# Bell Nonlocality from Wigner Negativity in Qudit Stabilizer States

Uta Isabella Meyer, Frédéric Grosshans, Damian Markham

Sorbonne Universite, CNRS, LIP6

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## Bell experiments

Measurement scenario for n parties

POVMs 
$$\{(A_{m_i})_k\}_i$$
 with  $\sum_i (A_{m_i})_k = \mathbb{1}_k, k = 1, ..., n$ 

### Experiment

Set of probabilities  $\{p(\mathbf{a}|\mathbf{m}) = p(a_i, i = 1, \dots, n|m_i, i = 1, \dots, n)\}$ 

Quantum Model  $\rightarrow$  Born rule

$$p(\mathbf{a}|\mathbf{m}) = \operatorname{Tr}\left(\Pi_{\mathbf{a}}^{\mathbf{m}}\rho\right) \tag{1}$$

Classical Model→ lhv model

$$p(\mathbf{a}|\mathbf{m}) = \sum_{\lambda} \mu(\lambda) \prod_{i=1}^{n} p_i(a_i|m_i, \lambda)$$
 (2)

# Bell Nonlocality

### Bell operator

$$\mathcal{B} = \sum_{m_i} b_{\{m_i\}_i} \bigotimes_{k=1}^n (A_{m_i})_k \tag{3}$$

### Bell Inequality

$$\langle \mathcal{B} \rangle_{lhv} = \sum_{m_i} b_{\{m_i\}_i} \sum_{\lambda} \mu(\lambda) \prod_i a_i \, p_i(a_i|m_i,\lambda) \leq C = \max_{lhv} \langle \mathcal{B} \rangle_{lhv} \qquad (4)$$

#### Bell Violation

$$\operatorname{tr}(\rho \mathcal{B}) = \sum_{m_i} b_{\{m_i\}_i} \left( \prod_i a_i \right) \operatorname{tr}\left(\rho \bigotimes_{k=1}^n \Pi_{a_k}^{m_i} \right) > C$$
 (5)

# Qudits & Gross' Wigner function

$Qubits \sim Qu2it$	Qudit (d prime)
$X k\rangle =  k+1\rangle, Z k\rangle = (-1)^k k\rangle$	$X k\rangle =  k+1\rangle$ , $Z k\rangle = \omega^k  k\rangle$ , $\omega = e^{2\pi i/d}$
ZX + XZ = 0	$ZX - \omega XZ = 0$
$\sigma_{(x,z)} = \mathrm{i}^{xz} X^x Z^z ,$	$D_{(x,z)} = \omega^{xz/2} X^x Z^z ,$
x, z = 0, 1	$x, z = 0, \dots, d-1$
$\sigma^{-1} = \sigma = \sigma^{\dagger}$	$D_{(x,z)}^{-1} = D_{(-x,-z)} = D_{(x,z)}^{\dagger}$

Wigner function (quasi-probability distribution)

$$W_{(x,z)}(\rho) = \frac{1}{d} \sum_{q,p=0}^{d-1} \omega^{xp-zq} \operatorname{Tr} \left( \rho D_{(q,p)} \right)$$
 (6)

## Pauli Stabilizer State & Measurements

## Pauli Stabilizer State $|S\rangle$

$$S_i|S\rangle = |S\rangle$$
 with  $S_i = \omega^{t(\mathbf{x}_i, \mathbf{z}_i)} D_{(\mathbf{x}_i, \mathbf{z}_i)}$ ,  $\{S_i\}_i$  abelian group.

Measurement description with Pauli operators

Operator Fourier transform

$$A_{(x,z)} = \sum_{q,p=0}^{d-1} \omega^{xp-zq} D_{(p,q)}$$
 (7)

 $\{A_{(x,z)}\}\$  are orthonormal basis of operators  $(A,B)=\operatorname{tr}\left(A^{\dagger}B\right)/d^n$ 

## Contextuality & Wigner Negativity

Contextuality, a generalization of nonlocality (Kochen and Specker 1975)

Non-contextual value assignment (Delfosse et al. 2017)

