Mathematical Methods 3
(Hilbert spaces)

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Mathematical Methods 3
(Hilbert spaces)

Bibliography (more in syllabus):

- L. Abellanas y A. Galindo, *Espacios de Hilbert*, Edema, 1987.
- A. Vera López y P. Alegría Ezquerra, *Un curso de Análisis Funcional. Teoría y problemas*, AVL, 1997.
- P. Roman, *Some modern mathematics for physicists and other outsiders* (vol. 2), Pergamon, 1975.
- G. Helmberg, *Introduction to spectral theory in Hilbert space*, Dover, 1997.
- …

Very schematic slides: examples, proofs and **relevant comments** in the blackboard or mentioned orally (recommended to take notes)

Motivation

Postulates of Quantum Mechanics:

Postulate I: Every physical system is represented by a **complex and separable Hilbert space**, and every pure state is described by a vector $|\Psi\rangle$ of said space

Postulate II: Every system observable is represented by a **linear and self-adjoint operator** of the Hilbert space. The **eigenvalues** of said operators are the possible values in a measurement of the observable.

Postulate III: The probability of obtaining a value (a) when measuring an observable (A) in a pure state ($|\Psi\rangle$) is given by $\langle\Psi|P_{A,a}|\Psi\rangle$ where $P_{A,a}$ is the **projector over the proper subspace of the eigenvalue.**

Not only useful in Quantum Mechanics: Differential equations, Fourier analysis,…

Why Hilbert spaces?

They generalize the properties of $\mathbb{R}^n$ to spaces of infinite dimension

Linear space

Finite linear combinations. Linear independence. Linear basis

Metric space

Infinite combinations require limits: the notion of distance

Distance invariant under translations: it suffices to know the distance from the origin (norm)

Normed space

Generalization of $\mathbb{R}^n$: geometry (orthogonality, angles). scalar product

(Pre-)Hilbert space

Course structure

Chapter 0: Linear and metric spaces [(quick) review]

Chapter 1: Normed and Banach spaces

Chapter 2: (Pre-)Hilbert spaces

Chapter 3: Function space and series expansions

Chapter 4: Linear operators in Hilbert spaces

Chapter 5: Functionals and dual space. Theory of distributions

Chapter 6: Spectral theory of operators

Linear spaces

Definition: Linear (or vector) space over a field Λ is a 3-tuple $(L, +, \cdot)$ formed by a non-empty set, $L \neq \emptyset$, and two laws of composition, $(+, \cdot)$, satisfying:

$+ :\ L \times L \to L$ (internal composition) $\qquad \cdot :\ \Lambda \times L \to L$ (external composition)

$i)$ $(L, +)$ Abelian group $\iff$ $\begin{cases} \text{ia) } (x+y)+z = x+(y+z) \\ \text{ib) } \exists \mathbf{0} \in L \,/\, x + \mathbf{0} = x \\ \text{ic) } \forall x \in L\ \exists (-x) \in L \,/\, x + (-x) = \mathbf{0} \\ \text{id) } x + y = y + x \end{cases}$

$ii)$ $\lambda \cdot (x+y) = \lambda \cdot x + \lambda \cdot y$

$iii)$ $\lambda \cdot x + \mu \cdot x = (\lambda + \mu) \cdot x$

$iv)$ $\lambda \cdot (\mu \cdot x) = (\lambda\mu) \cdot x$

$v)$ $1 \cdot x = x$

$\forall x, y, z \in L$

$\forall \lambda, \mu \in \Lambda$

Linear spaces

Properties:

$$i)\ \alpha \cdot \mathbf{0} = \mathbf{0}$$
$$ii)\ 0 \cdot x = \mathbf{0}$$
$$iii)\ -x = (-1) \cdot x$$
$$iv)\ x + y = x + z \Rightarrow y = z$$

$$v)\ \alpha \cdot x = \alpha \cdot y\,,\ \alpha \neq 0 \Rightarrow x = y$$
$$vi)\ \alpha \cdot x = \beta \cdot x\,,\ x \neq \mathbf{0} \Rightarrow \alpha = \beta$$
$$vii)\ \alpha \cdot x = \mathbf{0} \Rightarrow \alpha = 0 \text{ and/or } x = \mathbf{0}$$

