Query Complexity of Boolean Functions under Indefinite Causal Order

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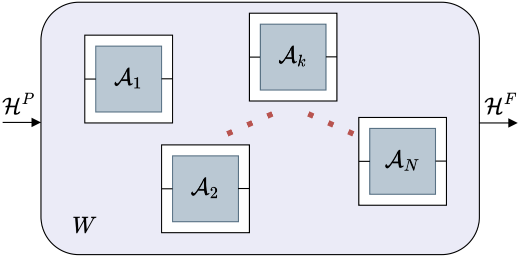
Quantum supermaps

Quantum supermaps

A quantum supermap $\mathcal{S}$ is an operator which transforms any N quantum channels $(\mathcal{A}_1, \ldots, \mathcal{A}_N)$ into a valid quantum channel $\mathcal{S}(\mathcal{A}_1, \ldots, \mathcal{A}_N)$.

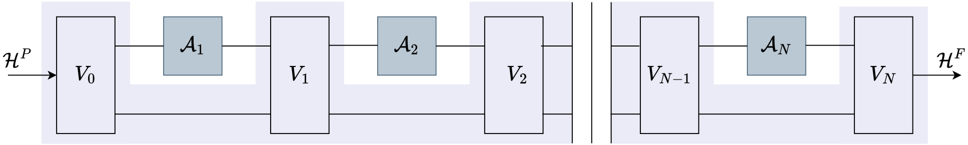
A general supermap:



A supermap $\mathcal{S}$ can be characterised by a process matrix $W \in \mathcal{L}(\mathcal{H}^{PA_1^{IO} \ldots A_N^{IO} F})$ *(Oreshkov, Costa, and Brukner 2012)*.

Different types of supermaps

FO-supermaps *(Chiribella, D'Ariano, Perinotti, and Valiron 2013)*:



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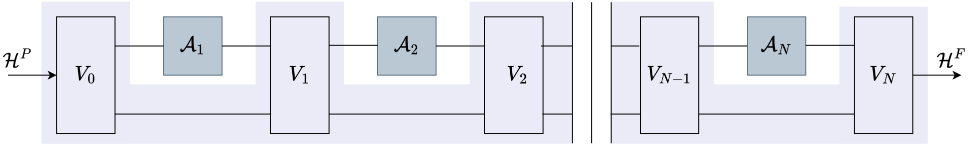
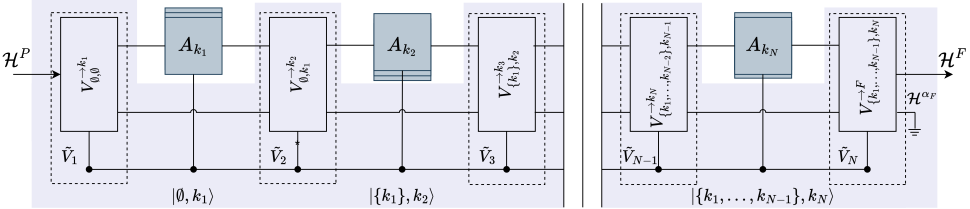
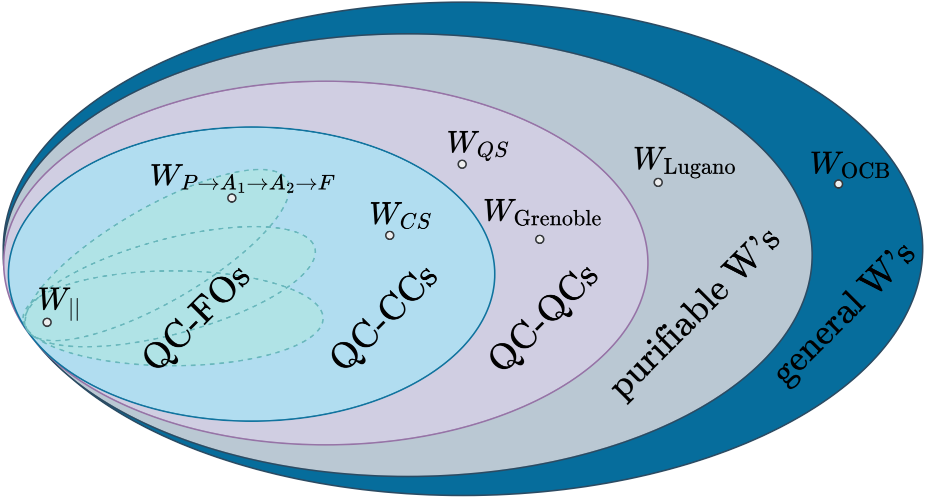


QC-supermaps *(Wechs, Dourdent, Abbott, and Branciard 2021)*:



Remark: the quantum switch is a QC-supermap.

