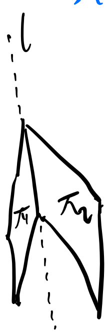
在仿射标架$[O;d_1,d_2,d_3]$中，若平面$\pi_1 \cap \pi_2 = l$

$$\begin{aligned}\pi_1: \\ \pi_2: \end{aligned}\begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_2=0 \end{cases},$$



若 $\vec{v} \parallel l$，$\vec{v}=(V_1, V_2, V_3)$.

考虑π_1, π_2的法向量 $\vec{n_1}, \vec{n_2}$.

则 $\vec{n_1}=(A_1, B_1, C_1)$

$\vec{n_2}=(A_2, B_2, C_2)$.

$\therefore l \perp \vec{n_1}$, $l \perp \vec{n_2}$

$\therefore l \parallel \vec{n_1} \times \vec{n_2}$

即 $\vec{v} \parallel \vec{n_1} \times \vec{n_2}$.

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} d_2 \times d_3 & A_1 & A_2 \\ d_3 \times d_1 & B_1 & B_2 \\ d_1 \times d_2 & C_1 & C_2 \end{vmatrix}$$

$$= \begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix} \cdot (d_2 \times d_3) - \begin{vmatrix} A_1 & A_2 \\ C_1 & C_2 \end{vmatrix} \cdot (d_3 \times d_1) + \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} \cdot (d_1 \times d_2).$$

又$\because \vec{V_1} \perp \vec{n_1}$, $\vec{V_2} \perp \vec{n_2}$

$\therefore \vec{V_1} \cdot \vec{n_1} = \vec{V_2} \cdot \vec{n_2} = 0$.

即：$\begin{cases} A_1V_1+B_1V_2+C_1V_3=0 \\ A_2V_1+B_2V_2+C_2V_3=0. \end{cases}$

但是由于这是一个齐次线性方程组，并且有无穷组解，故我们考虑：

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_1 & B_1 & C_1 \end{vmatrix} = A_1\cdot\underbrace{\begin{vmatrix} B_2 & C_2 \\ B_1 & C_1 \end{vmatrix}}_{V_1} - B_1\cdot\underbrace{\begin{vmatrix} A_2 & C_2 \\ A_1 & C_1 \end{vmatrix}}_{V_2} + C_1\cdot\underbrace{\begin{vmatrix} A_2 & B_2 \\ A_1 & B_1 \end{vmatrix}}_{V_3} = 0.$$

∴我找到了一个解 $(V_1,V_2,V_3)=\left(\begin{vmatrix} B_2 & C_2 \\ B_1 & C_1 \end{vmatrix}, -\begin{vmatrix} A_2 & C_2 \\ A_1 & C_1 \end{vmatrix}, \begin{vmatrix} A_2 & B_2 \\ A_1 & B_1 \end{vmatrix}\right)$.

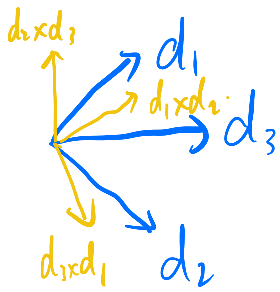
考虑 $\vec{v}$ 与 $\vec{n_1}\times\vec{n_1}$：　　$=(\alpha,\beta,\gamma)$

$$\vec{v}=\alpha d_1-\beta d_2+\gamma d_3$$

$$\vec{n_1}\times\vec{n_2}=\alpha(d_2\times d_3)-\beta(d_3\times d_1)+\gamma(d_1\times d_2).$$

又 α,β,γ 是完全自由的，那 $\vec{v}\,//\,\vec{n_1}\times\vec{n_2}$.

也就是说：对 $\forall\alpha,\beta,\gamma$，有：



$(d_1\alpha-d_2\beta+d_3\gamma)$ 与 $\left((d_2\times d_3)\alpha-(d_3\times d_1)\beta+(d_1\times d_2)\gamma\right)$

恒是共线的，我觉得不可能！

下面通过计算来验证：

将V的坐标表为以$\{d_2\times d_3, d_3\times d_1, d_1\times d_2\}$为基底的坐标:

记 $V = k_1(d_2\times d_3)+k_2(d_3\times d_1)+k_3(d_1\times d_2)$,

则 $V\cdot d_1 = k_1\cdot(d_1,d_2,d_3)$.

$$\therefore V = \frac{V\cdot d_1}{(d_1,d_2,d_3)}\cdot(d_2\times d_3)+\frac{V\cdot d_2}{(d_1,d_2,d_3)}\cdot(d_3\times d_1)+\frac{V\cdot d_3}{(d_1,d_2,d_3)}\cdot(d_1\times d_2)$$

$\because V \parallel n_1\times n_2 \Longleftrightarrow (V\cdot d_1, V\cdot d_2, V\cdot d_3)$ 与 (α,β,γ) 成比例,

记比例为λ,

则:
$$\lambda = \frac{V\cdot d_1}{\alpha} = \frac{\alpha d_1^2+\beta d_2 d_1+\gamma d_3 d_1}{\alpha} = t_1$$

$$\lambda = \frac{V\cdot d_2}{\beta} = \frac{\alpha d_1 d_2+\beta d_2^2+\gamma d_3 d_2}{\beta} = t_2$$

$$\lambda = \frac{V\cdot d_3}{\gamma} = \frac{\alpha d_1 d_3+\beta d_2 d_3+\gamma\cdot d_3^2}{\gamma} = t_3.$$

令 $t_1=t_2$: $\beta(\alpha d_1^2+\beta d_1 d_2+\gamma d_1 d_3) = \alpha(\alpha d_1 d_2+\beta d_2^2+\gamma d_2 d_3)$.

$\Longleftrightarrow (\beta^2-\alpha^2)d_1 d_2+\beta\alpha(d_1^2-d_2^2) = \alpha\gamma d_2 d_3-\beta\gamma d_1 d_3$.

随便取 $\vec{\alpha_1}=(1,1,0)$, $\vec{\alpha_2}=(1,0,1)$, $\vec{\alpha_3}=(1,2,2)$

则 $LHS=(\beta^2-\alpha^2)\cdot 1 = 3(\alpha\gamma-\beta\gamma)$.

这对于 $\forall\alpha,\beta,\gamma$ 明显不可能恒成立!

请问以上推导及思考的问题在哪?