All pure multipartite entangled states of qubits can be self-tested

Ivan Šupić

joint work with Maria Balanzo, Andrea Coladangelo, Remigiusz Augusiak and Antonio Acín

JFQI2023 , Tokyo, December 13th 2023

Joint work with:

Maria Balanzo-Juando
ICFO
Barcelona

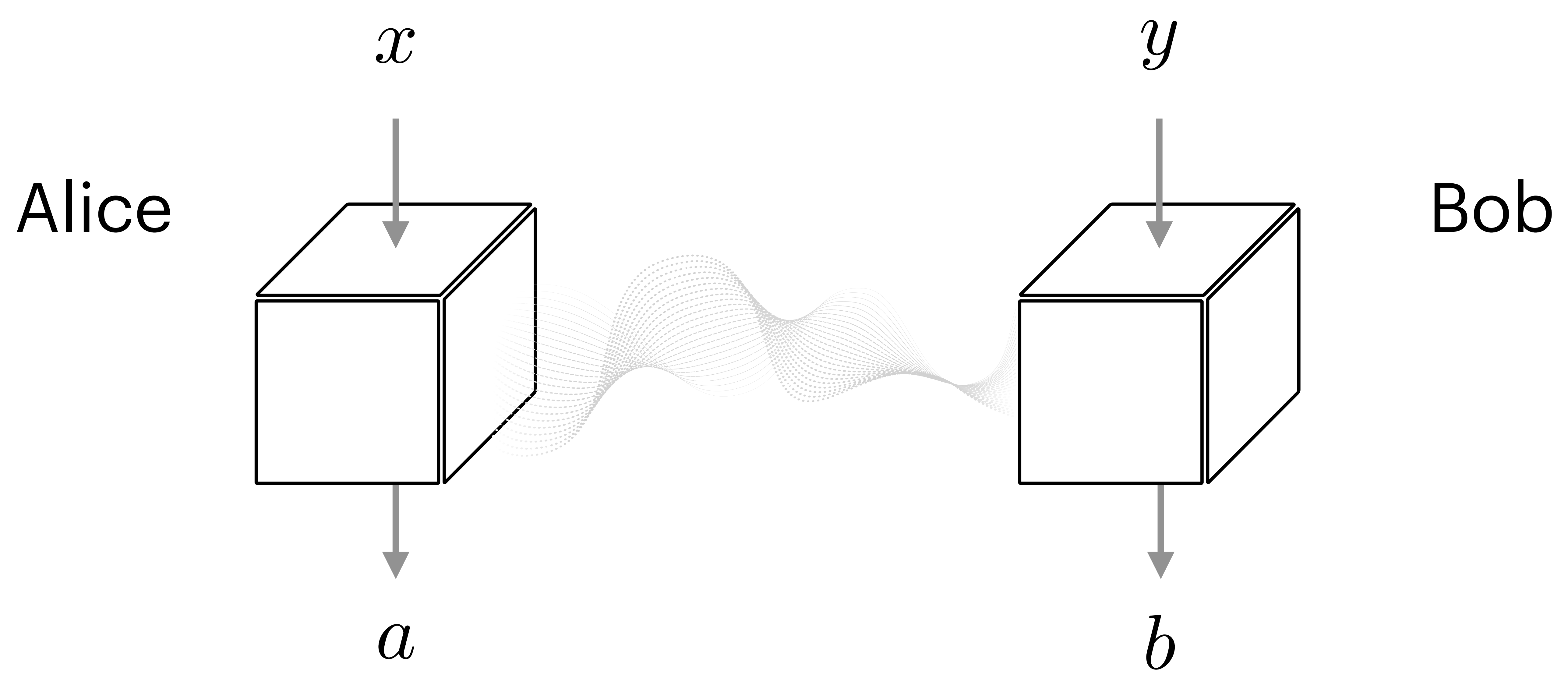
Andrea Coladangelo
University of Washington
Seattle

Remigiusz Augusiak
Center for Theoretical Physics
Warsaw

Antonio Acin
ICFO
Barcelona

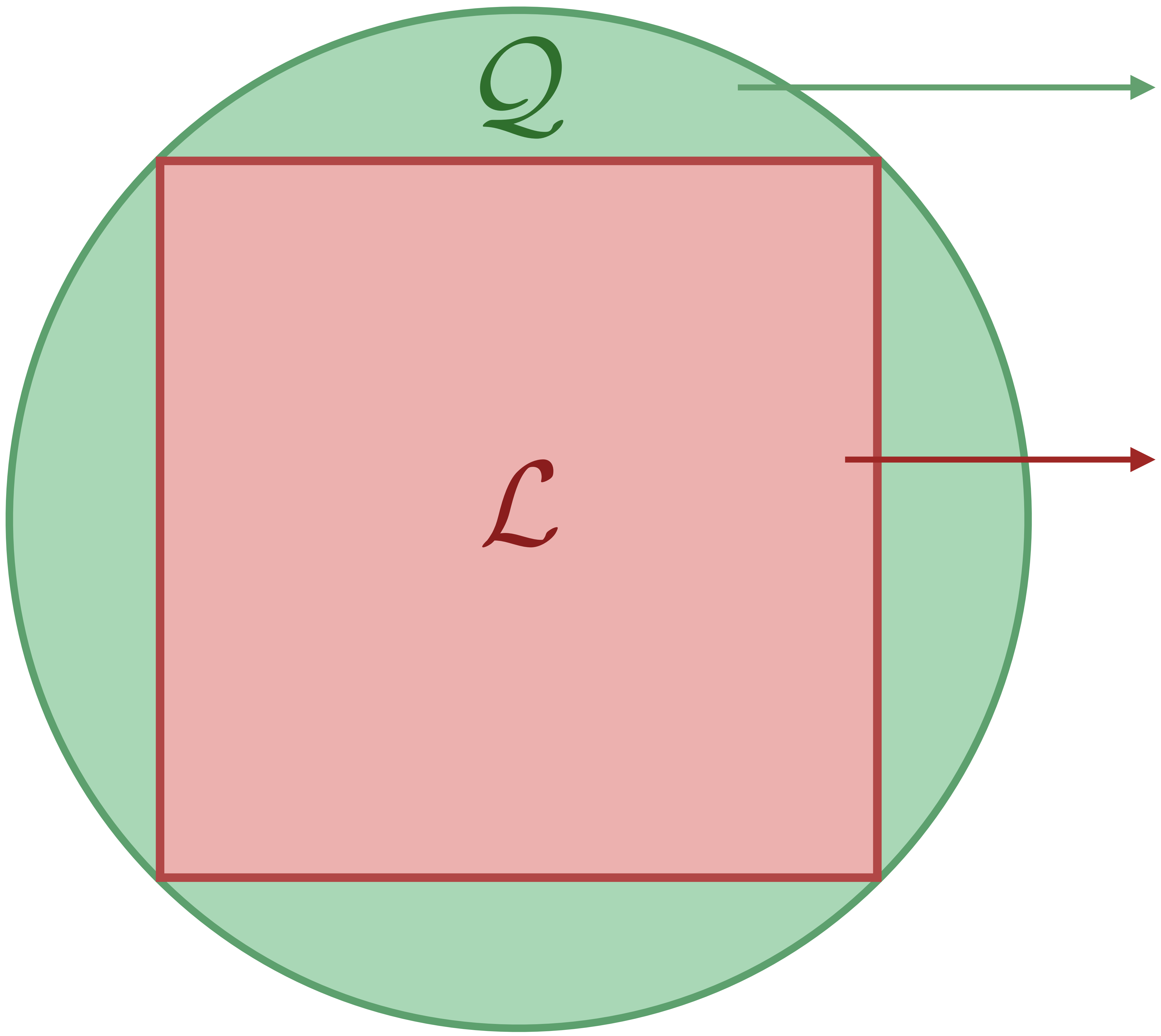
All pure multipartite qubit states can be self-tested
M. Balazo-Juando, A. Coladangelo, R. Augusiak, A. Acin, IŠ
arXiv: 2401.xxxxx

