Improving social welfare in non-cooperative games with different types of quantum resources

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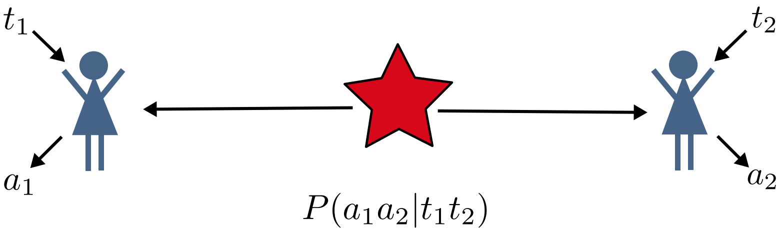
Games and quantum strategies

Games and quantum strategies

Nonlocal games:

- E.g. CHSH game: players win if $a_1 \oplus a_2 = t_1 t_2$



- How well can the players do given different resources?
- Cooperative game: all players win and lose together, goals are aligned

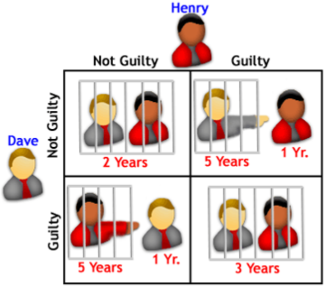
Outline

- Non-cooperative games and equilibria
- Two different quantum resources
  - Shared quantum correlations (classical “black box” access)
  - Shared quantum states (quantum access)
- Comparing different resources
  - What equilibria from different resources?
  - Maximising the social welfare

Non-cooperative game theory

Reality: Players' objectives often not aligned

- Players get different rewards depending on their choices and those of others
- Examples:
  - Zero-sum games
  - Prisoner's dilemma



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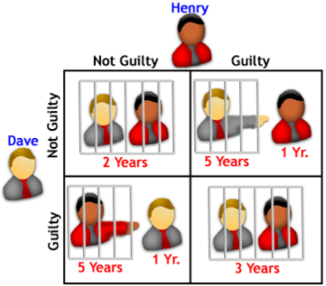
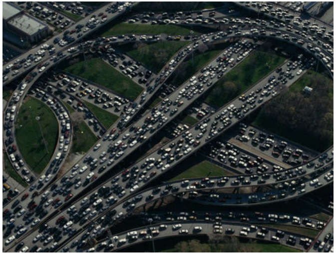
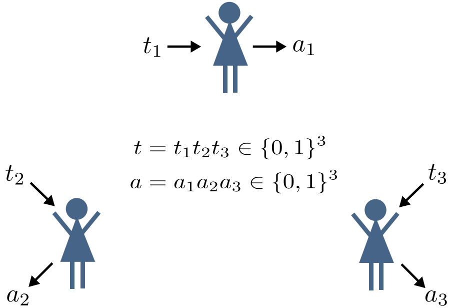


Extensively studied in game theory

- Complex behaviour, Nash equilibria, . . .
- Widely applicable



Example: A three-player game



$$t = t_1 t_2 t_3 \in \{0,1\}^3$$
$$a = a_1 a_2 a_3 \in \{0,1\}^3$$

Question $t_1 t_2 t_3$	Winning conditions
100	$a_1 \oplus a_2 \oplus a_3 = 0$
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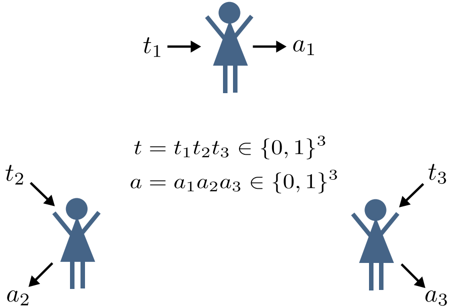

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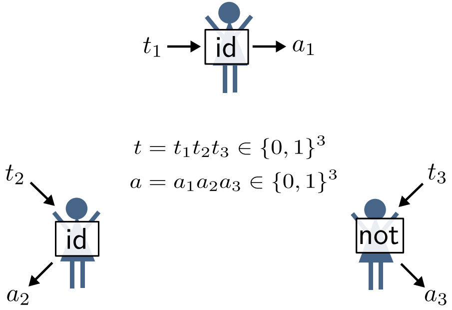

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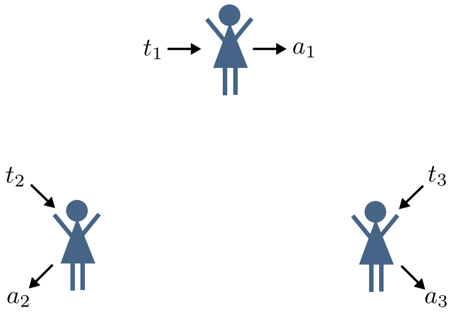
- The strategy $(\text{id}, \text{id}, \text{not})$ wins $3/4$ of the time
- Can a player increase their expected gain, potentially at the expense of the others?
- What strategy maximises the overall (or average) payoff?

