determine the natural solution $v_{6n}(t)$. Afterwards we computed the Forced Solution by applying the same method as the one used for $t < 0$ replacing, however, the voltages and currents with their respective phasors and the conductances of the resistors with the corresponding admittances, this resulted in a vector of the voltage phasors at every node, which we then used to create the forced solution (using the magnitude and phase of $\widetilde{V_6}$). We finally obtained the Total solution simply by superimposing the natural and forced solutions. To finalize the theoretical anlysis we analysed the frequency response of $v_c(f) = v_6(f) - v_8(f)$ and $v_6(f)$.

To get the simulation in Ngspice (section 3) we begin with an operating point analysis, getting the values for the currents and the voltages for t¡0 and for vs=0 (for this last one we replaced the capacitor with a voltage source $V_x = V_6 - V_8$ where V_6 and V_8 were obtained in the previous section). We then go on to the Transient Analysis, in which we computed and plotted the Natural Response and the Total Response of the voltage in node 6. Finally, for the Frequency Response, we do a frequency sweep so that we're able to plot both the magnitude and phase of v_6, v_s and $v_c = v_6 - v_8$.

Finally, in section 4, we analysed the tables obtained from the theoretical analysis and the ones obtained from the simulation software and came to the conclusion that there are little to no differences between these two methods.

2 Theoretical Analyses

2.1 Node Analysis

Firstly we begin by analising the circuit for $t < 0$. Using the Kirchoff's Current Law we can determine the currents in branches and voltages in nodes. The following equations were used in the node analysis:

Node 2:

$$\begin{gathered} I_1 + I_2 - I_3 = 0 \iff \\ \iff G_1(V_1 - V_2) + G_2(V_3 - V_2) - G_3(V_2 - V_5) = 0 \iff \\ \iff G_1V_1 - (G_1 + G_2 + G_3)V_2 + G_2V_3 + G_3V_5 \end{gathered} \tag{1}$$

Node 3:

$$\begin{gathered} -I_2 + I_b = 0 \iff \\ \iff -G_2(V_3 - V_2) + K_b(V_2 - V_5) \iff \\ \iff (K_b + G_2)V_2 - G_2V_3 - K_bV_5 = 0 \end{gathered} \tag{2}$$

Node 6:

$$\begin{gathered} -I_b + I_5 - I_c = 0 \iff \\ \iff -K_b(V_2 - V_5) + G_5(V_5 - V_6) - I_c = 0 \iff \\ \iff -K_bV_2 - G_5V_6 + (K_b + G_5)V_5 = 0 \end{gathered} \tag{3}$$

Node 7:

$$\begin{gathered} I_c - I_7 = 0 \iff \\ \iff G_6(V_4 - V_7) - G_7(V_7 - V_8) = 0 \iff \\ \iff G_6V_4 - (G_6 + G_7)V_7 + G_7V_8 = 0 \end{gathered} \tag{4}$$

We can conclude by observation:

$$V_1 - V_4 = V_s \tag{5}$$

The current controlled voltage source also gives:

$$\begin{gathered} V_d = K_dI_d \iff \\ \iff V_5 - V_8 = K_dG_6(V_4 - V_7) \iff \\ \iff -V_8 + K_dG_6V_7 + V_5 - K_dG_6V_4 = 0 \end{gathered} \tag{6}$$

We get the last equation analysing the supernode:

$$\begin{gathered} I_3 - I_4 - I_5 + I_7 + I_c = 0 \iff \\ \iff G_3(V_2 - V_5) - G_4(V_5 - V_4) - G_5(V_5 - V_6) + G_7(V_7 - V_8) + 0 = 0 \iff \\ \iff G_3V_2 + G_4V_4 - (G_3 + G_4 + G_5)V_5 + G_5V_6 + G_7V_7 - G_7V_8 = 0 \end{gathered} \tag{7}$$

Using the 8 equations above, we have the following linear equations system:

$$\begin{bmatrix} G1 & -(G1+G2+G3) & G2 & 0 & G3 & 0 & 0 & 0 \\ 0 & Kb+G2 & -G2 & 0 & -Kb & 0 & 0 & 0 \\ 0 & Kb & 0 & 0 & -(Kb+G5) & G5 & 0 & 0 \\ 0 & 0 & 0 & G6 & 0 & 0 & -(G6+G7) & G7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -KdG6 & -1 & 0 & KdG6 & 1 \\ 0 & G3 & 0 & G4 & -(G3+G4+G5) & G5 & G7 & -G7 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ V_s \\ 0 \\ 0 \end{bmatrix}$$