(Bipartite) Bell scenario



Correlation $p(a, b|x, y)$

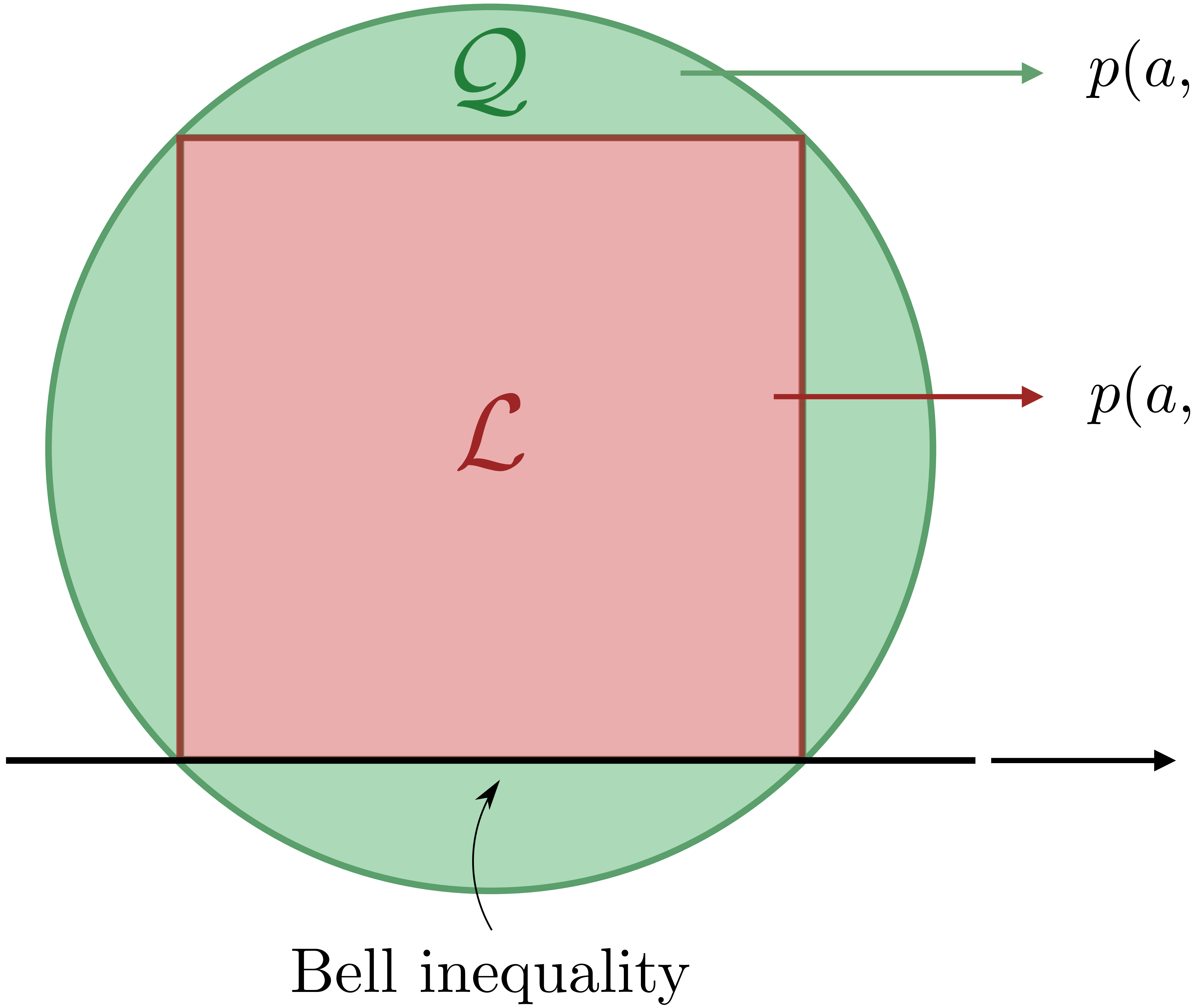
Bell inequalities



$$p(a,b|x,y) = \mathrm{tr}(\rho_{AB}\, M_{a|x} \otimes N_{b|y})$$

$$p(a,b|x,y) = \int d\lambda\, q(\lambda)\; p(a|x\lambda)\, p(b|y,\lambda)$$

Bell inequalities



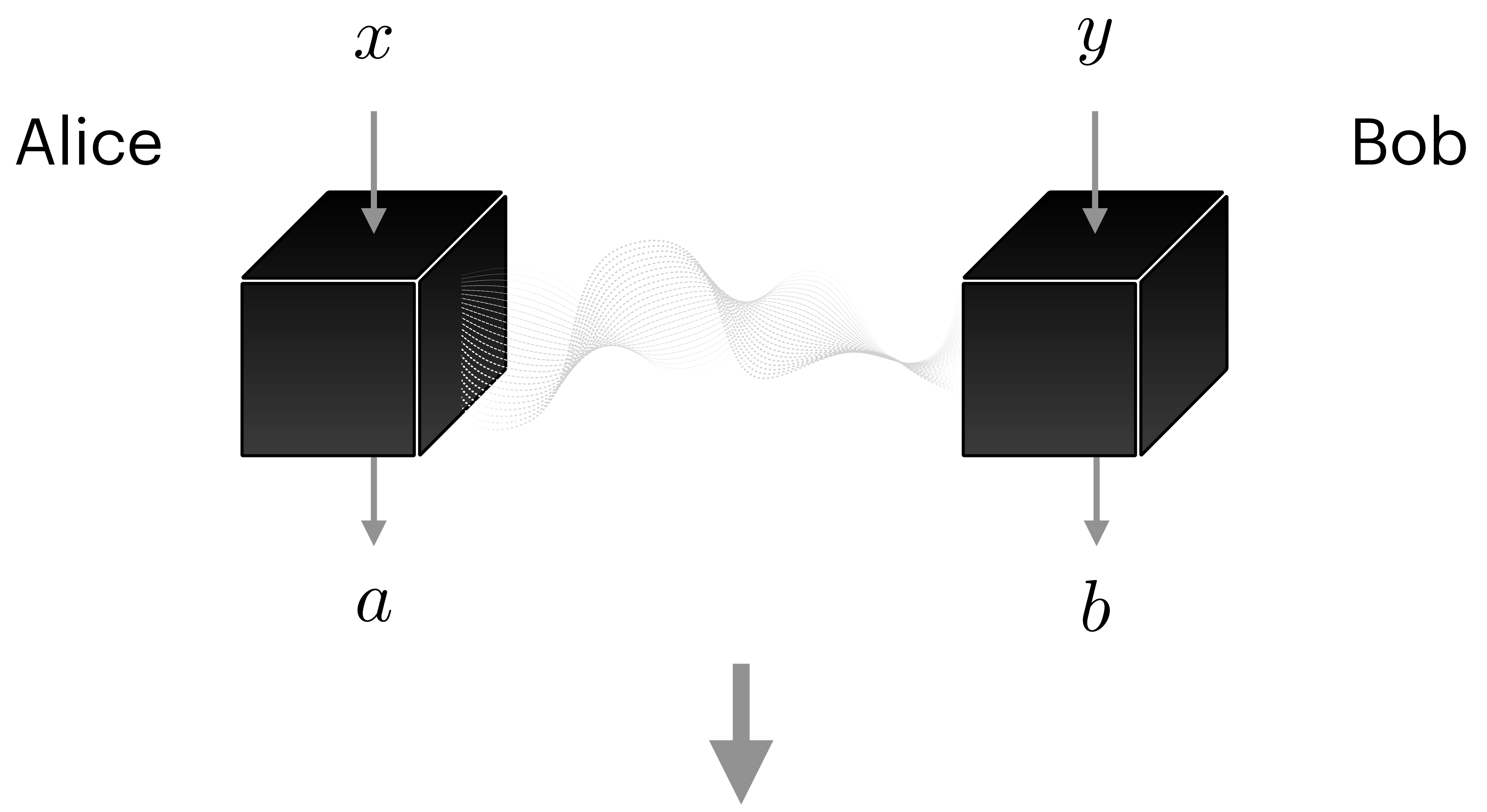
$$p(a,b|x,y) = \mathrm{tr}(\rho_{AB}\, M_{a|x} \otimes N_{b|y})$$