Notation: Let $A, B \subset L$ (L linear space over Λ)

$$A \pm B = \{z \in L \,/\, z = x \pm y, x \in A\,, y \in B\}$$
$$\lambda A = \{z \in L \,/\, z = \lambda \cdot x, x \in A\}\ (\lambda \in \Lambda)$$
$$\Lambda x = \{z \in L \,/\, z = \lambda \cdot x, \lambda \in \Lambda\}\ (x \in L)$$
$$\Lambda A = \{z \in L \,/\, z = \lambda \cdot x, x \in A\,, \lambda \in \Lambda\}$$

$$A \times B = \{(x, y) \,/\, x \in A\,, y \in B\} \subset L \times L$$
$$A \backslash B = \{x \in L \,/\, x \in A\,, x \notin B\}$$
$$A^C = L \backslash A = \{x \in L \,/\, x \in L\,, x \notin A\}$$

Linear spaces

Definition: Linear subspace. Non-empty subset with the structure of linear space

$M \subset L$ (L linear space, $M \neq \emptyset$) linear subspace, $M < L$, iff

$$\alpha x + \beta y \in M \qquad\qquad \forall\, \alpha, \beta \in \Lambda\,,\, \forall\, x, y \in M$$

Properties:

Let $\{M_\alpha\}_{\alpha \in A}$ (arbitrary A) is a family of linear spaces $\Rightarrow \cap_\alpha M_\alpha < L$

If $M_1, M_2, \ldots, M_n < L \Rightarrow M_1 + M_2 + \ldots + M_n < L$

Let $M < L \Rightarrow \sum_{i=1}^{n} \alpha_i\, x_i \in M\,, \forall n \in \mathbb{N}\,, \forall\, x_1, \ldots, x_n \in M$

Linear spaces

Definition: Linear span. Let $S \subset L$

$$[S] = \text{span}(S) = \{ \sum_{i=1}^{n} \alpha_i x_i \,,\, \forall n \in \mathbb{N}, \forall x_i \in S, \forall \alpha_i \in \Lambda \}$$

Properties:

$$[S] < L$$

$[S]$ is the smallest linear subspace containing S

$[S] = \cap_i M_i$, where $\{M_i\}$ is the set of linear subspaces containing S

Linear spaces

Definition: Linear independence

$X \subset L$ is linearly independent (l.i.) iff

$\sum_{i=1}^{n} \alpha_i x_i = \mathbf{0}\,,\; x_i \in X\,,\; \alpha_i \in \Lambda \Rightarrow \alpha_1 = \ldots = \alpha_n = 0$

Definition: Hamel basis. Maximal l.i. set (not contained in any other l.i. set)

Properties:

$\forall\, L \neq \{\mathbf{0}\}$ there exist a Hamel basis. Every l.i. set is extendable to a Hamel basis

Every Hamel basis of L has the same cardinality $(\dim L = \operatorname{card} B)$

$L = [B]\,,\; \forall\, B$ Hamel basis of L

B Hamel basis of $L \Rightarrow x = \sum_{i=i}^{n} \alpha_i x_i\,,\; \alpha_i \in \Lambda\,,\; x_i \in B$ is unique

Linear spaces

Definition: Direct sum of subspaces. Let $\{M_i\}_{i=1}^n$ be a family of subspaces of L

$L = M_1 \overrightarrow{\oplus} \dots \overrightarrow{\oplus} M_n$ (L direct sum of M_i) iff

$$\forall x \in L \; \exists ! \, x_1 \in M_1, \dots, x_n \in M_n \, / \, x = x_1 + \dots + x_n$$

Theorem: Let $L = M_1 + M_2$

$L = M_1 \overrightarrow{\oplus} M_2 \Leftrightarrow M_1 \cap M_2 = \{\mathbf{0}\}$ [M_2 linear complement of M_1 in L]

in general if $L = M_1 + \dots + M_n$

$$L = M_1 \overrightarrow{\oplus} \dots \overrightarrow{\oplus} M_n \Leftrightarrow M_i \cap \sum_{j \neq i} M_j = \{\mathbf{0}\}$$