[Groisman, Mc Gettrick, Mhalla, Pawłowski, IEEE JIT (2020)]

Different types of resources



- Base scenario: independent local strategies

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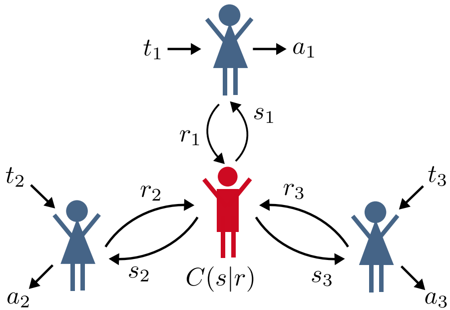

- Base scenario: independent local strategies
- Shared resources: correlated advice

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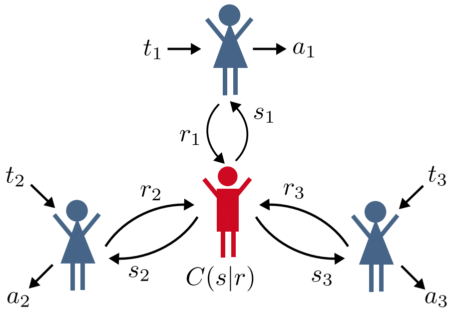

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Different class of correlations $\mathcal{C}$:

- Classical shared random variables
- Belief-invariant (non-signalling) correlations
- Full communication
- n-partite quantum correlations ($\mathcal{C}_Q$)

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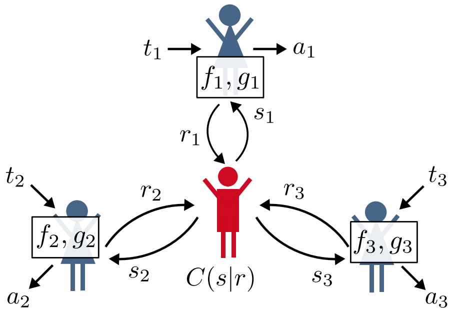

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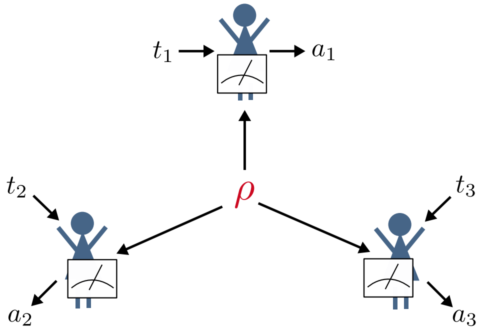
Definition (Solution)

A solution is a tuple $(f_1, \ldots, f_n, g_1, \ldots, g_n, C)$ and induces a correlation

$$P(a|t) = \sum_s C(s|f(t))\delta_{g(t,s),a}$$

Quantum resources: quantum states as advice



Players receive part of a shared quantum state as "advice", and can measure it directly.

Definition (Quantum solution)

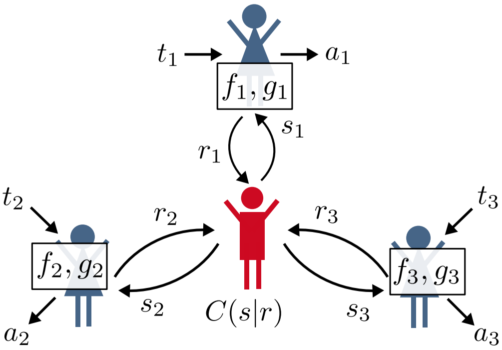
A quantum solution is a tuple $(\rho, \mathcal{M}^{(1)}, \dots, \mathcal{M}^{(n)})$, with $\mathcal{M}^{(i)}$ sets of POVMs $\{M^{(i)}_{a_i|t_i}\}_{a_i,t_i}$.
It induces a correlation:

$$P(a|t) = \mathrm{Tr}\Big[\rho\,\Big(M^{(1)}_{a_1|t_1} \otimes \cdots \otimes M^{(n)}_{a_n|t_n}\Big)\Big]$$

[Auletta, Ferraioli, Rai, Scarpa, Winter, JTCS (2021)]

Nash equilibria

In game theory, we are interested in equilibrium solutions, where no player can increase their payoff by unilaterally deviating from a solution.



Player i payoff:

$\sum_{a,t} u_i(a,t)P(a|t)\Pi(t)$

Definition (Nash equilibrium (informal))

A solution is a Nash equilibrium if no player can increase their payoff $\sum_{a,t} u_i(a,t)P(a|t)\Pi(t)$ by changing their local strategy (f_i, g_i) to (ν_i, μ_i).