Classes of quantum supermaps



(Wechs, Dourdent, Abbott, and Branciard 2021)

A resource for computation?

Examples:

- *Quantum switch*: commutation/anti-commutation of pair of unitaries[1],
- *N-switch*: property testing; unitary permutation problem[2],
- Channel discrimination[3],
- Transformation of unitary operators[4],
- ...

These are rather ad hoc computational tasks.

[1] Chiribella 2012.
[2] Araújo, Costa, and Brukner 2014; Facchini and Perdrix 2015; Taddei et al. 2021.
[3] Bavaresco, Murao, and Quintino 2021, 2022.
[4] Quintino and Ebler 2022; Quintino, Dong, Shimbo, Soeda, and Murao 2019.

Query complexity of Boolean functions

Query complexity of Boolean functions

For a Boolean function $f : \{0,1\}^n \to \{0,1\}$, how many queries to $x \in \{0,1\}^n$ are needed to compute $f(x)$, in the worst case.

A standard computational problem:

- it captures the complexity of *decision problems* in the query model of computation,
- it can be used to compare the relative power of classical, randomised, and quantum computers,
- there exist methods to find *non-trivial lower bounds* on the quantum query complexity (polynomial[5] and adversarial methods[6]).

[5] Beals, Buhrman, Cleve, Mosca, and Wolf 2001.
[6] Ambainis 2002.

Quantum query complexity

In the quantum case, the query (phase) oracle is: $O_x \left| i \right\rangle = \begin{cases} (-1)^{x_i} \left| i \right\rangle & \text{if } i \neq 0, \\ \left| i \right\rangle & \text{otherwise.} \end{cases}$

A T-query quantum algorithm is then:

$$\left| \psi_x^T \right\rangle = U_T O_x U_{T-1} \dots U_1 O_x U_0 \left| 0 \dots 0 \right\rangle .$$

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Definition: quantum query complexity of a Boolean function f

It is the *minimal* T for which there exists a T-query quantum algorithm such that, for all x, measuring the left-most qubit of $\left| \psi_x^T \right\rangle$ in the computational basis outputs $f(x)$ with probability 1 (or with a bounded error $\varepsilon < 1/3$).

These query complexities are written: $Q_E(f)$ and $Q_2(f)$.

Query complexity under indefinite causal order

Let us write different types of supermaps:

- $\mathcal{S}^{\text{FO}}$: Fixed Order supermaps,
- $\mathcal{S}^{\text{QC}}$: Quantum Controlled supermaps,
- $\mathcal{S}^{\text{Gen}}$: General supermaps.

Definition: $\mathcal{C}$ quantum query complexity of a Boolean function f

For a class $\mathcal{C} \in \{\text{FO, QC, Gen}\}$ of supermaps, it is the *minimal* T for which there exist a supermap $\mathcal{S}^{\mathcal{C}}$ such that for all x, measuring $\mathcal{S}^{\mathcal{C}}(\underbrace{O_x, \ldots, O_x}_{T \text{ copies}}) = \rho_x$, in the computational basis gives $f(x)$ with probability 1 (or with a bounded error $\varepsilon < 1/3$).

One can define new query complexities:

$$Q_E^{\text{Gen}}(f) \leq Q_E^{\text{QC}}(f) \leq Q_E^{\text{FO}}(f) = Q_E(f),$$
$$Q_2^{\text{Gen}}(f) \leq Q_2^{\text{QC}}(f) \leq Q_2^{\text{FO}}(f) = Q_2(f).$$

The polynomial method

Examples of polynomial representations:

- $f_{\mathrm{OR}}(x_1, x_2) = x_1 + x_2 - x_1 x_2$, *represents* the OR function.
- $f_{\mathrm{AND}}(x_1, x_2) = \frac{1}{3}(x_1 + x_2)$, *approximates* the AND function.