$$p(a,b|x,y) = \int d\lambda\, q(\lambda)\ p(a|x\lambda)\, p(b|y,\lambda)$$

$$\sum_{abxy} w_{abxy}\, p(a,b|x,y) \leq \eta$$

Device-independent formalism

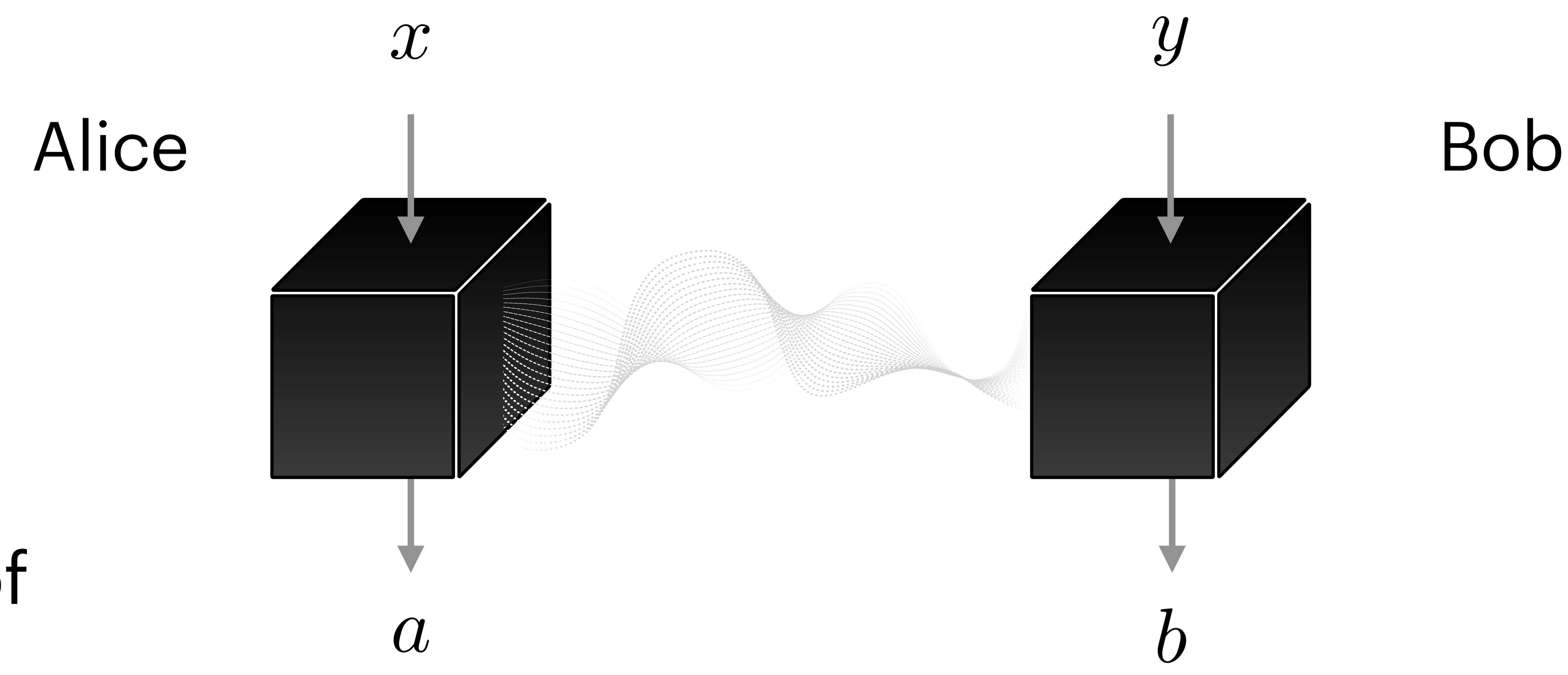
Devices treated as black boxes



Correlation $p(a, b|x, y)$

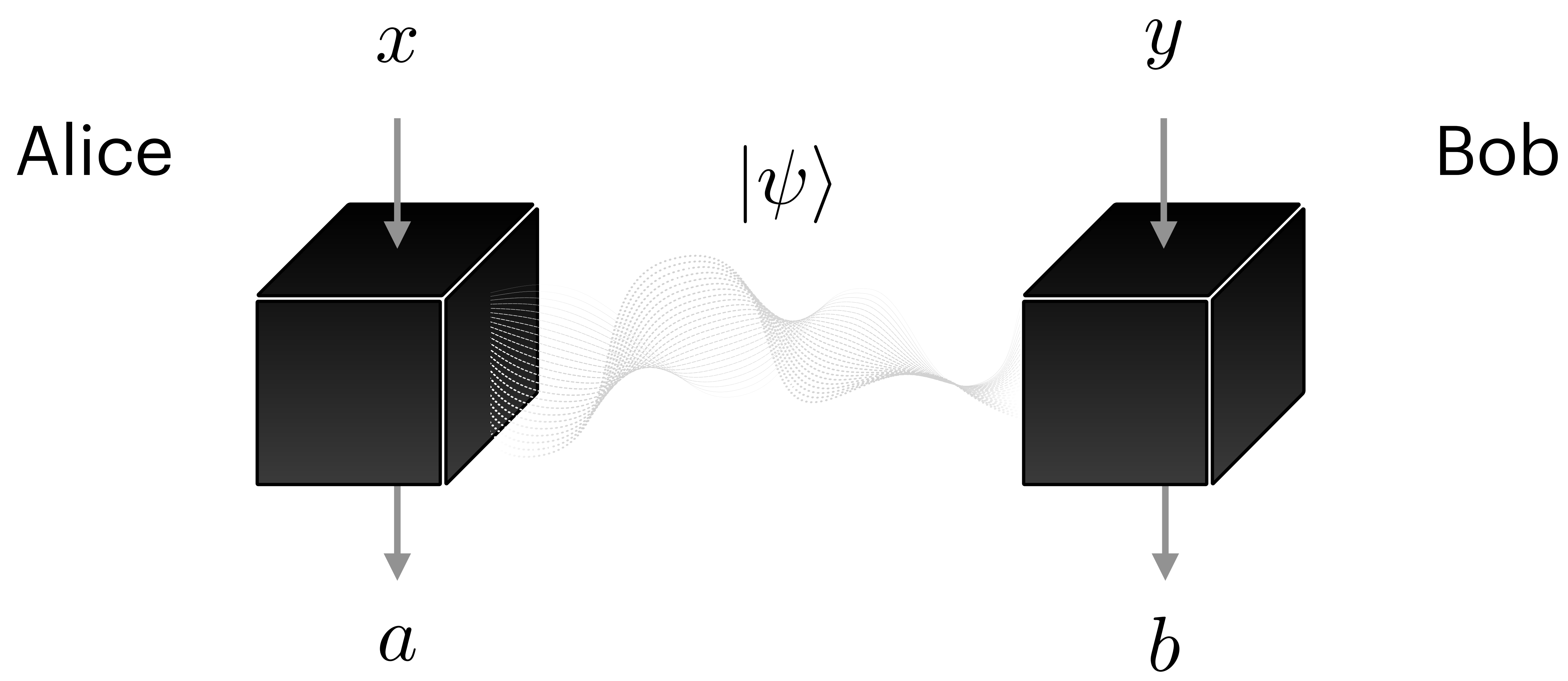
Device-independent paradigm

- Observing violation larger than η witnesses entanglement and incompatibility
- This witness is based only on the observed probability distribution $p(a, b \mid x, y)$
- There is no trust whatsoever in any of the devices used in the experiment
- This form of witnessing is known as **device-independent**.



Self-testing

Strongest form of certification



What is the shared state? What are the measurements?

Rigidity / self-testing

- Given that the violation larger than η witnesses entanglement, does η_q witness anything more?