Using Octave to calculate the solution of the system we obtain:

V_1	5.00662
V_2	4.75586
V_3	4.22864
V_4	0
V_5	4.79054
V_6	5.55906
V_7	-1.90293
V_8	-2.84482

Figure 2: Electric potential of the 8 nodes

We can now use Ohm's Law to calculate the current flowing through each resistor which, consequently, allows us to reach the values for the currents in the 4 meshes.

I_a	0.000242748
I_b	-0.000254098
I_c	0
I_d	0.000308297

Figure 3: Current values for the different meshes

2.2 Calculating Equivalent Resistance

In order to calculate the equivalent resistance seen from the terminals of the capacitor, we use the Thévenin's Theorem by replacing the capacitor with a voltage source with an equivalent value of Vx = V(6)-V(8) = 8.4039 V, this is done in order to simulate the capacitor being fully charged. Now the equivalent resistance is the value of the hypotetical resistor in series with the voltage source Vx. We now assume Vs = 0V, and since the voltage source has an interior resistance of 0, it is equivalent to short-circuit that source. We can now perform a node analysis using KCL and other restrains to determine the voltages in each node, and then the currents in each mesh.

Equations (1), (2), (4), (5), (6) and (7) from the previous section are equivalent in this one. We can get the remaining 2 equations from:

$$V_x = V_6 - V_8 \tag{8}$$

And from node 4, which has been short-circuited with node 1:

$$\begin{gathered} -I_1 + I_4 - I_6 = 0 \iff \\ \iff -G_1(V_1 - V_2) + G_4(V_5 - V_4) - G_6(V_4 - V_7) = 0 \iff \\ \iff -G_1V_1 + G_1V_2 - (G_4 + G_6)V_4 + G_4V_5 + G_6V_7 = 0 \end{gathered} \tag{9}$$

With all the equations we can now form the following linear equation system:

$$\begin{bmatrix} G1 & -(G1+G2+G3) & G2 & 0 & G3 & 0 & 0 & 0 \\ 0 & Kb+G2 & -G2 & 0 & -Kb & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & G6 & 0 & 0 & -(G6+G7) & G7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -KdG6 & -1 & 0 & KdG6 & 1 \\ -G1 & G1 & 0 & -(G4+G6) & G4 & 0 & G6 & 0 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ Vx \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Octave to calculate the solution of the system we obtain:

V_1	0
V_2	0
V_3	0
V_4	0
V_5	0
V_6	8.40388
V_7	0
V_8	0

Figure 4: Electric potential of the 8 nodes

Now that we have the values for the voltages in every node, we can use Ohm's Law to calculate the current flowing through V_x: $V_6 - V_8 = R_5 \cdot I_x \iff I_x = \frac{V_6 - V_8}{R_5} = 0.0027786$ A, and with the current I_x, the equivalent resistance becomes $R_{eq} = \frac{V_x}{I_x} = 3024.499\ \Omega$.

2.3 Natural Solution

We will now analyse the natural solution V6n(t) when t varies in the interval [0,20] ms. To do so, we will use the equivalent resistance calculated before.

We start with the equation of a linear capacitor:

$$Q(t) = Cv(t) \iff Q(t) = -CR_{eq}i(t) \iff Q(t) = -CR_{eq}\frac{dQ(t)}{dt} \iff \frac{dQ(t)}{Q(t)} = -\frac{dt}{CR_{eq}} \iff$$
$$\iff \int_{Q_0}^{Q(t)} \frac{1}{Q(t)}dQ(t) = \int_0^t -\frac{1}{CR_{eq}}dt \iff ln(\frac{Q(t)}{Q_0}) = -\frac{t}{CR_{eq}} \iff e^{ln(\frac{Q(t)}{Q_0})} = e^{-\frac{t}{CR_{eq}}} \iff$$
$$\iff \frac{Q(t)}{Q_0} = e^{-\frac{t}{CR_{eq}}} \iff Q(t) = Q_0 e^{-\frac{t}{CR_{eq}}} \iff \frac{Q(t)}{C} = \frac{Q_0}{C}e^{-\frac{t}{CR_{eq}}} \iff$$
$$\iff V_{6n}(t) = V_0 e^{-\frac{t}{CR_{eq}}} \tag{10}$$