Linear spaces

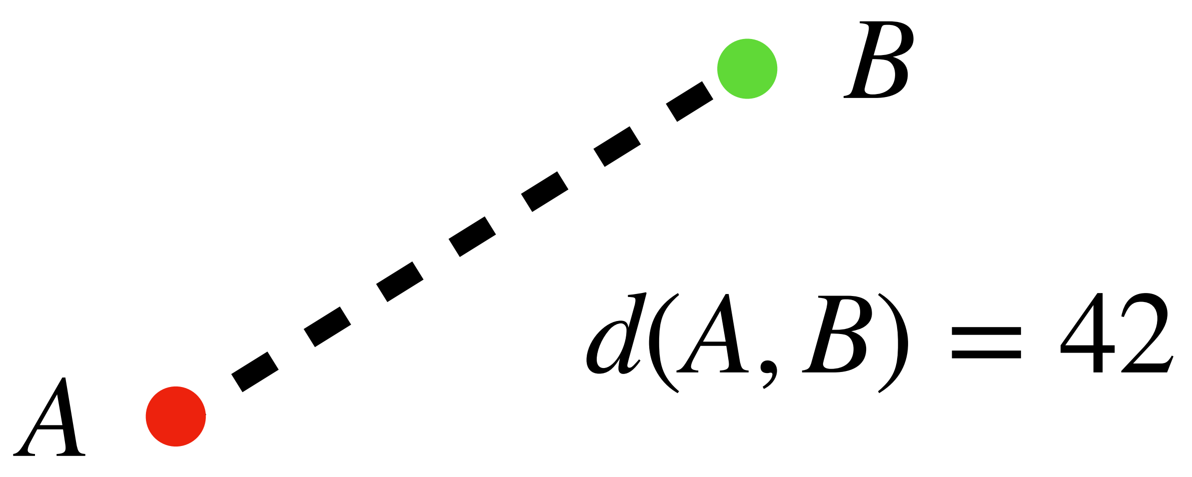
Summary of results:

- Linear (sub)space : $(L, + \, , \cdot \,)$
- Linear span: $[S] = \{$**finite** linear combinations of elements of $S\}$
- Linear independence: finite linear combinations $= \mathbf{0} \Rightarrow$ all coefficients $= 0$
- (Hamel) basis: Maximal l.i. set. Unique cardinal (linear dimension).
  The decomposition of elements of L in terms of elements of B is unique.
- Direct sum of subspaces: sum of subspaces of L with null intersection

Other results and definitions (linear applications, isomorphisms, projectors,…) could be introduced now but we will discuss them after the introduction of Hilbert spaces

Metric spaces

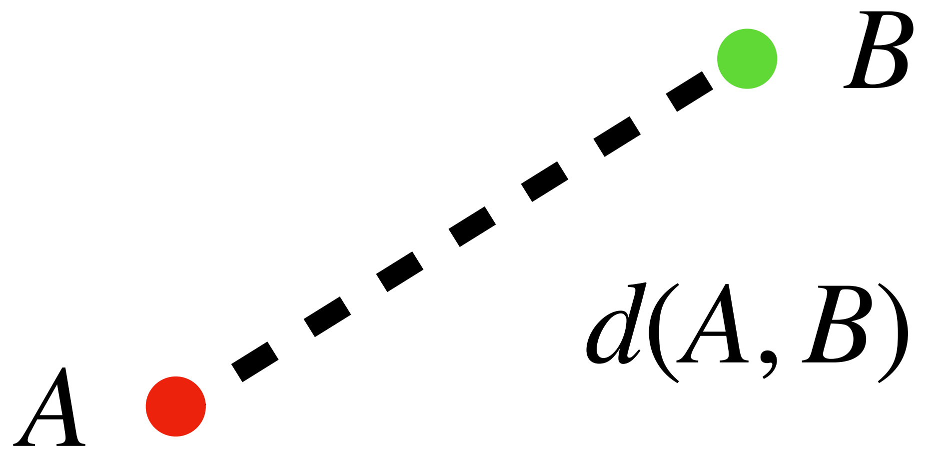
Introduction: Metric spaces generalize the notion of distance between "objects"