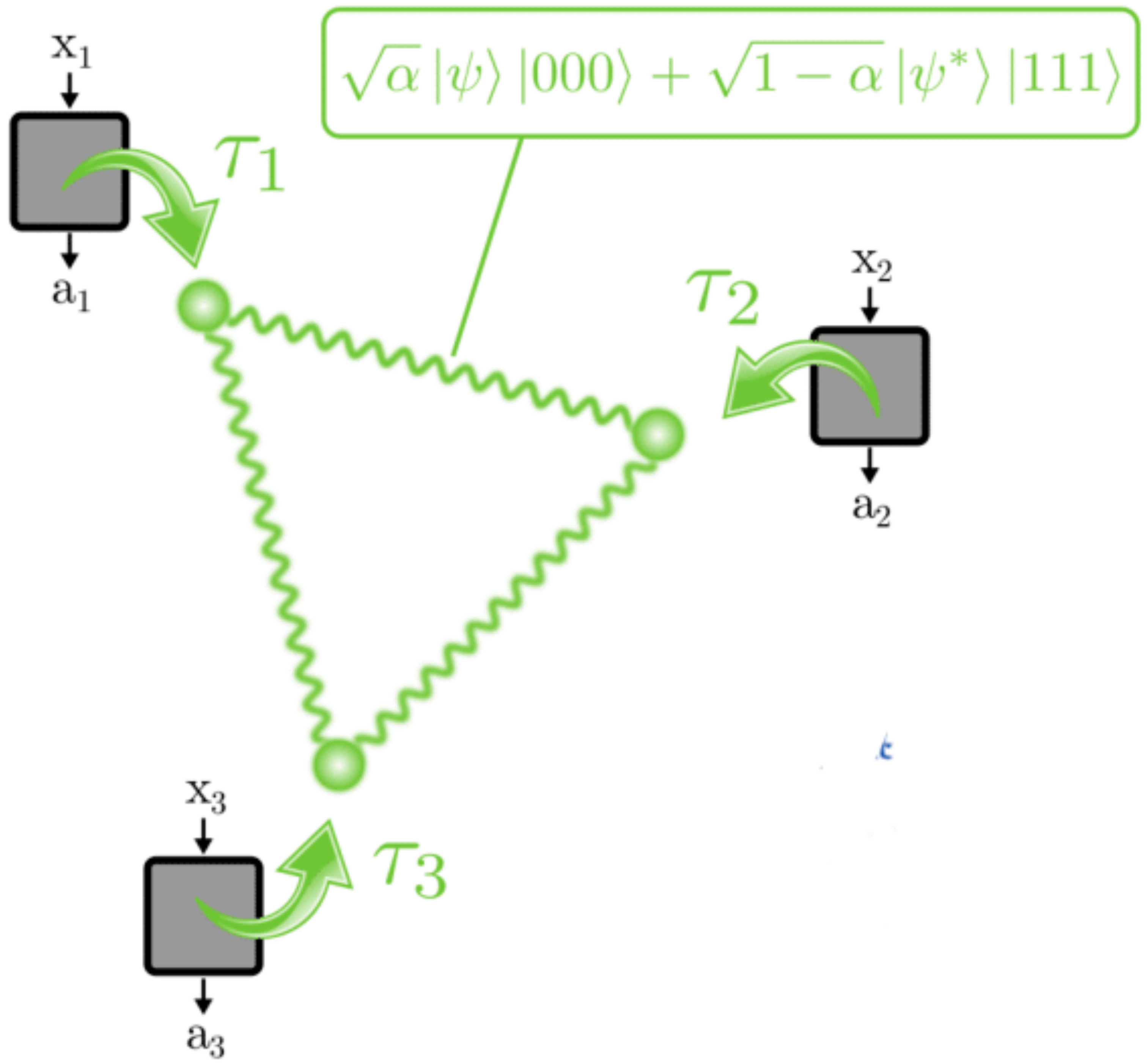
Rigidity / self-testing

- Given that the violation larger than η witnesses entanglement, does η_q witness anything more?
- Maximal score in the CHSH game can be achieved only by measuring maximally entangled pair of qubits. [Tsirelson87, Popescu&Rohrlich92]
- Rigidity: a rigid collection C of mathematical objects (for instance sets of functions) is one in which every $c \in C$ is uniquely determined by less information about c than one would expect.

Rigidity / self-testing

- Given that the violation larger than η witnesses entanglement, does η_q witness anything more?
- Maximal score in the CHSH game can be achieved only by measuring maximally entangled pair of qubits. [Tsirelson87, Popescu&Rohrlich92]
- Rigidity: a rigid collection C of mathematical objects (for instance sets of functions) is one in which every $c \in C$ is uniquely determined by less information about c than one would expect.

- Mayers-Yao (04): A state $|\psi_N\rangle$ can be self-tested if there exist quantum correlations $P(a_1, \cdots, a_N | x_1, \cdots, x_N)$ such that for any quantum realisation of them, there exists a set of N local quantum maps $\tau_1, \cdots, \tau_N$ that when applied to the unknown state of the system extract the state $|\psi_N^\alpha\rangle = \sqrt{\alpha}\,|\psi_N\rangle \otimes |0\rangle^N + \sqrt{1-\alpha}\,|\psi_N^*\rangle \otimes |1\rangle^N$ for some unknown $\alpha \in [0,1]$