Assuming the initial condition of $V_0 = V_x$ the natural solution for the capacitor is:

$V_{6n}(t) = V_x e^{-\frac{t}{CR_{eq}}}$

We can now plot the graph of the solution for t $\in$[0,20] ms:

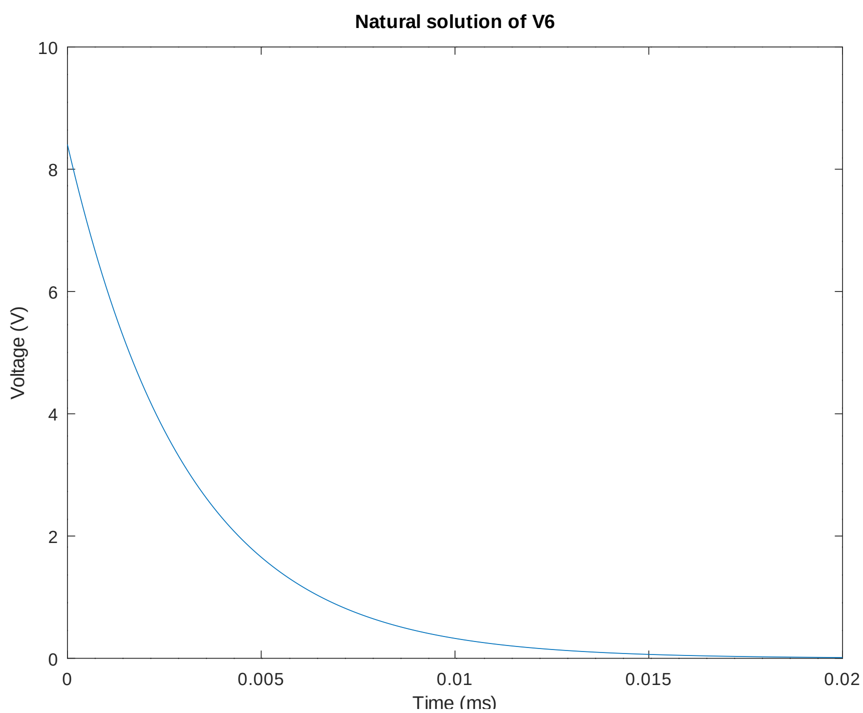


Figure 5: Natural solution of the capacitor

2.4 Forced Solution

Now we will analyse the forced solution v6f(t) when t varies in [0,20] ms. To do so we will do once more nodal analysis. Using the phasor $\widetilde{V_S} = V_s e^{-j\varphi_s}$, and using equations (1), (2), (4), (5), (6) and (7), we can now get the final 2 equations from:

Node 6:

$$\begin{gathered} -\widetilde{I_b} + \widetilde{I_5} - \widetilde{I_c} = 0 \iff \\ \iff -K_b(\widetilde{V_2} - \widetilde{V_5}) + G_5(\widetilde{V_5} - \widetilde{V_6}) - Y_c(\widetilde{V_6} - \widetilde{V_8}) = 0 \iff \\ \iff Kb\widetilde{V_2} - (Kb + G5)\widetilde{V_5} + (Yc + G5)\widetilde{V_6} - Yc\widetilde{V_8} = 0 \end{gathered} \tag{11}$$

Super Node:

$$\begin{gathered} \widetilde{I_3} - \widetilde{I_4} - \widetilde{I_5} + \widetilde{I_7} + \widetilde{I_c} = 0 \iff \\ \iff G_3(\widetilde{V_2} - \widetilde{V_5}) - G_4(\widetilde{V_5} - \widetilde{V_4}) - G_5(\widetilde{V_5} - \widetilde{V_6}) + G_7(\widetilde{V_7} - \widetilde{V_8}) + Y_c(\widetilde{V_6} - \widetilde{V_8}) = 0 \iff \\ \iff G3\widetilde{V_2} + G4\widetilde{V_4} - (G3 + G4 + G5)\widetilde{V_5} + (G5 + Yc)\widetilde{V_6} + G7\widetilde{V_7} - (G7 + Yc)\widetilde{V_8} = 0 \end{gathered} \tag{12}$$