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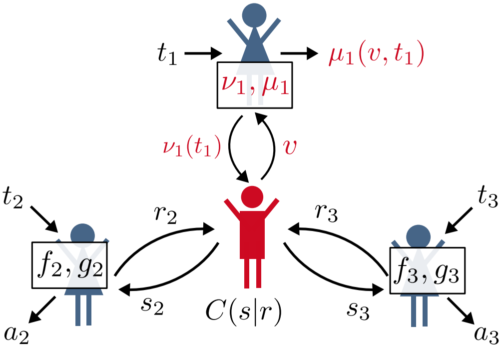

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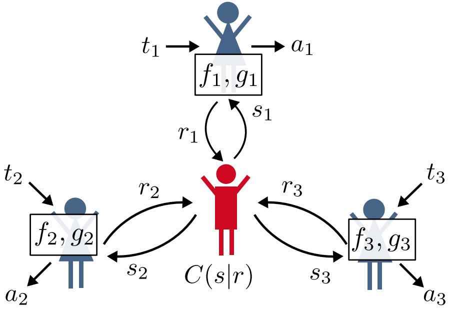
Simplifying things

It turns out that for most classes of correlations $\mathcal{C}$, we can restrict ourselves to canonical solutions:

- Each player sends t_i to the mediator and outputs what they receive as a_i
- $P(a|t) = C(a|t)$



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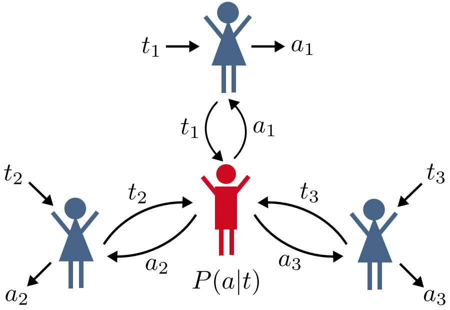

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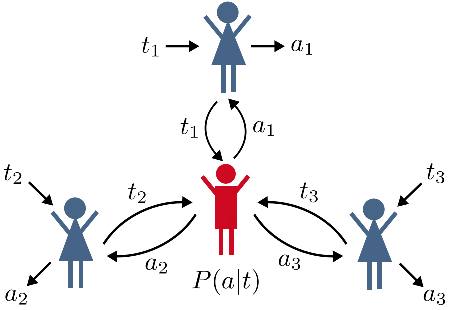

Definition (Nash equilibrium)

A solution is a Nash equilibrium if, for all players i, all $t_i, r_i \in T_i$, and all functions $\mu_i : T_i \times A_i \to A_i$:

$$\sum_{t_{-i},a} u_i(a,t)P(a|t) \geq \sum_{t_{-i},a} u_i(\mu_i(a_i,t_i)a_{-i}, t_i t_{-i})P(a|r_i t_{-i}).$$

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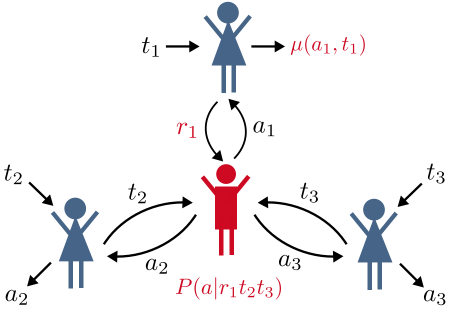

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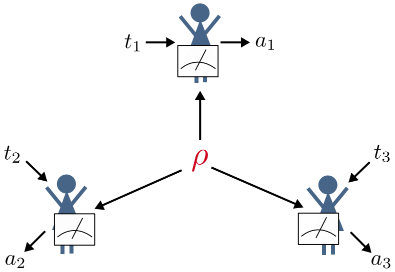


Definition (Quantum equilibrium)

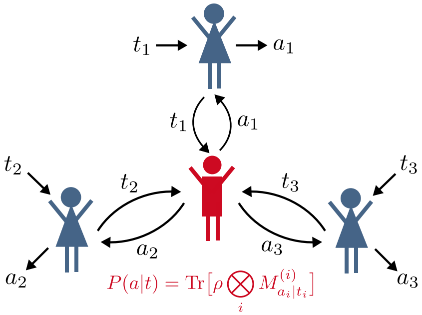
A quantum solution $\left(\rho, \mathcal{M}^{(1)}, \ldots, \mathcal{M}^{(n)}\right)$, is a *quantum equilibrium* if, for every player i, for any type t_i and any POVM $N^{(i)} = \{N_{a_i}^{(i)}\}_{a_i \in A_i}$:

$$\sum_{t_{-i},a} u_i(a,t) \operatorname{Tr}\Big[\rho \left(M_{a_1|t_1}^{(1)} \otimes \cdots \otimes M_{a_n|t_n}^{(n)}\right)\Big]\Pi(t)$$

$$\geq \sum_{t_{-i},a} u_i(a,t) \operatorname{Tr}\Big[\rho \left(M_{a_1|t_1}^{(1)} \otimes \cdots \otimes M_{a_{i-1}|t_{i-1}}^{(i-1)} \otimes N_{a_i}^{(i)} \otimes M_{a_{i+1}|t_{i+1}}^{(i+1)} \otimes \cdots \otimes M_{a_n|t_n}^{(n)}\right)\Big]\Pi(t).$$

Two types of quantum resources

Classical access: advice $P \in \mathcal{C}_Q$



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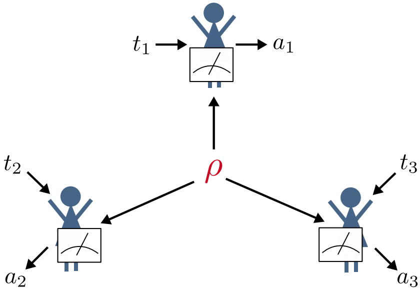

How should we compare these different resources?

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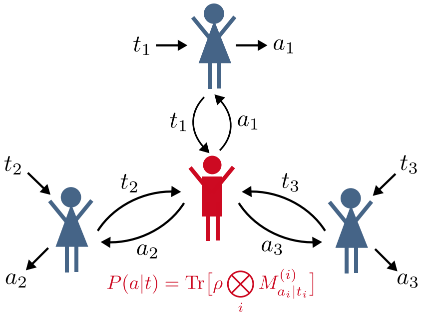

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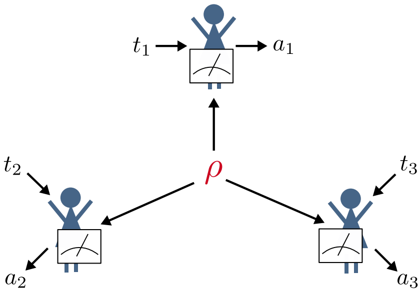

How should we compare these different resources?

- Two different levels of access to quantum resources leads to two different notions of equilibria
- Two corresponding sets of equilibrium correlations:

$$Q_{\mathrm{corr}}(G) = \{P \mid P \text{ defines a canonical Nash equilibrium and } P \in \mathcal{C}_Q\} \subseteq \mathcal{C}_Q$$
$$Q(G) = \{P \mid \text{there exists } (\rho, \mathcal{M}) \text{ a quantum equilibrium inducing } P\} \subseteq \mathcal{C}_Q$$

Social Welfare

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- Can one obtain different equilibria using these different resources?
- How *good* are the equilibria one can obtain in each case?

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Definition (Social welfare)

For a game G, the *social welfare* of a solution inducing a distribution P is

$$SW_G(P) = \frac{1}{n} \sum_i \sum_{a,t} u_i(a,t) P(a|t) \Pi(t).$$

- In cooperative games, no difference in power between these resources
- What about non-cooperative games?

Quantum access restricts equilibria

Counter-intuitively, allowing the players more control restricts the equilibria they can reach

Theorem

For any game G, $Q(G) \subseteq Q_{\text{corr}}(G)$.

Proof idea.

Any modification of a classical strategy can be represented by an equivalent change of quantum strategy by relabelling the POVMs used to obtain the correlations. $\square$

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The quantum families fit within a hierarchy of equilibrium correlations:

$$\mathsf{Nash}(G) \subset \mathsf{Corr}(G) \subset Q(G) \subseteq Q_{\mathsf{corr}}(G) \subset \mathsf{B.I.}(G) \subset \mathsf{Comm}(G)).$$

[Auletta, Ferraioli, Rai, Scarpa, Winter, JTCS (2021)]

- Classical access to quantum devices at least as powerful as quantum access
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Pseudo-telepathic solution for the $\mathrm{NC}(C_3)$ games

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Quantum solutions from graph states:

- Share a C_3 graph state: $|\Psi\rangle = CZ^{(1,2)}CZ^{(2,3)}CZ^{(3,1)}(|+\rangle \otimes |+\rangle \otimes |+\rangle)$
- Players measure in Z-basis if $t_i = 0$, X-basis if $t_i = 1$
- Solution wins the game deterministically
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- Induced distribution both a quantum and quantum-correlated equilibrium (in $Q_{\text{corr}}(G)$, $Q(G)$)

[Groisman, McGettrick, Mhalla, Pawlowski, IEEE JSAIT (2020)]

Tilted Graph-state Solution

Let's modify the pseudo-telepathic solution a bit:

- Share the state $|\Psi_{\mathsf{tilt}(\theta)}\rangle = CZ^{(1,2)}CZ^{(2,3)}CZ^{(3,1)}\left((\cos(\tfrac{\theta}{2})\,|0\rangle + \sin(\tfrac{\theta}{2})\,|1\rangle) \otimes |+\rangle \otimes |+\rangle\right)$
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For $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4})$ there is an interval of values of v_0 (around $v_0 = 1$) such that:

1. the tilted solution gives a quantum correlated equilibrium
2. but isn't a quantum equilibrium (Player 1 can do better by measuring closer to X and Z)

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Doesn't quite show $Q(G) \subsetneq Q_{\mathsf{corr}}(G)$

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Tilted Graph-state Solution

Let's modify the pseudo-telepathic solution a bit:

- Share the state $|\Psi_{\text{tilt}(\theta)}\rangle = CZ^{(1,2)}CZ^{(2,3)}CZ^{(3,1)}\left((\cos(\frac{\theta}{2})|0\rangle + \sin(\frac{\theta}{2})|1\rangle) \otimes |+\rangle \otimes |+\rangle\right)$
- Player 1 measures $(X+Z)/\sqrt{2}$ if $t_1 = 0$, and $(X-Z)/\sqrt{2}$ if $t_1 = 1$
- Players 2 and 3 measure Z if $t_i = 0$ and X if $t_i = 1$

For $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4})$ there is an interval of values of v_0 (around $v_0 = 1$) such that:

1. the tilted solution gives a quantum correlated equilibrium
2. but isn't a quantum equilibrium (Player 1 can do better by measuring closer to X and Z)

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- Could a different quantum solution $(\rho, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ induce the same distribution $P_{\text{tilt}(\theta)}(a|t)$ and be a quantum equilibrium?

Approach: use self-testing

Self-testing quantum solutions

Intuition: Any solution $(\rho, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ reproducing the correlations $P_{\mathsf{tilt}(\theta)}$ must be equivalent up to local isometries to the tilted solution.

- The self-testing isometries must preserve the equilibrium condition

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Self-testing the tilted solution

CENSORED

Proof similar to graph state self-test of [Baccari, Augusiak, Šupić, Tura, Acín, PRL (2020)]

Comparing social welfare

Does more equilibria mean *better* equilibria?

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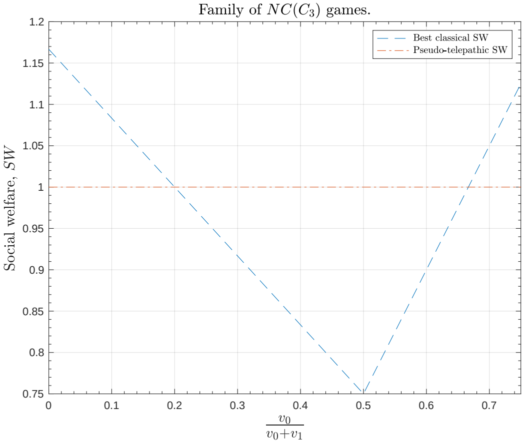

- Graph state solution better than tilted solution for all θ
- Can one do better?

Improving social welfare

- Pseudo-telepathy: Graph state solution wins all the time
- Can we do better by losing some of the time?
- What is the maximal social welfare obtainable by the different types of equilibria?

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Maximising social welfare

$$\max_P SW_G(P) = \frac{1}{n} \sum_{a,t} \sum_i u_i(a,t) P(a|t) \Pi(t),$$

where the maximisation is either over $Q_{\mathrm{corr}}(G) \subseteq \mathcal{C}_Q$ or $Q(G) \subseteq \mathcal{C}_Q$

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- Question: how to characterise these sets of equilibria?
- Use numerical and SDP methods to compute upper and lower bounds on the maximum social welfare.

Lower bounds: See-saw optimisation

- Key observation: checking if $(\rho, \mathcal{M}_1, \ldots, \mathcal{M}_n)$ is a quantum equilibrium is an SDP

See-saw iteration over $\mathcal{C}_Q$

$$\max_{\mathcal{M}_n} \cdots \max_{\mathcal{M}_1} \max_{\rho} SW_G(P) = \frac{1}{n} \sum_{a,t} \sum_i u_i(a,t) \operatorname{Tr}\left[\rho \left(M^{(1)}_{a_1|t_1} \otimes \cdots \otimes M^{(n)}_{a_n|t_n}\right)\right] \Pi(t)$$

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To converge to an equilibrium, we then add:

Quantum equilibria: $Q(G)$

Each player tries to optimise their own payoff

$$\max_{\mathcal{M}^{(N)}} \cdots \max_{\mathcal{M}^{(1)}} \sum_{a,t} u_i(a,t) \operatorname{Tr}\Big[\rho \left(M^{(1)}_{a_1|t_1} \otimes \cdots \otimes M^{(n)}_{a_n|t_n}\right)\Big] \Pi(t).$$

Nash equilibria: $Q_{\text{corr}}(G)$

The (finite) inequalities constraining Nash equilibria.