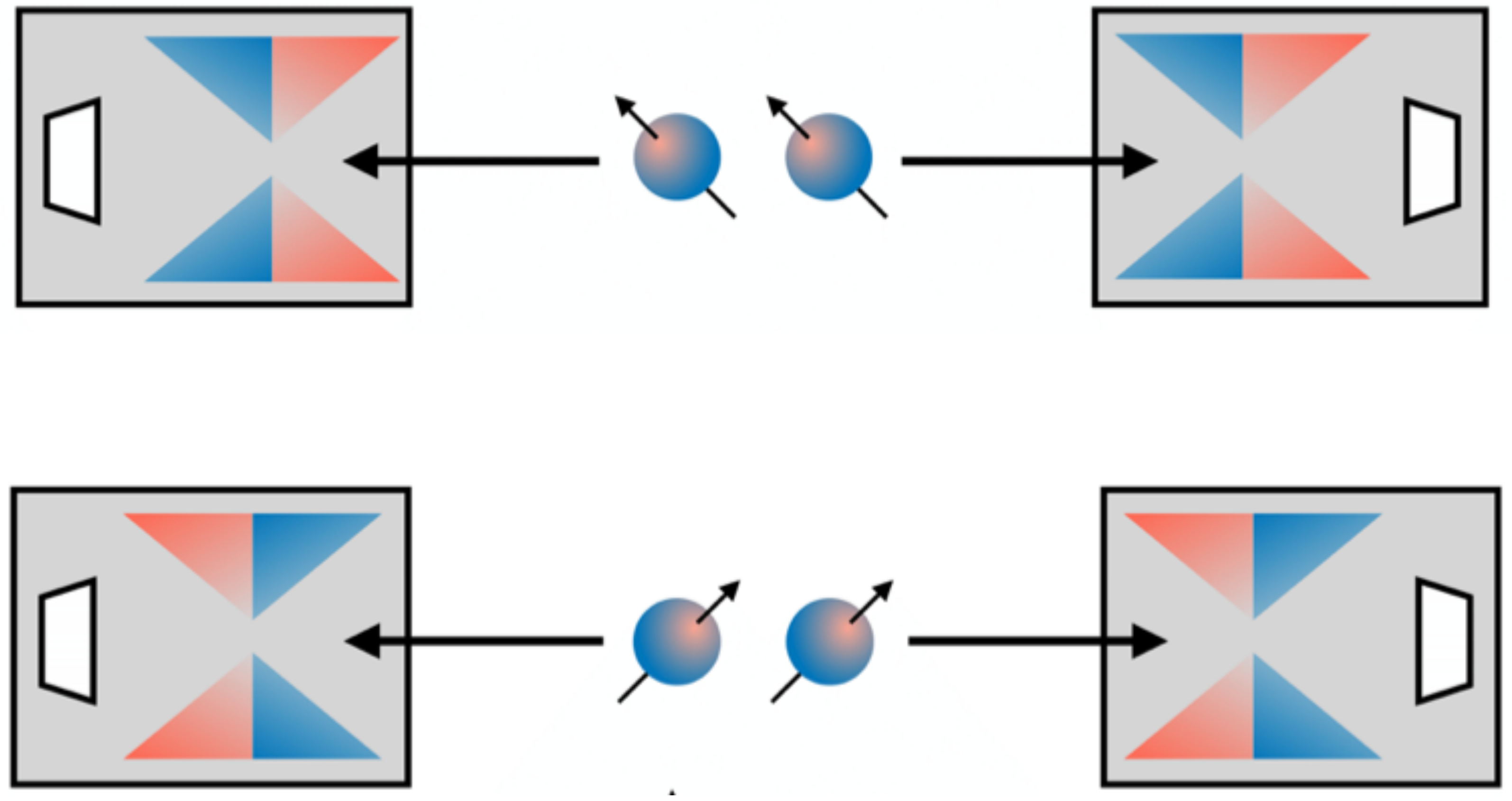


Self-testing: isometries and complex conjugation

- There is a whole class of state and measurement transformations which leave the probability distributions invariant
- Local change of basis
  $\{U_A \otimes U_B |\psi\rangle, U_A M_{a|x} U_A^\dagger, U_B N_{b|y} U_B^\dagger\} \Leftrightarrow \{|\psi\rangle, M_{a|x}, N_{b|y}\}$
- Uncorrelated degrees of freedom to which measurements act trivially
  $\{|\psi\rangle^{AB} \otimes |\phi\rangle^{A'B'}, M_{a|x}^{A} \otimes \mathbb{I}^{A'}, N_{b|y}^{B} \otimes \mathbb{I}^{B'}\} \Leftrightarrow \{|\psi\rangle, M_{a|x}, N_{b|y}\}$
- These transformations are are supposed to be encompassed by the local extraction maps
  $\tau_A \otimes \tau_B(|\psi'\rangle) \Leftrightarrow |\psi\rangle$

Self-testing: isometries and complex conjugation

- There is a whole class of state and measurement transformations which leave the probability distributions invariant
- Local change of basis
  $\{U_A \otimes U_B |\psi\rangle, U_A M_{a|x} U_A^\dagger, U_B N_{b|y} U_B^\dagger\} \Leftrightarrow \{|\psi\rangle, M_{a|x}, N_{b|y}\}$
- Uncorrelated degrees of freedom to which measurements act trivially
  $\{|\psi\rangle^{AB} \otimes |\phi\rangle^{A'B'}, M_{a|x}^A \otimes \mathbb{I}^{A'}, N_{b|y}^B \otimes \mathbb{I}^{B'}\} \Leftrightarrow \{|\psi\rangle, M_{a|x}, N_{b|y}\}$
- These transformations are are supposed to be encompassed by the local extraction maps
  $\tau_A \otimes \tau_B(|\psi'\rangle) \Leftrightarrow |\psi\rangle$



- Simultaneous complex conjugation of states and measurements does not change the correlation probabilities
- $\tau(|\psi'\rangle) = |\psi\rangle \otimes |\mathrm{junk}_0\rangle + |\psi^*\rangle \otimes |\mathrm{junk}_1\rangle$