Theorem: polynomial bound on quantum circuits *(Beals, Buhrman, Cleve, Mosca, and Wolf 2001)*

For any Boolean function f, $\frac{\deg(f)}{2} \leq Q_E(f)$ and $\frac{\widetilde{\deg}(f)}{2} \leq Q_2(f)$.

Theorem: polynomial bound on general supermaps

$$\frac{\deg(f)}{2} \leq Q_E^{\mathrm{Gen}}(f) \leq Q_E^{\mathrm{QC}}(f) \leq Q_E^{\mathrm{FO}}(f) = Q_E(f),$$
$$\frac{\widetilde{\deg}(f)}{2} \leq Q_2^{\mathrm{Gen}}(f) \leq Q_2^{\mathrm{QC}}(f) \leq Q_2^{\mathrm{FO}}(f) = Q_2(f).$$

Proof

We have $\mathsf{S}^{\mathcal{C}}(\mathsf{O}_x, \ldots, \mathsf{O}_x) = \rho_x$,
Measuring in the computational basis, the probability of obtaining the outcome 1 result is

$$g(x) = \mathrm{Tr}\big[\mathsf{S}^{\mathcal{C}}(\mathsf{O}_x, \ldots, \mathsf{O}_x) \cdot |1\rangle\langle 1|\big].$$

In the process matrix formalism, it is written

$$g(x) = \mathrm{Tr}\Big[(\mathsf{O}_x^{\otimes T} \otimes |1\rangle\langle 1|) \cdot W^{\mathcal{C}}\Big].$$

By linearity of the trace and because $|1\rangle\langle 1|$ and $W^{\mathcal{C}}$ are independent of x, g is a multivariate polynomial of degree at most $2T$

$$g(x) = g(x_1, \ldots, x_n).$$

Finally, if $\mathcal{S}^{\mathcal{C}}$ computes f, so does g, implying: $2T \geq \deg(f)$.

Reminder: a general lower bound

Theorem: polynomial bound on general supermaps

We have for all Boolean functions f:

$$\frac{\deg(f)}{2} \le Q_E^{\mathrm{Gen}}(f) \le Q_E^{\mathrm{QC}}(f) \le Q_E^{\mathrm{FO}}(f) = Q_E(f),$$

$$\frac{\widetilde{\deg}(f)}{2} \le Q_2^{\mathrm{Gen}}(f) \le Q_2^{\mathrm{QC}}(f) \le Q_2^{\mathrm{FO}}(f) = Q_2(f).$$

Note that $Q_E(f) = \tilde{O}(\deg(f)^3)$ and $Q_2(f) = \tilde{O}(\widetilde{\deg}(f)^4)$.[7]

[7] Aaronson, Ben-David, Kothari, Rao, and Tal 2021.

QC-supermaps on T copies of the same unitary

Theorem

For any T-input QC-supermap $\mathcal{S}^{\mathrm{QC}}$, there exists a T-input FO-supermap $\mathcal{S}^{\mathrm{FO}}$ such that for all unitary channels $\mathcal{U}$, $\mathcal{S}^{\mathrm{QC}}(\mathcal{U},\ldots,\mathcal{U}) = \mathcal{S}^{\mathrm{FO}}(\mathcal{U},\ldots,\mathcal{U})$.

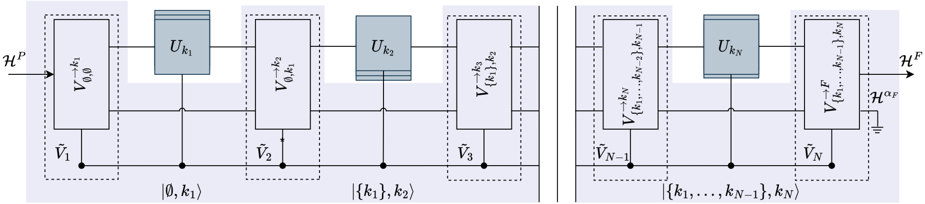
It was already known to be the case for N-switch *(Bavaresco, Murao, and Quintino 2022)*.