~ Deterministic lhv model

$$D_{(\mathbf{x},\mathbf{z})} \sim (\mathbf{a}_x, \mathbf{a}_z) \equiv \omega^{\mathbf{x} \mathbf{a}_z - \mathbf{z} \mathbf{a}_x}$$

& probability distributions  $\{q\}$  such that

$$\operatorname{tr}\left(\rho\,D_{(\mathbf{x},\mathbf{z})}\right) = \sum_{\mathbf{a}_x,\mathbf{a}_z} q_{(\mathbf{a}_x,\mathbf{a}_z)} \omega^{\mathbf{x}\mathbf{a}_z - \mathbf{z}\mathbf{a}_x}$$

Nonlocality ⇒ Contextuality ⇔ Wigner negativity (Delfosse et al. 2017)

Pauli Stabilizer States and Clifford operators are non-contextual (Gross 2006)

& efficiently classically simulatable

(Howard, Brennan, and Vala 2013; Howard et al. 2014)

### Previous Works & This Work

### Nonlocality in Singlet states, GHZ states, graph states

- d even: Cerf, Massar, and Pironio 2002, Tang, Yu, and Oh 2013
- Qutrits: Lawrence 2017, Chen et al. 2002, Kaszlikowski et al. 2002a;
   Kaszlikowski et al. 2002b, Acín et al. 2004, Gruca, Laskowski, and Żukowski 2012, Li and Chen 2011, Mackeprang et al. 2023
- <u>Kaniewski et al. 2019</u>, Collins et al. 2001, Ji et al. 2008; Liang, Lim, and Deng 2009

### This Work

- Apply Wigner negativity straightaway
- Where to find Wigner negativity?
- No composite systems
- No slicing or subdivising of Qudit into d' < d Qud'it
- Study of d-intrinsic character polynomials

## Qudit $\pi/8$ -Gate

Clifford unitary operator (Appleby 2009):

$$C \sim D_{(x,z)} \sum_{j,k=0}^{d-1} \omega^{f_2(j,k)} |j\rangle\langle k| \qquad (f_2 \in poly(\mathbb{Z}_d), \frac{\deg(f_2) = 2}{})$$

## Qudit $\pi/8$ -Gate (Howard and Vala 2012)

$$U_{\nu} = \sum_{k=0}^{d-1} \omega^{\nu_k} |k\rangle\langle k| \tag{8}$$

maps Pauli operators to Clifford operators for d > 3 with

$$\nu \in poly(\mathbb{Z}_d)$$
,  $deg(\nu) = 3$ 

$$\nu_k = 12^{-1}k(\gamma + k(6z + (2k+3)\gamma)) + \epsilon k \text{ for } z, \epsilon \in \mathbb{Z}_d \text{ and } \gamma \in \mathbb{Z}_d^*$$

# Bell Inequalities with Qudit $\pi/8$ gates

Wigner function (d > 3)

of 
$$|\Psi_{\nu}\rangle\langle\Psi_{\nu}|:=(U_{\nu}\otimes\mathbb{1})\,|\Psi\rangle\langle\Psi|\left(U_{\nu}^{\dagger}\otimes\mathbb{1}\right)$$
, singlet state  $|\Psi\rangle=\sum_{k=0}^{d-1}|k\,k\rangle/\sqrt{d}$ :

$$W_{(u_1,u_2)}(|\Psi_{\nu}\rangle\langle\Psi_{\nu}|) = \frac{1}{d^3} \delta_{(u_1)_x,(u_2)_x} \sum_{k=0}^{d-1} \omega^{\frac{1}{3}} a_3 k^3 + a_1 k , \qquad (9)$$

$$a_3 = 24^{-1}\gamma$$
,  $a_1 = \epsilon + (u_1)_z + (u_2)_z + z(u_1)_x + 2^{-1}\gamma((u_1)_x^2 - (u_1)_x + 6^{-1})$ 

$$\exists \gamma, \epsilon, \text{ s.t. } W_{(u_1,u_2)}(|\Psi_{\nu}\rangle\langle\Psi_{\nu}|) < 0$$

$$C(\nu) := W_{(0,0)}(|\Psi_{\nu}\rangle\langle\Psi_{\nu}|) < 0,$$

 $|C(\nu)| < 2\sqrt{d}$  (Weil's Theorem)