- $\langle \mathrm{junk}_0 | \mathrm{junk}_0\rangle + \langle \mathrm{junk}_1 | \mathrm{junk}_1\rangle = 1$
- $\langle \mathrm{junk}_0 | \mathrm{junk}_1\rangle = 0$

Self-testing from Bell inequalities

In some cases, self-testing can be proven from the maximal violation of some Bell inequality.

Maximally entangled state $|\phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \rightarrow$ CHSH inequality

$$I_{\mathrm{CHSH}} = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle$$

Partially entangled states $|\phi_\theta\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle \rightarrow$ tilted CHSH inequality

$$I_{\mathrm{tilted}} = \alpha \langle A_0 \rangle + \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle$$

Self-testing
Known results

- ✓ Partially entangled qubits *[Bamps, Pironio, 2015]*
- ✓ Qudit pure bipartite entangled *[Coladangelo, Goh, Scarani, 2017]*
- ✓ Graph states *[McKague, 2014]*
- ✓ GHZ states *[Pál, Vértesi, Navascués, 2014]*
- ✓ Dicke states *[Šupić, Coladangelo, Augusiak, Acín, 2018]*

$\Longrightarrow$ All multipartite states? *Quantum networks self-test all entangled states*

[Šupić, Bowles, Renou, Acín, Hoban, 2023]

Motivation

- Networks solve the problem, but what can we say in the standard Bell scenario
- Understanding the relation between correlations on the level of density matrices and correlations in probability distributions

- Aims to verify that a quantum device operates on a certain quantum state, and performs certain measurements on it, only from the generated correlations.

Motivation

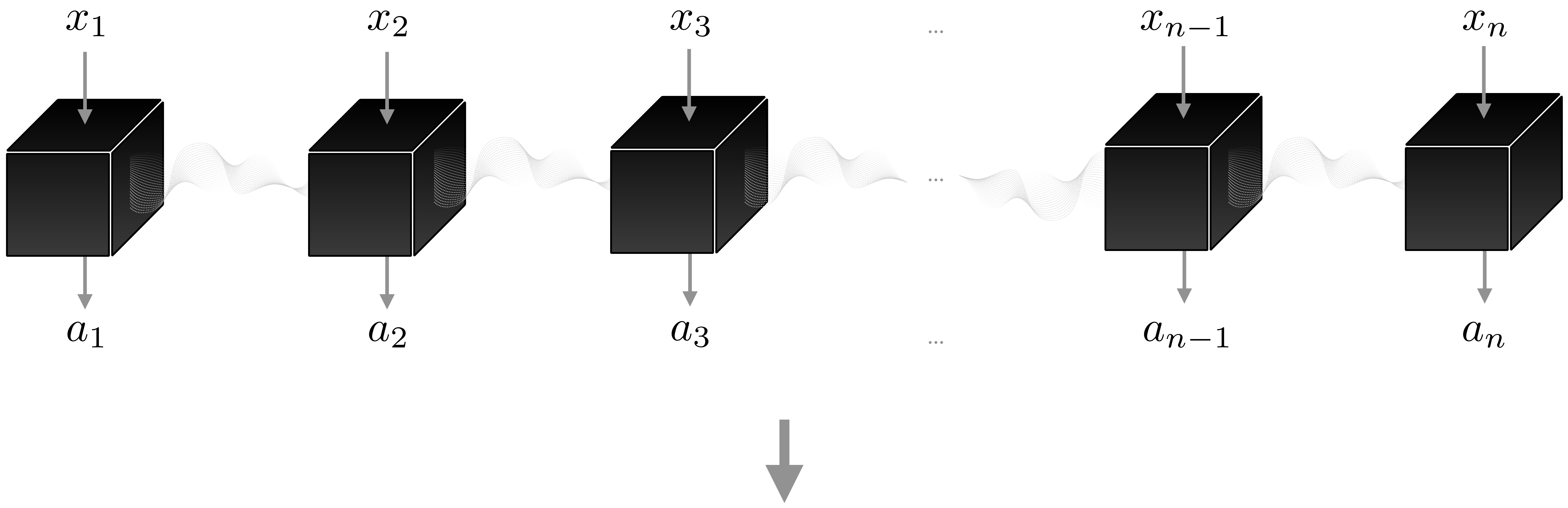
- Only results for bipartite states and some multipartite.
- Self-test is tailored to the state, or a class of states sharing some symmetry
- Multipartite is hard → the structure of multipartite entanglement is very diverse

