Bell Nonlocality from Wigner Negativity in Qudit Stabilizer States

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Bell experiments

Measurement scenario for n parties

POVMs $\{(A_{m_i})_k\}_i$ with $\sum_i (A_{m_i})_k = \mathbb{1}_k$, $k = 1, \ldots, n$

Experiment

Set of probabilities $\{p(\mathbf{a}|\mathbf{m}) = p(a_i, i = 1, \ldots, n | m_i, i = 1, \ldots, n)\}$

Quantum Model $\rightarrow$ Born rule

$$p(\mathbf{a}|\mathbf{m}) = \mathrm{Tr}\left(\Pi_{\mathbf{a}}^{\mathbf{m}} \rho\right) \tag{1}$$

Classical Model$\rightarrow$ lhv model

$$p(\mathbf{a}|\mathbf{m}) = \sum_{\lambda} \mu(\lambda) \prod_{i=1}^{n} p_i(a_i | m_i, \lambda) \tag{2}$$

Bell Nonlocality

Bell operator

$$\mathcal{B} = \sum_{m_i} b_{\{m_i\}_i} \bigotimes_{k=1}^{n} (A_{m_i})_k \tag{3}$$

Bell Inequality

$$\langle \mathcal{B} \rangle_{lhv} = \sum_{m_i} b_{\{m_i\}_i} \sum_{\lambda} \mu(\lambda) \prod_i a_i \, p_i(a_i|m_i, \lambda) \leq C = \max_{lhv} \langle \mathcal{B} \rangle_{lhv} \tag{4}$$

Bell Violation

$$\mathrm{tr}\,(\rho\,\mathcal{B}) = \sum_{m_i} b_{\{m_i\}_i} \left(\prod_i a_i \right) \mathrm{tr} \left(\rho \bigotimes_{k=1}^{n} \Pi_{a_k}^{m_i} \right) > C \tag{5}$$

Qudits & Gross' Wigner function

Qubits $\sim$ Qu2it	Qudit (d prime)
$X\|k\rangle = \|k+1\rangle\,,\ Z\|k\rangle = (-1)^k\|k\rangle$	$X\|k\rangle = \|k+1\rangle\,,\ Z\|k\rangle = \omega^k\|k\rangle\,,\ \omega = e^{2\pi \mathrm{i}/d}$
$ZX + XZ = 0$	$ZX - \omega XZ = 0$
$\sigma_{(x,z)} = \mathrm{i}^{xz} X^x Z^z\,,$ $x, z = 0, 1$	$D_{(x,z)} = \omega^{xz/2} X^x Z^z\,,$ $x, z = 0, \ldots, d-1$
$\sigma^{-1} = \sigma = \sigma^\dagger$	$D^{-1}_{(x,z)} = D_{(-x,-z)} = D^\dagger_{(x,z)}$

Wigner function (quasi-probability distribution)

$$W_{(x,z)}(\rho) = \frac{1}{d}\sum_{q,p=0}^{d-1} \omega^{xp-zq}\,\mathrm{Tr}\left(\rho\, D_{(q,p)}\right) \tag{6}$$

Pauli Stabilizer State & Measurements

Pauli Stabilizer State $|S\rangle$

$S_i|\mathcal{S}\rangle = |\mathcal{S}\rangle$ with $S_i = \omega^{t(\mathbf{x}_i, \mathbf{z}_i)} D_{(\mathbf{x}_i, \mathbf{z}_i)}$, $\{S_i\}_i$ abelian group.