# Bell Inequalities with Qudit $\pi/8$ gates

$$D_{(s,r)} \otimes D_{(s,-r)} |\Psi\rangle = |\Psi\rangle : \mathcal{B}_2 = \frac{C(\nu)}{d^3} \sum_{s,r,q=0}^{d-1} U_{\nu} D_{(s,q-r)} U_{\nu}^{\dagger} \otimes D_{(s,r)}$$
$$D_{(\mathbf{s},\Gamma\mathbf{s})} |G_{\Gamma}\rangle = |G_{\Gamma}\rangle : \mathcal{B}_G = \frac{C(\nu)}{d^{n+1}} \sum_{s,r,q=0} U T_{s_1,(\Gamma\mathbf{s})_1+r} U^{\dagger} \bigotimes^{n} T_{s_i,(\Gamma\mathbf{s})_i}$$

$$S|\mathcal{S}\rangle = |\mathcal{S}\rangle: \ \mathcal{B}_S = \frac{C(\nu)}{d^{n+1}} \sum_{\mathbf{u} \in V(S)} \left( \sum_{r=0}^{d-1} \omega^{-(u_1)_x r/2} U \, S_{u_1} Z_1^r \, U^\dagger \right) \bigotimes_{i=2}^n S_{u_i}$$

$$\langle \mathcal{B} \rangle_{\mathrm{lhv}^*} = C(\nu) \, \delta_{(a_x)_1,0} \delta_{\mathbf{a} \in V(S)} \leq 0 \,, \quad \Rightarrow \quad \langle \mathcal{B} \rangle_{\mathrm{lhv}} \leq 0$$

$$\operatorname{Tr}(\mathcal{B}|\mathcal{S}\rangle\langle\mathcal{S}|) = C(\nu)^2/d^2 > 0.$$

## **Bell Inequalities Beyond Characters**

$$V_{\phi} = \sum_{k \in \mathbb{Z}_d} \omega^{k\phi} |k\rangle\langle k| \,, \quad X_{\phi} = V_{\phi} \, X \, V_{\phi}^{\dagger} \,, \quad \boxed{\phi \in \mathbb{Q} \setminus \mathbb{N}}$$

#### Deterministic Bell violation

$$\mathcal{B}_{\phi} = \frac{1}{d} \sum_{k \in \mathbb{Z}_d} \omega^{-k[\phi]} X^k \otimes X_{\phi}^k, \quad \langle \Psi | \mathcal{B}_{\phi} | \Psi \rangle = 1, \quad |\langle \mathcal{B}_{\phi} \rangle_{\text{lhv}}| < 1$$

$$\phi = 1/2$$
:  $|\langle \mathcal{B}_{\phi} \rangle_{\text{lhv}}| \approx 0.647 (d = 5), 0.642 (d = 7), 0.639 (d = 11)$ 

### Fixed number of measurement operators

$$\begin{split} 2\mathcal{B}_\phi' &= \omega^{[\phi]} X^\dagger \otimes X_\phi^\dagger + \omega^{[\phi]} X_\phi^\dagger \otimes X^\dagger + X \otimes X + \omega^{-2[\phi]} X_\phi \otimes X_\phi + h.c. \ . \\ &\langle \mathcal{B}_\phi' \rangle_{\text{lhv}^*} < 4 \,, \quad \langle \Psi | \mathcal{B}_\phi' | \Psi \rangle = 4 \end{split}$$

$$\phi = 2/3 \colon \langle \mathcal{B}_\phi' \rangle_{\text{lhv}} \approx 3.496 (d=5), 3.737 (d=7), 3.892 (d=11)$$

### **Discussion**

\* Higher degree polynomials  $U = \sum_{k=0}^{d-1} \omega^{f_q(k)}, \ q = deg(f) \le d-1$ 

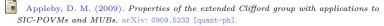
$$\left[\frac{d-1}{\deg(f)}\right] + 1 \le |V_f|$$

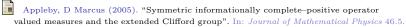
- \* Self-testing mutually unbiased bases
- \* Extension to continuous variables

Thank you very much for your attention!

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