With the 8 equations we have the following linear equation system:

$$\begin{bmatrix} G1 & -(G1+G2+G3) & G2 & 0 & G3 & 0 & 0 & 0 \\ 0 & Kb+G2 & -G2 & 0 & -Kb & 0 & 0 & 0 \\ 0 & Kb & 0 & 0 & -Kb-G5 & Yc+G5 & 0 & -Yc \\ 0 & 0 & 0 & G6 & 0 & 0 & -(G6+G7) & G7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -KdG6 & -1 & 0 & KdG6 & 1 \\ 0 & G3 & 0 & G4 & -G3-G4-G5 & G5+Yc & G7 & -G7-Yc \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Using Octave to calculate the solution of the system we obtain:

V_1	1
V_2	0.949914
V_3	0.84461
V_4	0
V_5	0.95684
V_6	0.570346
V_7	0.380083
V_8	0.568212

Figure 6: Electric potential of the 8 nodes

2.5 Total Solution

The total solution v6t(t) can be obtained by superimposing the natural and forced solutions. In section 2.2 we determined the function of the natural solution, now we will determine the function of the forced solution before proceding with the superimposing.

The complex equation of the phaser is:

$$\widetilde{V_s} = V_s e^{-j\varphi_s} \implies v_{6f}(t) = V_6 sin(\omega t + \varphi_s) \tag{13}$$

Where $\omega = 2\pi f = 2000\pi$ and φ is the argument of $\widetilde{V_S}$

Now we can superimpose both equations and obtain the final total solution:

$$v_6(t) = v_{6n}(t) + v_{6f}(t) \iff v_6(t) = V_6 sin(\omega t + \varphi_s) + V_x e^{-\frac{t}{R_5 C}} \tag{14}$$

If we plot this function for t $\in$[-5,20] ms we get the following graph:

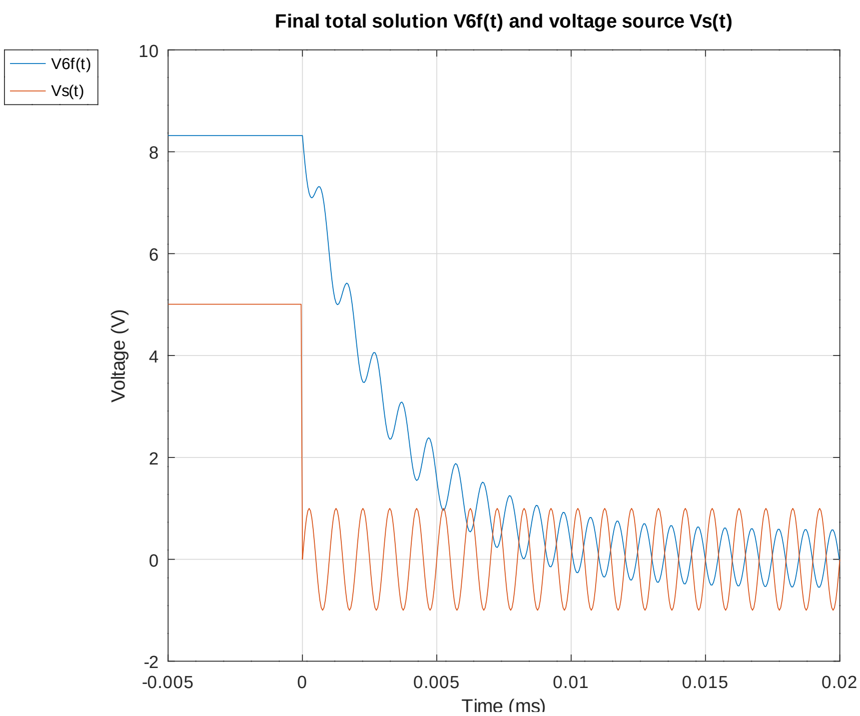


Figure 7: Total solution $v_6(t)$

2.6 Frequency Response

Finally we will analyse the frequency response of each phasor. To do so we will simply use the same method we used in section 4, however now Y_c will remain as a variable.