Measurement description with Pauli operators

Operator Fourier transform

$$A_{(x,z)} = \sum_{q,p=0}^{d-1} \omega^{xp-zq} D_{(p,q)} \tag{7}$$

$\{A_{(x,z)}\}$ are orthonormal basis of operators $(A, B) = \mathrm{tr}\left(A^\dagger B\right)/d^n$

Contextuality & Wigner Negativity

Contextuality, a generalization of nonlocality (Kochen and Specker 1975)

Non-contextual value assignment (Delfosse et al. 2017)

$\sim$ Deterministic lhv model

$$D_{(\mathbf{x},\mathbf{z})} \sim (\mathbf{a}_x, \mathbf{a}_z) \equiv \omega^{\mathbf{x}\mathbf{a}_z - \mathbf{z}\mathbf{a}_x}$$

& probability distributions $\{q\}$ such that

$$\operatorname{tr}\left(\rho\, D_{(\mathbf{x},\mathbf{z})}\right) = \sum_{\mathbf{a}_x, \mathbf{a}_z} q_{(\mathbf{a}_x, \mathbf{a}_z)} \omega^{\mathbf{x}\mathbf{a}_z - \mathbf{z}\mathbf{a}_x}$$

Nonlocality $\Rightarrow$ Contextuality $\Leftrightarrow$ Wigner negativity (Delfosse et al. 2017)

Pauli Stabilizer States and Clifford operators are non-contextual (Gross 2006)

& efficiently classically simulatable
(Howard, Brennan, and Vala 2013; Howard et al. 2014)

Previous Works & This Work

Nonlocality in Singlet states, GHZ states, graph states

- **d even:** Cerf, Massar, and Pironio 2002, Tang, Yu, and Oh 2013
- **Qutrits:** Lawrence 2017, Chen et al. 2002, Kaszlikowski et al. 2002a; Kaszlikowski et al. 2002b, Acín et al. 2004, Gruca, Laskowski, and Żukowski 2012, Li and Chen 2011, Mackeprang et al. 2023
- Kaniewski et al. 2019, Collins et al. 2001, Ji et al. 2008; Liang, Lim, and Deng 2009

This Work

- Apply Wigner negativity straightaway
- Where to find Wigner negativity?
- No composite systems
- No slicing or subdividing of Qu$d$it into $d' < d$ Qud'it
- Study of d-intrinsic character polynomials

Qudit $\pi/8$-Gate

Clifford unitary operator (Appleby 2009):

$$C \sim D_{(x,z)} \sum_{j,k=0}^{d-1} \omega^{f_2(j,k)} |j\rangle\langle k| \qquad (f_2 \in poly(\mathbb{Z}_d)\,,\ \boxed{deg(f_2) = 2}\,)$$

Qudit $\pi/8$-Gate (Howard and Vala 2012)

$$U_\nu = \sum_{k=0}^{d-1} \omega^{\nu_k} |k\rangle\langle k| \tag{8}$$

maps Pauli operators to Clifford operators for $d > 3$ with

$$\nu \in poly(\mathbb{Z}_d)\,,\ \boxed{deg(\nu) = 3}$$

$\nu_k = 12^{-1}k(\gamma + k(6z + (2k+3)\gamma)) + \epsilon k$ for $z, \epsilon \in \mathbb{Z}_d$ and $\gamma \in \mathbb{Z}_d^*$

Bell Inequalities with Qudit $\pi/8$ gates

Wigner function $(d > 3)$

of $|\Psi_\nu\rangle\langle\Psi_\nu| := (U_\nu \otimes \mathbb{1})\,|\Psi\rangle\langle\Psi|\left(U_\nu^\dagger \otimes \mathbb{1}\right)$, singlet state $|\Psi\rangle = \sum_{k=0}^{d-1} |k\,k\rangle/\sqrt{d}$:

$$W_{(u_1,u_2)}(|\Psi_\nu\rangle\langle\Psi_\nu|) = \frac{1}{d^3}\delta_{(u_1)_x,(u_2)_x} \sum_{k=0}^{d-1} \omega^{a_3 k^3 + a_1 k}, \tag{9}$$

$a_3 = 24^{-1}\gamma$, $a_1 = \epsilon + (u_1)_z + (u_2)_z + z(u_1)_x + 2^{-1}\gamma((u_1)_x^2 - (u_1)_x + 6^{-1})$

$$\exists\,\gamma, \epsilon, \text{ s.t. } W_{(u_1,u_2)}(|\Psi_\nu\rangle\langle\Psi_\nu|) < 0$$

$$C(\nu) := W_{(0,0)}(|\Psi_\nu\rangle\langle\Psi_\nu|) < 0,$$

$|C(\nu)| < 2\sqrt{d}$ (Weil's Theorem)