For the problem of query complexity of Boolean function, it implies:

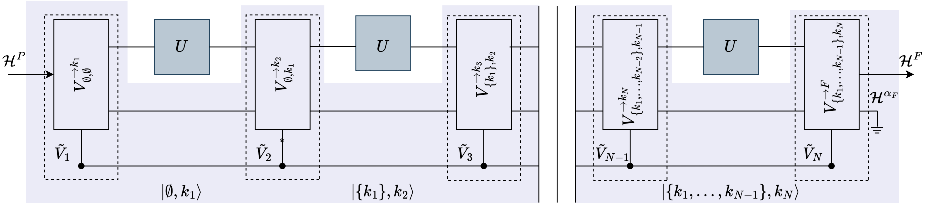
$$\frac{\deg(f)}{2} \leq Q_E^{\mathrm{Gen}}(f) \leq Q_E^{\mathrm{QC}}(f) = Q_E^{\mathrm{FO}}(f) = Q_E(f),$$

$$\frac{\widetilde{\deg}(f)}{2} \leq Q_2^{\mathrm{Gen}}(f) \leq Q_2^{\mathrm{QC}}(f) = Q_2^{\mathrm{FO}}(f) = Q_2(f).$$

Sketch of the proof



Decoupling the U's from the control:



Reminder: two limiting results

Polynomial bound + equivalence of QC-supermaps and FO-supermaps

We have for all Boolean functions f:

$$\frac{\deg(f)}{2} \leq Q_E^{\mathrm{Gen}}(f) \leq Q_E^{\mathrm{QC}}(f) = Q_E^{\mathrm{FO}}(f) = Q_E(f),$$
$$\frac{\widetilde{\deg}(f)}{2} \leq Q_2^{\mathrm{Gen}}(f) \leq Q_2^{\mathrm{QC}}(f) = Q_2^{\mathrm{FO}}(f) = Q_2(f).$$

Exhaustive search for an advantage

Can we still expect some advantage from causally indefinite supermaps?

Definition

Let us denote by $\varepsilon_T^{\mathcal{C}}(f)$ the minimum error ε for which there exists a T-query supermap of class $\mathcal{C} \in \{\text{FO, QC, Gen}\}$ computing f with a bounded error ε.

Semi-definite program:

$$
\begin{aligned}
\varepsilon_T^{\mathcal{C}}(f) = \max_{\varepsilon, W^{[0]}, W^{[1]}} \; & 1-\varepsilon \\
\text{s.t. } & \forall x \in F^{(0)},\; \mathrm{Tr}\left[W^{[0]} \cdot \mathrm{O}_x^{\otimes T}\right] \geq 1-\varepsilon, \\
& \forall x \in F^{(1)},\; \mathrm{Tr}\left[W^{[1]} \cdot \mathrm{O}_x^{\otimes T}\right] \geq 1-\varepsilon, \\
& W^{[0]} \geq 0,\; W^{[1]} \geq 0,\; \varepsilon \geq 0, \\
& W^{[0]} + W^{[1]} \in \mathcal{W}^{\mathcal{C}}.
\end{aligned}
$$

For 4 bit Boolean functions and 2 queries:

- gap between $\varepsilon_2^{\text{FO}}$ and $\varepsilon_2^{\text{Gen}}$ for 179 representatives out of the 222 NPN (Negate-Permute-Negate) representatives.

Advantage of general supermaps

For a mathematical proof, lower and upper bounds are obtained from the primal and dual representation of the SDP, by extracting solutions that rigorously satisfy the constraints *(Bavaresco, Murao, and Quintino 2021)*.

Theorem

There exists some function f for which $\varepsilon_2^{\text{Gen}}(f) < \varepsilon_2^{\text{FO}}(f)$.

The largest separation is obtained for the 4-bit Boolean function,
$f(x) = x_1 + x_2x_3 + x_2x_4 + x_3x_4 + x_2x_3x_4$, for which:

$$0.0324 \leq \varepsilon_2^{\text{Gen}}(f) \leq 0.0377 < 0.0452 \leq \varepsilon_2^{\text{FO}}(f) \leq 0.0467.$$

Conclusion

Extension of the notion of query complexity to supermaps beyond FO-supermaps.
Two limiting results:

- a *general lower* bound with the polynomial method,
- *no advantage* from coherent control of the order of application of the queries,

but

- general supermaps can reduce the probability of error for the computation of some Boolean functions.

Open questions

1. Can one find an asymptotic separation?
2. Can purifiable processes reduce the query complexity of Boolean functions?
3. Are there interesting settings with multiple different oracles? Links with communication?