GOAL

Find a general way to self-test multipartite qubit entangled states

Scenario

Multipartite scenario



Correlation $p(a_1, \ldots, a_n | x_1 \ldots x_n)$

Multipartite scenario
The idea behind

Use successive self-testing of bipartite states for all possible bipartitions

→ different sub-tests.

To do this, project the rest of the parties + assume the bipartite state is entangled*.

Self-test the bipartite entangled state and the measurements.

[*] M. Zwerger, W. Dür, J-D. Bancal, and P. Sekatski. PRL, 122(6):060502, (2019).

Tripartite states

Tripartite states

Example

Consider the target state $|\psi\rangle = \boxed{|0\rangle_A} \otimes |\phi^+\rangle_{BC} + \boxed{|1\rangle_A} \otimes |\psi^-\rangle_{BC}$

Obtain 0 → project to $|\phi^+\rangle$

→ Max violates CHSH_1

Obtain 1 → project to $|\psi^-\rangle$

→ Max violates CHSH_4

Recipe for self-testing: Ask the first party to measure in the computational basis and make the two remaining parties play the CHSH game.

If they win with certainty for both outcomes of the measurement on the first party certain entanglement structure is revealed.

Tripartite states
Example

However it is not enough: the state $\rho = |0\rangle\langle 0| \otimes |\phi^+\rangle\langle\phi^+| + |1\rangle\langle 1| \otimes |\psi^-\rangle\langle\psi^-|$ would also pass the test.

This state is separable across the bipartition A|BC → Not the target state.

We also need to establish that there is entanglement across A|BC.

Tripartite states

Example

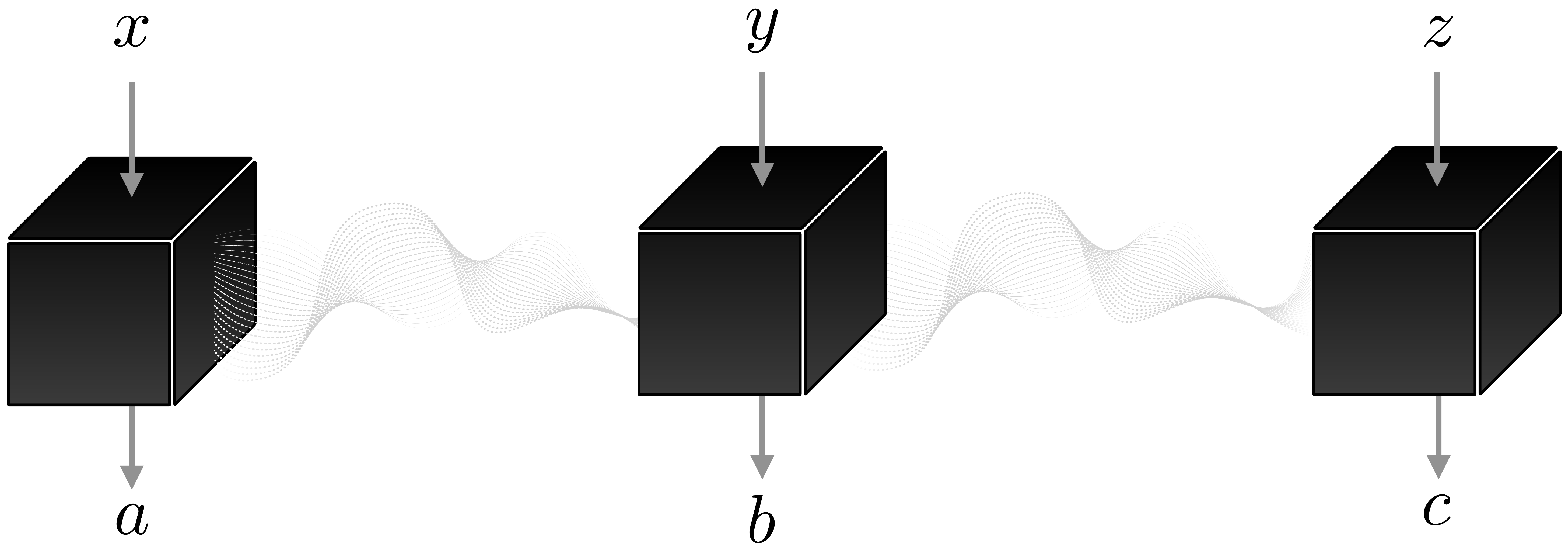
The target state can also be written as: $|\psi\rangle = |\phi^-\rangle_{AB} \otimes |0\rangle_C + |\psi^+\rangle_{AB} \otimes |1\rangle_C$

In this sub-test we can do the same: Charlie measures in the computational basis and the other parties perform CHSH games.

If they win every time, there is enough information to self-test the whole state.

Tripartite states

The simplest multipartite case



What is the shared state? $|\psi\rangle = \sum_{ijk} \lambda_{ijk} |ijk\rangle$

Tripartite states

The idea behind

There are 3 possible sub-tests: A|BC, B|AC, C|AB.

Each of the remaining bipartite state is entangled*.

Self-test the states and the measurements for every sub-test.

Show that all the bases have to be the same to construct the isometry.

[*] M. Zwerger, W. Dür, J-D. Bancal, and P. Sekatski. Physical review letters, 122(6):060502, (2019).