Upper bounds: NPA hierarchy

Main difficulty computing upper bounds: there is no easy way to characterise the set of quantum correlations $\mathcal{C}_Q$.

NPA hierarchy

Convergent hierarchy of SDP constraints to test if a distribution is in $\mathcal{C}_Q$, approximating it from the outside (upper bounds).

$$+$$

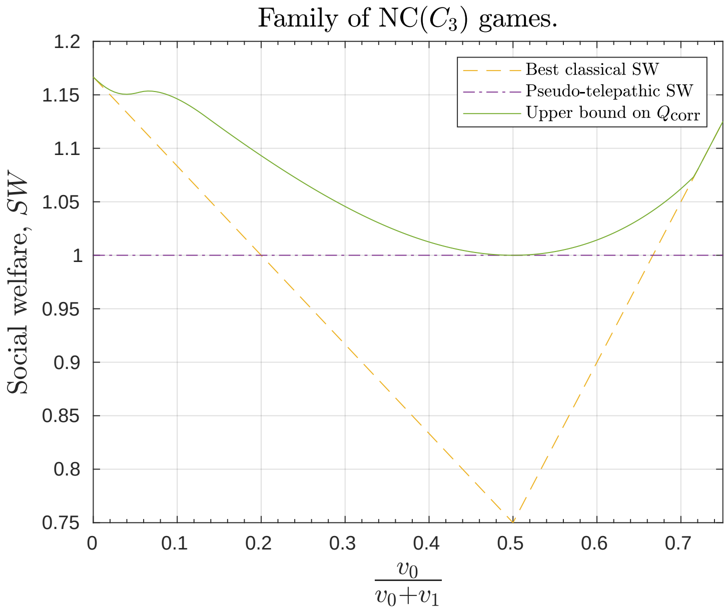
Nash equilibrium

Finite number of linear constraint to test if a probability distribution is a Nash equilibrium.

$$\underbrace{\max_{P \in Q_{\text{corr}}(G)}} SW_G(P) = \frac{1}{n} \sum_{a,t} \sum_i u_i(a,t) P(a|t) \Pi(t).$$

[Navascues, Pironio, Acin, NJP (2008)]

Social Welfare in NC(C_3) games



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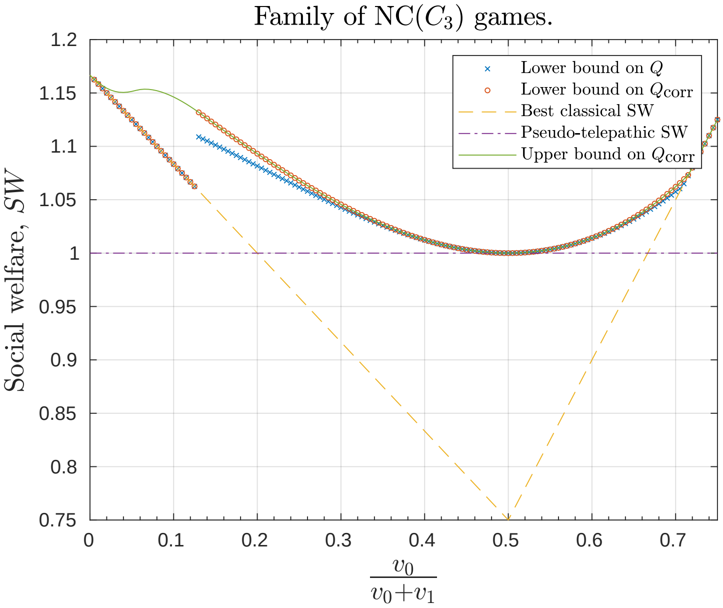
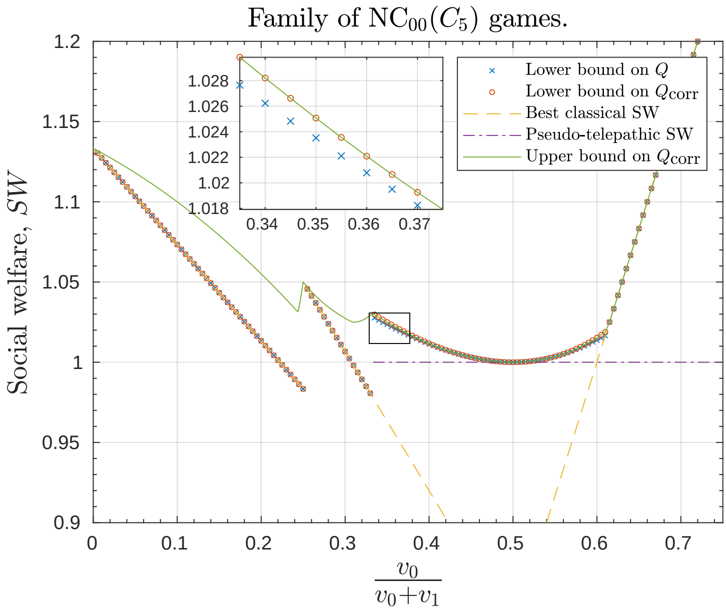