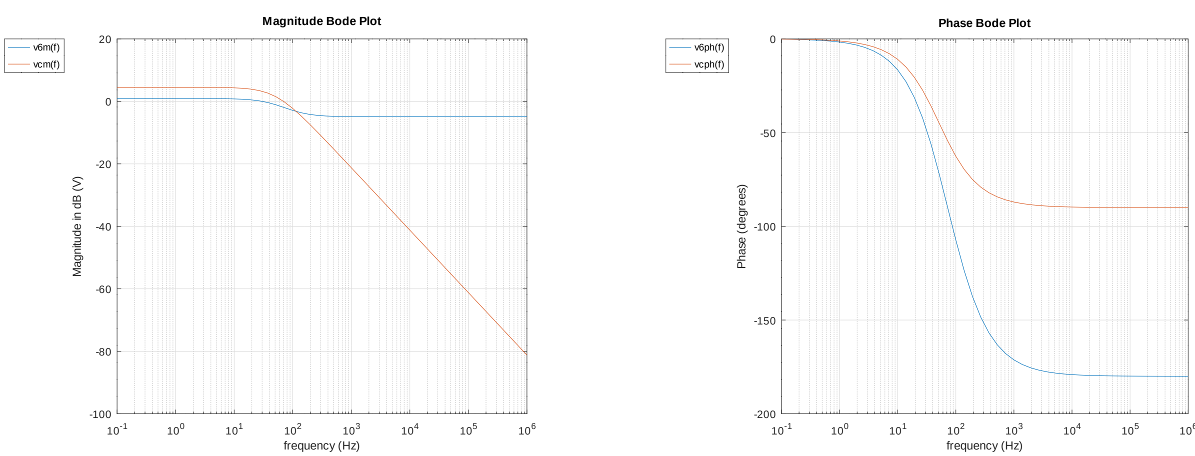
$Y_c(f) = j2\pi fC$

This way, the result that Octave gives us (after solving the matrix equation) for the different voltages (now as phasors) will depend on f. By defining $\widetilde{V_c}$ as the difference between the phasors $\widetilde{V_6}$ and $\widetilde{V_8}$, we can plot both the magnitude and the phase of $\widetilde{V_6}$ and $\widetilde{V_c}$ as functions of f $\in [10^{-1}\ ,\ 10^{6}]$ Hz, which we will define in a logarithmic scale.

The magnitude plot will have values of the magnitude in dB, which we define in the following way:

$V_{dB} = 20log_{10}(V)$

We can now plot the resulting functions $v_c(f)$ and $v_6(f)$:



(a) (b)

Figure 8: Magnitude (a) and phase plots (b) of $v_c(f)$ and $v_6(f)$

As we can see in the magnitude plot above (Figure 8a), for low frequencies, the magnitude of v_c is close to 1V (the magnitude of the source v_s) and as the frequency increases, this magnitude drops (as seen by the descending slope) for this reason, we can confirm that this circuit can be classified as a low-pass filter, this results from the nature of the linear capacitor: it takes time to discharge! and if the frequency of the source is too high it will not have enough time to do so and thus the voltage difference between the positive and negative plates will be nearly constant = sinusoidal with amplitude of 0 (is dB 0 does not exist but it can be expressed as a limit torward negative infinity which is exactly what we see in the magnitude plot of $v_c(f)$).

However, in the case of v_6, we see the same behavior at the lower half of the frequency space (constant followed by a descending slope) but for the higher half the slope dies down and the magnitude of v_6 approaches another constant (close to $10^{-5}V$).

3 Simulation

3.1 Operating Point Analysis for $t < 0$ and v_S(t) = 0

Using Ngspice to perform an operating point analysis for $t < 0$, we obtain Figure 9 and after setting $v_S = 0$ and replacing the capacitor with a voltage source $V_x = V_6 - V_8$ (with V_6 and V_8 obtained from the nodal analysis), we obtain the results for the voltages and the currents in Figure 10:

Name	Value (A or V)
@c1[i]	0.000000e+00
@g1[i]	-2.54098e-04
@r1[i]	2.427474e-04
@r2[i]	-2.54098e-04
@r3[i]	-1.13501e-05
@r4[i]	1.154213e-03
@r5[i]	-2.54098e-04
@r6[i]	9.114659e-04
@r7[i]	9.114659e-04
v(1)	5.006622e+00
v(2)	4.755862e+00
v(3)	4.228644e+00
v(5)	4.790538e+00
v(6)	5.559056e+00
v(7)	-1.90293e+00
v(8)	-2.84482e+00
v(9)	-1.90293e+00

Figure 9: Currents and voltages obtained for $t < 0$

Name	Value (A or V)
@g1[i]	-4.17770e-18
@r1[i]	3.991093e-18
@r2[i]	-4.17770e-18
@r3[i]	-1.86611e-19
@r4[i]	-8.55977e-19
@r5[i]	-2.77861e-03
@r6[i]	-8.67362e-19
@r7[i]	-1.68559e-18
v(1)	0.000000e+00
v(2)	-4.12284e-15
v(3)	-1.27910e-14
v(5)	-3.55271e-15
v(6)	8.403900e+00
v(7)	1.810851e-15
v(8)	3.552714e-15
v(9)	1.810851e-15

Figure 10: Currents and voltages obtained for $v_S = 0$

It is important to note that Node 9 is only used to define a 0V voltage source between R_6 and R_7 in order to use the current flowing through it to fully define the current controlled voltage source.

3.2 Transient Analysis - Natural Response

In order to compute the natural response of v_6, we first need to set $v_S(t) = 0$ and set initial conditions so that v_6 and v_8 are what we obtained in Figure 10, this ensures that in the start of thetransient analysis the capacitor is fully charged. By performing a transient analysis for $t \in [0, 20]ms$ we obtain the plot represented in Figure 11.

We can see that v_6 starts with a voltage of a little over 8V and quickly drops to almost 0 after only 20ms, this is explained by the fact that the capacitor (which started fully charged) is now only discharging and since there are no independent sources in the circuit, there is nothing to provide v_6 with voltage.

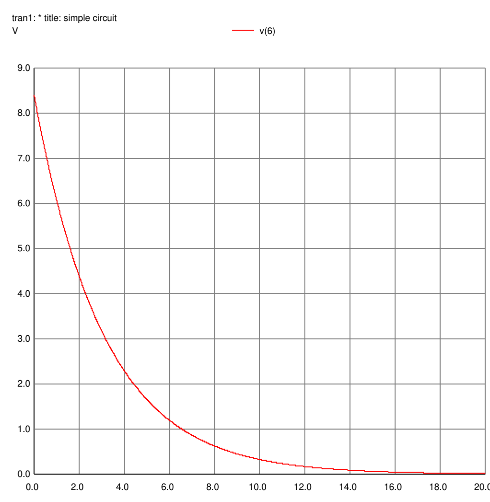


Figure 11: Natural response

3.3 Transient Analysis - Total Response

The total response is obtained by performing a transient analysis whilst $v_S(t)$ is given by $v_S(t) = sin(2\pi f t)$ with f = 1000Hz, by plotting both $v_S(t)$ and $v_6(t)$ we obtain the Figure 12 in which we can clearly see both the decaying exponential nature of the natural response and the sinusoidal of the excitation voltage.

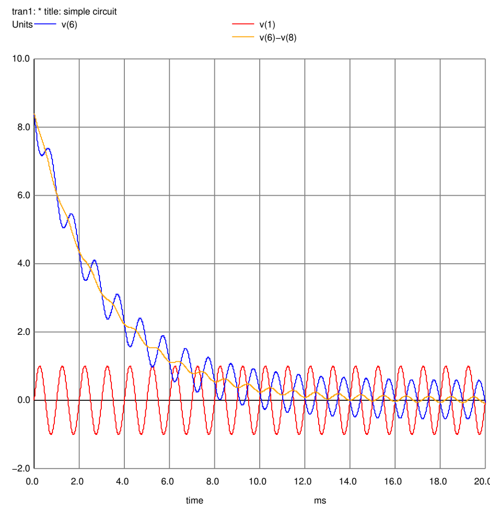


Figure 12: Stimulus voltage $v_S(t)$ (red) and total response of $v_6(t)$ (blue)