"Usual" (or Euclidean) distance

Metric spaces

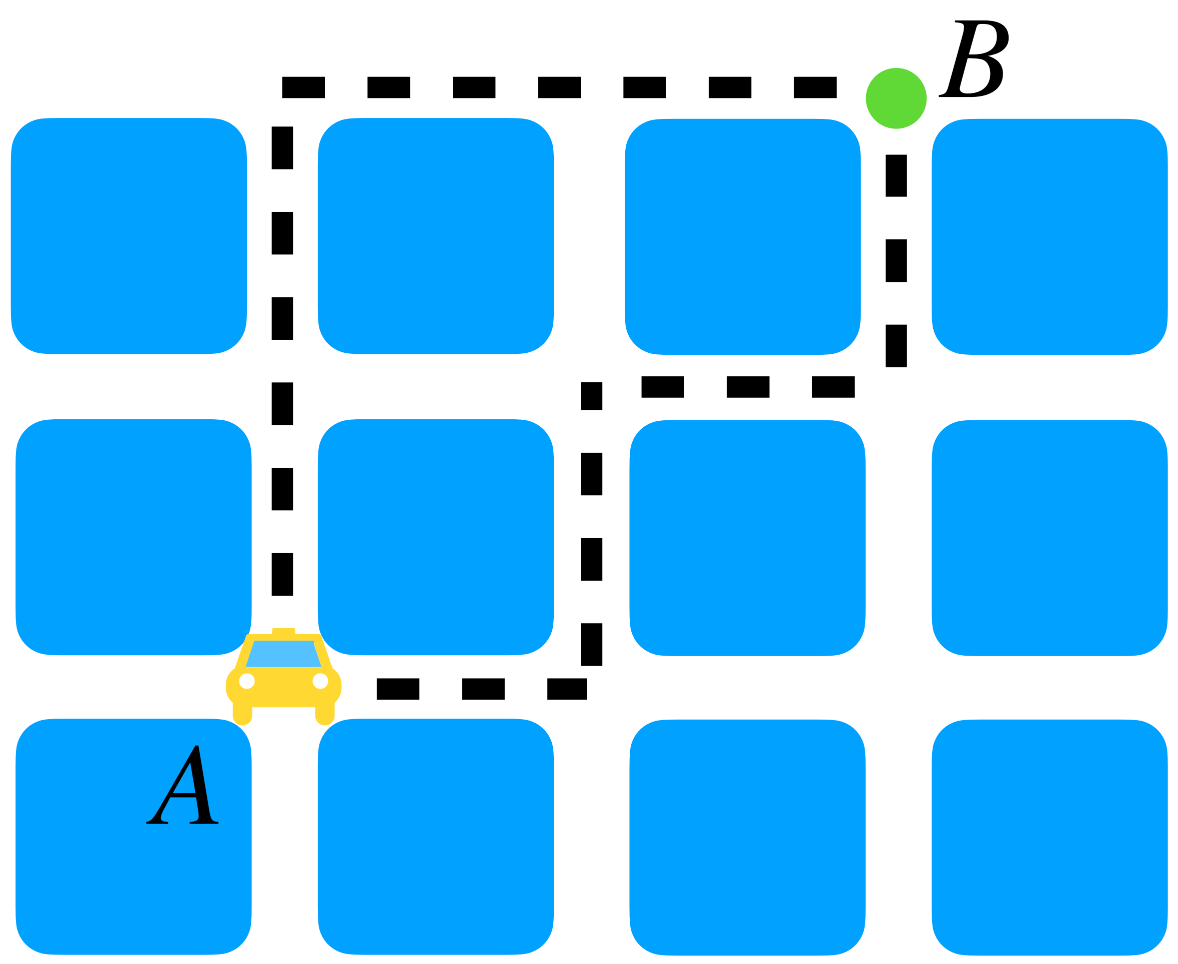
Introduction: Metric spaces generalize the notion of distance between “objects”