Tool 1: Tilted CHSH inequality
The idea behind

Set of 3 Bell inequalities with 3 inputs on Alice's side and 6 inputs on Bob's side in total → 2 tilted CHSH + 1 CHSH.

It can be used to self-test the partially entangled pair of qubits and three Pauli observables on Alice's side (up to the complex conjugation uncertainty).

E. Woodhead, J. Kaniewski, B. Bourdoncle, A. Salavrakos, J. Bowles, R. Augusiak, and A. Acín. Phys. Rev. Research 2, 042028(R), (2020).

What do we expect?

Making the 3 sub-tests we should obtain:

- When party A makes ***computational basis measurement*** the remaining two parties are projected to $\sum_{j,k} \lambda_{0,j,k} |jk\rangle$ and $\sum_{j,k} \lambda_{1,j,k} |jk\rangle$

- When party B makes ***computational basis measurement*** the remaining two parties are projected to $\sum_{i,k} \lambda_{i,0,k} | ik\rangle$ and $\sum_{i,k} \lambda_{i,1,k} | ik\rangle$

- When party C makes ***computational basis measurement*** the remaining two parties are projected to $\sum_{i,j} \lambda_{i,j,0} | ij\rangle$ and $\sum_{i,j} \lambda_{i,j,1} | ij\rangle$.

What we get?

Making the 3 sub-tests we know that:

- When party A makes some ***unknown measurement*** the remaining two parties are projected to $\sum_{\hat{j},\hat{k}} \lambda_{0,\hat{j},\hat{k}} |\hat{j}\hat{k}\rangle$ and $\sum_{\hat{j},\hat{k}} \lambda_{1,\hat{j},\hat{k}} |\hat{j}\hat{k}\rangle$ in an ***unknown basis.***

- When party B makes some ***unknown measurement*** the remaining two parties are projected to $\sum_{\bar{j},\bar{k}} \lambda_{\bar{j},0,\bar{k}} |\bar{j}\bar{k}\rangle$ and $\sum_{\bar{j},\bar{k}} \lambda_{\bar{j},1,\bar{k}} |\bar{j}\bar{k}\rangle$ in an ***unknown basis.***

- When party C makes some ***unknown measurement*** the remaining two parties are projected to $\sum_{\tilde{j},\tilde{k}} \lambda_{\tilde{j},\tilde{k},0} |\tilde{j}\tilde{k}\rangle$ and $\sum_{\tilde{j},\tilde{k}} \lambda_{\tilde{j},\tilde{k},1} |\tilde{j}\tilde{k}\rangle$ in an ***unknown basis.***

Tripartite states

If we knew the bases this marginal problem would be easy to solve, the only state that could pass all the sub-tests would be our target state → but we don't know the bases.

How do we combine the three tests and make sure that the six marginal states correspond to the target state?

Need to understand the relation between the measurements the parties use to project and those they use when they are maximally violating the tilted CHSH inequality.

Tool 2: The measurement lemma

Let $|\psi\rangle$ be a bipartite state and $A_0^{(j)}, A_1^{(j)}, A_2^{(j)}$ dichotomic observables.

Suppose there exists unitaries U_j and U_k such that:

$$U_j \otimes U_k|\psi\rangle = |\psi_\theta\rangle \otimes |\xi\rangle \qquad U_j A_1^{(j)} U_j^\dagger = \sigma_X \otimes \mathbb{I}$$

$$U_j A_0^{(j)} U_j^\dagger = \sigma_Z \otimes \mathbb{I} \qquad U_j A_2^{(j)} U_j^\dagger = \sigma_Y \otimes A_Y$$

Suppose that

$$\langle\psi|A_0^{(j)} \otimes A^{(k)}|\psi\rangle = \alpha \qquad \langle\psi|A_2^{(j)} \otimes A^{(k)}|\psi\rangle = \gamma \sin 2\theta$$

$$\langle\psi|A_1^{(j)} \otimes A^{(k)}|\psi\rangle = \beta \sin 2\theta \qquad \langle\psi|\mathbb{I}^{(j)} \otimes A^{(k)}|\psi\rangle = \alpha \cos 2\theta$$

where $\alpha^2 + \beta^2 + \gamma^2 = 1$, then we have

$$U_k A^{(k)} U_k^\dagger = \alpha\sigma_Z \otimes \mathbb{I} + \beta\sigma_X \otimes \mathbb{I} + \gamma\sigma_Y \otimes \bar{B}_Y$$

Allows us to fix a “measurement reference frame”

Tripartite states

How do we proceed then?

Measurement lemma: allowing us to fix a “measurement reference frame”.

Each of the measurements used to project are also self-tested: reveals the relation between measurements used to self-test and measurements used to project, or ensuring that they are related in the way we expect.

Tripartite states

Final statement

In the end, we are able to certify that

$$|\psi\rangle_{ABCP} = \boxed{|\psi\rangle_{A'B'C'}} \otimes |\phi^+\rangle_{A''B''C''P} + \boxed{|\psi^*\rangle_{A'B'C'}} \otimes |\phi^-\rangle_{A''B''C''P}$$

$$|\psi\rangle = \sum_{ijk} \lambda_{ijk} |ijk\rangle$$

LU-equivalent to its complex conjugate

$\Longrightarrow$ It is possible to self-test every pure entangled state of 3 qubits locally unitarilly equivalent to its complex conjugate

Multipartite states

Multipartite states

How to generalize the tripartite case

Consider the multipartite state:

$$|\Psi\rangle = \sum_{\vec{x}\in\{0,1\}^{\otimes N}} \lambda_{\vec{x}} \; |\vec{x}\rangle$$

Perform the sub-tests for all possible partitions.

Problem: the total number of measurements (at least for 1 party) will grow exponentially.

Multipartite states

How to generalize the tripartite case

All the bases are the same → We can construct the isometry

$$\Phi[|\psi\rangle] = |\Psi\rangle \otimes |\text{junk}_0\rangle + |\Psi^*\rangle \otimes |\text{junk}_1\rangle$$

We can self-test multipartite entangled state of qubits which is equivalent to its complex conjugate.

For any state which is not equivalent to its complex conjugate, we exhibit a correlation that singles out the state up to complex conjugation.

Summary

Summary

- Self-testing: way of certifying the state and the measurements only from the correlations → device-independent.
- There was no generic way of self-test any multipartite state.
- We can self-test multipartite states using a set of sub-tests.

Thank you!