Social Welfare in some five-player games



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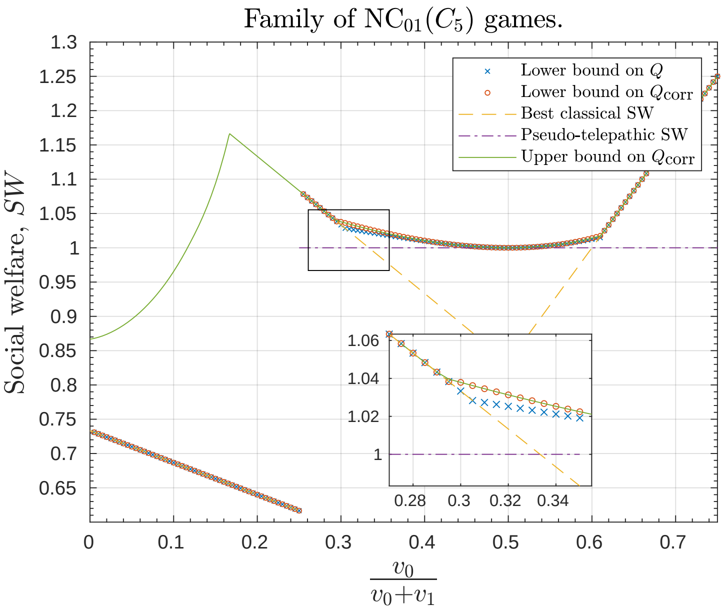
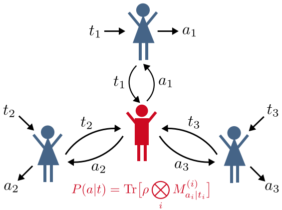
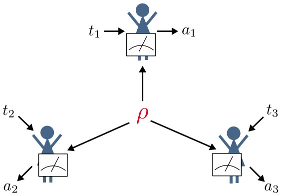


Discussion

- Two distinct ways to consider quantum resources in non-cooperative games:





- "Quantum resources improve social welfare"

Open questions:

- Can the NPA hierarchy be adapted to give upper bounds on $Q(G)$?
- Intermediate settings (with classical or quantum access for different players)
- What applications for quantum advantages in non-cooperative games?
- Understanding the power of delegated quantum measurements

arXiv:2211.01687

Self-testing quantum solutions

Intuition: Any solution $(\rho, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ reproducing the correlations $P_{\mathsf{tilt}(\theta)}$ must be equivalent up to local isometries to the tilted solution.

- The self-testing isometries must preserve the equilibrium condition

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Self-testing the tilted solution

Let $(\,|\tilde{\psi}\rangle\langle\tilde{\psi}|\,, \tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2, \tilde{\mathcal{M}}_3)$ be an uncharacterised solution inducing $P_{\text{tilt}(\theta)}$ with $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4})$, and defining $\tilde{A}^{(i)}_{t_i} = \tilde{M}^{(i)}_{0|t_i} - \tilde{M}^{(i)}_{1|t_i}$ and

$$\tilde{X}_1 = \frac{\tilde{A}^{(1)}_0 + \tilde{A}^{(1)}_1}{\sqrt{2}},\ \tilde{Z}_1 = \frac{\tilde{A}^{(1)}_0 - \tilde{A}^{(1)}_1}{\sqrt{2}},\ \tilde{X}_2 = \tilde{A}^{(2)}_1,\ \tilde{Z}_2 = \tilde{A}^{(2)}_0,\ \tilde{X}_3 = \tilde{A}^{(3)}_1,\ \tilde{Z}_3 = \tilde{A}^{(3)}_0.$$