$d(A, B) = 42$

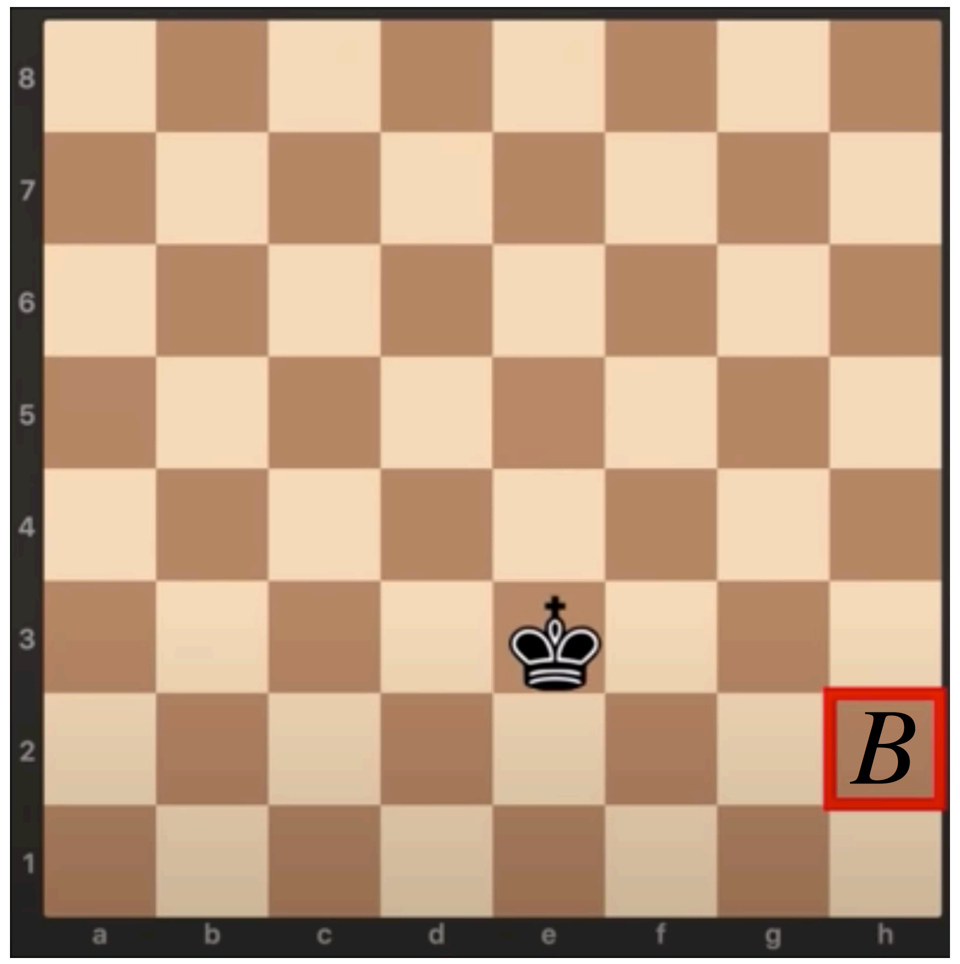
“Usual” (or Euclidean) distance

but there are many other distances, such as



Manhattan distance

Chebyshev (or infinity) distance



Metric spaces

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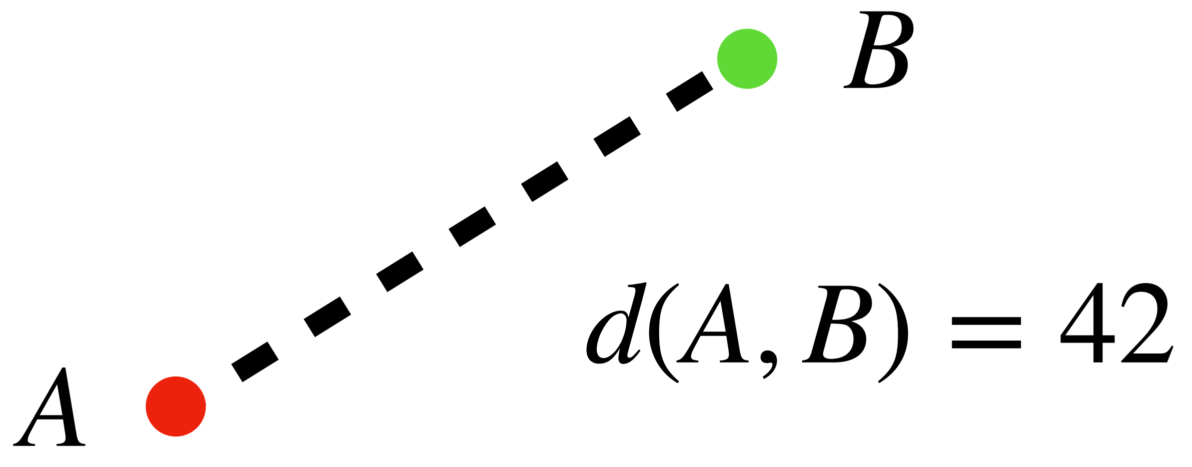