Bell Inequalities with Qudit $\pi/8$ gates

$$D_{(s,r)} \otimes D_{(s,-r)}|\Psi\rangle = |\Psi\rangle : \ \mathcal{B}_2 = \frac{C(\nu)}{d^3} \sum_{s,r,q=0}^{d-1} U_\nu D_{(s,q-r)} U_\nu^\dagger \otimes D_{(s,r)}$$

$$D_{(\mathbf{s},\Gamma\mathbf{s})}|G_\Gamma\rangle = |G_\Gamma\rangle : \ \mathcal{B}_G = \frac{C(\nu)}{d^{n+1}} \sum_{\mathbf{s}\in\mathbb{Z}_d^n, r\in\mathbb{Z}_d} U\, T_{s_1,(\Gamma\mathbf{s})_1+r}\, U^\dagger \bigotimes_{i=2}^{n} T_{s_i,(\Gamma\mathbf{s})_i}$$

$$S|\mathcal{S}\rangle = |\mathcal{S}\rangle : \ \mathcal{B}_S = \frac{C(\nu)}{d^{n+1}} \sum_{\mathbf{u}\in V(S)} \left(\sum_{r=0}^{d-1} \omega^{-(u_1)_x r/2} U\, S_{u_1} Z_1^r\, U^\dagger \right) \bigotimes_{i=2}^{n} S_{u_i}$$

$$\langle\mathcal{B}\rangle_{\mathrm{lhv}^*} = C(\nu)\, \delta_{(a_x)_1,0}\delta_{\mathbf{a}\in V(S)} \leq 0\,, \quad \Rightarrow \quad \langle\mathcal{B}\rangle_{\mathrm{lhv}} \leq 0$$

$$\mathrm{Tr}\left(\mathcal{B}|\mathcal{S}\rangle\langle\mathcal{S}|\right) = C(\nu)^2/d^2 > 0\,.$$

Bell Inequalities Beyond Characters

$$V_\phi = \sum_{k\in\mathbb{Z}_d} \omega^{k\phi}|k\rangle\langle k|\,, \quad X_\phi = V_\phi X V_\phi^\dagger\,, \quad \phi \in \mathbb{Q}\setminus\mathbb{N}$$

Deterministic Bell violation

$$\mathcal{B}_\phi = \frac{1}{d}\sum_{k\in\mathbb{Z}_d} \omega^{-k[\phi]} X^k \otimes X_\phi^k\,, \quad \langle\Psi|\mathcal{B}_\phi|\Psi\rangle = 1\,, \quad |\langle\mathcal{B}_\phi\rangle_{\text{lhv}}| < 1$$

$\phi = 1/2$: $|\langle\mathcal{B}_\phi\rangle_{\text{lhv}}| \approx 0.647(d=5), 0.642(d=7), 0.639(d=11)$

Fixed number of measurement operators

$$2\mathcal{B}'_\phi = \omega^{[\phi]} X^\dagger \otimes X_\phi^\dagger + \omega^{[\phi]} X_\phi^\dagger \otimes X^\dagger + X \otimes X + \omega^{-2[\phi]} X_\phi \otimes X_\phi + h.c.\ .$$

$$\langle\mathcal{B}'_\phi\rangle_{\text{lhv}^*} < 4\,, \quad \langle\Psi|\mathcal{B}'_\phi|\Psi\rangle = 4$$

$\phi = 2/3$: $\langle\mathcal{B}'_\phi\rangle_{\text{lhv}} \approx 3.496(d=5), 3.737(d=7), 3.892(d=11)$

Discussion

* Higher degree polynomials $U = \sum_{k=0}^{d-1} \omega^{f_q(k)}$, $q = deg(f) \leq d-1$

$$\left[\frac{d-1}{\deg(f)}\right] + 1 \leq |V_f|$$

* Self-testing mutually unbiased bases

* Extension to continuous variables

Thank you very much for your attention!

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