Then there exists a local isometry $\Phi = \Phi_1 \otimes \Phi_2 \otimes \Phi_3$ such that

$$\begin{aligned}
\Phi[\,|\tilde{\psi}\rangle] &= |\Psi_{\text{tilt}(\theta)}\rangle \otimes |\text{junk}\rangle & \Phi[\tilde{X}_i\,|\tilde{\psi}\rangle] &= (X_i\,|\Psi_{\text{tilt}(\theta)}\rangle) \otimes |\text{junk}\rangle \\
\Phi[\tilde{Z}_i\,|\tilde{\psi}\rangle] &= (Z_i\,|\Psi_{\text{tilt}(\theta)}\rangle) \otimes |\text{junk}\rangle & \Phi[\tilde{X}_i\tilde{Z}_i\,|\tilde{\psi}\rangle] &= (X_iZ_i\,|\Psi_{\text{tilt}(\theta)}\rangle) \otimes |\text{junk}\rangle\,.
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Proof similar to graph state self-test of [Baccari, Augusiak, Šupić, Tura, Acín, PRL (2020)]

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Self-testing: Preserving equilibria

We can reduce question of whether $P_{\text{tilt}(\theta)} \in Q(G)$ to whether the tilted solution is a quantum equilibrium:

Theorem

Let G be a tripartite game and $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4})$. Then $P_{\textit{tilt}(\theta)} \in Q(G)$ if and only if the tilted solution $(\,|\Psi_{\textit{tilt}(\theta)}\rangle\langle\Psi_{\textit{tilt}(\theta)}|\,, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ is a quantum equilibrium.

Nontrivial direction to prove: If some solution $(\rho, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ inducing $P_{\text{tilt}(\theta)} \in Q(G)$ is a quantum equilibrium, then the tilted solution must be too.

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- Assume for contradiction that tilted solution not an equilibrium: player i can improve their payoff by choosing POVM $\{N_{a_i}^{(i)}\}$ on input t_i.
- We can decompose $N_{a_i}^{(i)} = \alpha \mathbb{1}_i + \beta X_i + \gamma Z_i + \epsilon \mathrm{i} X_i Z_i$
- Then $\tilde{N}_{a_i}^{(i)} = \alpha \tilde{\mathbb{1}}_i + \beta \tilde{X}_i + \gamma \tilde{Z}_i + \epsilon \mathrm{i} \tilde{X}_i \tilde{Z}_i$ gives a POVM in uncharacterised scenario
- From self testing, $\{\tilde{N}_{a_i}^{(i)}\}$ also improves payoff, so initial solution not an equilibrium either.

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Preservation of equilibria when self-testing

Assuming that the tilted solution is not an equilibrium but $P_{\mathsf{tilt}(\theta)} \in Q(G)$:

$$
\begin{aligned}
&\sum_{t_{-i},a} u_i(a,t)\,\mathrm{tr}\Big[(\tilde{M}^{(1)}_{a_1|t_1} \otimes \cdots \otimes \tilde{M}^{(n)}_{a_n|t_n})\,\tilde{\rho}\Big]\Pi(t) \\
&= \sum_{t_{-i},a} u_i(a,t)\,\mathrm{tr}\Big[(M^{(1)}_{a_1|t_1} \otimes \cdots \otimes M^{(n)}_{a_n|t_n})\,\rho_{\mathsf{tilt}(\theta)}\Big]\Pi(t) \\
&< \sum_{t_{-i},a} u_i(a,t)\mathsf{tr}\Big[(M^{(1)}_{a_1|t_1} \otimes \cdots \otimes M^{(i-1)}_{a_{i-1}|t_{i-1}} \otimes N^{(i)}_{a_i} \otimes M^{(i+1)}_{a_{i+1}|t_{i+1}} \otimes \\
&\qquad\qquad \otimes \cdots \otimes M^{(n)}_{a_n|t_n})\,\rho_{\mathsf{tilt}(\theta)} \otimes |\xi\rangle\langle\xi|\Big]\Pi(t) \\
&= \sum_{t_{-i},a} u_i(a,t)\ \mathrm{tr}\Big[\Phi[(\tilde{M}^{(1)}_{a_1|t_1} \otimes \cdots \otimes \tilde{M}^{(i-1)}_{a_{i-1}|t_{i-1}} \otimes \tilde{N}^{(i)}_{a_i} \otimes \tilde{M}^{(i+1)}_{a_{i+1}|t_{i+1}} \otimes \cdots \otimes \tilde{M}^{(n)}_{a_n|t_n})\,\tilde{\rho}]\Big]\Pi(t) \\
&= \sum_{t_{-i},a} u_i(a,t)\,\mathrm{tr}\Big[(\tilde{M}^{(1)}_{a_1|t_1} \otimes \cdots \otimes \tilde{M}^{(i-1)}_{a_{i-1}|t_{i-1}} \otimes \tilde{N}^{(i)}_{a_i} \otimes \tilde{M}^{(i+1)}_{a_{i+1}|t_{i+1}} \otimes \cdots \otimes \tilde{M}^{(n)}_{a_n|t_n})\,\tilde{\rho}\Big]\Pi(t),
\end{aligned}
$$

a contradiction. □