3.4 Frequency Response

By doing a frequency sweep (analysing v_S and v_6 while changing the value of f) over the interval $[10^{-1}, 10^6]Hz$, we can plot both the **magnitude** (Figure 13) and the **phase** (Figure 14)

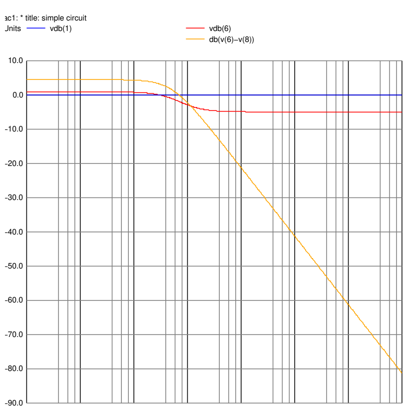


Figure 13: Magnitude plot

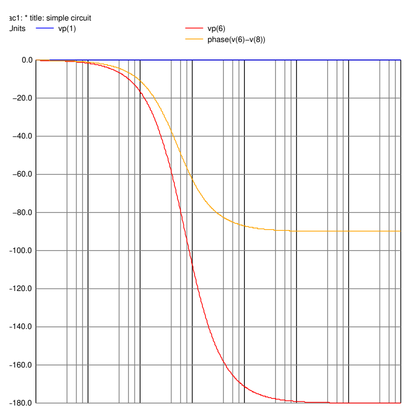


Figure 14: Phase plot

4 Conclusion

Now that we've obtained all the results from both the theoretical and the simulation analysis, we will compare them ...

4.1 V_S constant and non zero

Performing nodal analysis (octave table on the left) and operating point analysis (ngspice table on the right) for a dc voltage source V_S, we obtain the following results (Figure 15) in which we can clearly see the only difference being matters of rounding and the number of significant figures the Octave table provides.

V_1	5.00662
V_2	4.75586
V_3	4.22864
V_4	0
V_5	4.79054
V_6	5.55906
V_7	-1.90293
V_8	-2.84482

Name	Value (A or V)
@c1[i]	0.000000e+00
@g1[i]	-2.54098e-04
@r1[i]	2.427474e-04
@r2[i]	-2.54098e-04
@r3[i]	-1.13501e-05
@r4[i]	1.154213e-03
@r5[i]	-2.54098e-04
@r6[i]	9.114659e-04
@r7[i]	9.114659e-04
v(1)	5.006622e+00
v(2)	4.755862e+00
v(3)	4.228644e+00
v(5)	4.790538e+00
v(6)	5.559056e+00
v(7)	-1.90293e+00
v(8)	-2.84482e+00
v(9)	-1.90293e+00

Figure 15: Theoretical results using Octave (left) and simulation results from Ngspice (right)

4.2 $V_S = 0$ and capacitor replaced by voltage source

After setting the voltage source V_S off and replacing the capacitor by a dc voltage source $V_x = V_6 - V_8$ as obtained by the methods above (this is used in order to simulate the charged capacitor) and performing the same analysis as in Figure 15, we arrive at results in Figure 16. In the Ngspice table, we can see a repeated pattern of voltages whose order of magnite is around 10^{-15}, this can be explained as approximations that the software makes while solving the respective systems os linear equations. Nonetheless, these voltages are so incredibly small that they are negligible.

V_1	0
V_2	0
V_3	0
V_4	0
V_5	0
V_6	8.40388
V_7	0
V_8	0

Name	**Value (A or V)**
@g1[i]	-4.17770e-18
@r1[i]	3.991093e-18
@r2[i]	-4.17770e-18
@r3[i]	-1.86611e-19
@r4[i]	-8.55977e-19
@r5[i]	-2.77861e-03
@r6[i]	-8.67362e-19
@r7[i]	-1.68559e-18
v(1)	0.000000e+00
v(2)	-4.12284e-15
v(3)	-1.27910e-14
v(5)	-3.55271e-15
v(6)	8.403900e+00
v(7)	1.810851e-15
v(8)	3.552714e-15
v(9)	1.810851e-15

Figure 16: Theoretical results using Octave (left) and simulation results from Ngspice (right)

We can thus see that in both the analysed cases, the voltage results from both methods agree with each other within a margin of $10^{-14}V$, we can then conclude that the Ngspice software does appear to use the same methods as the ones introduced in the theoretical analysis section.