"Usual" (or Euclidean) distance

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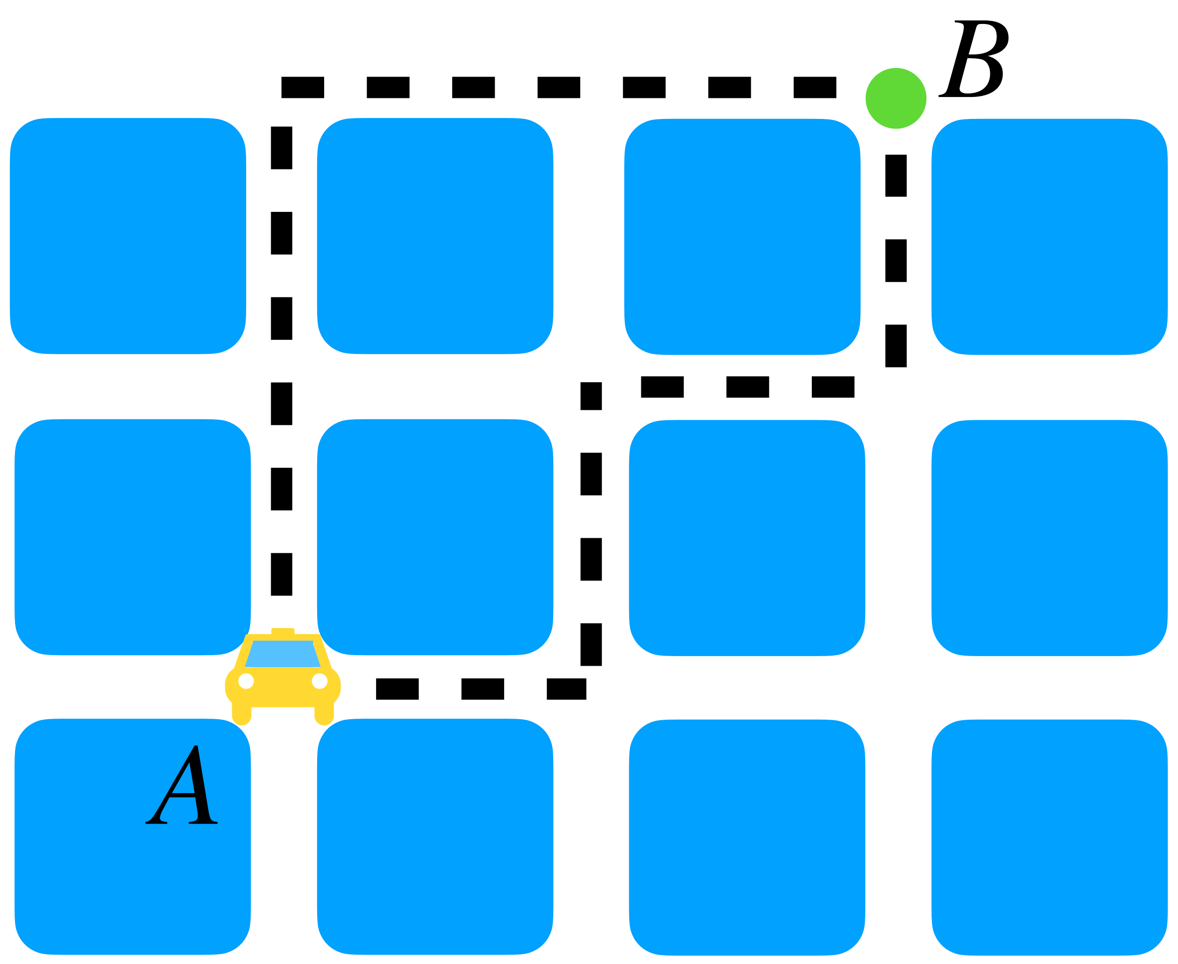
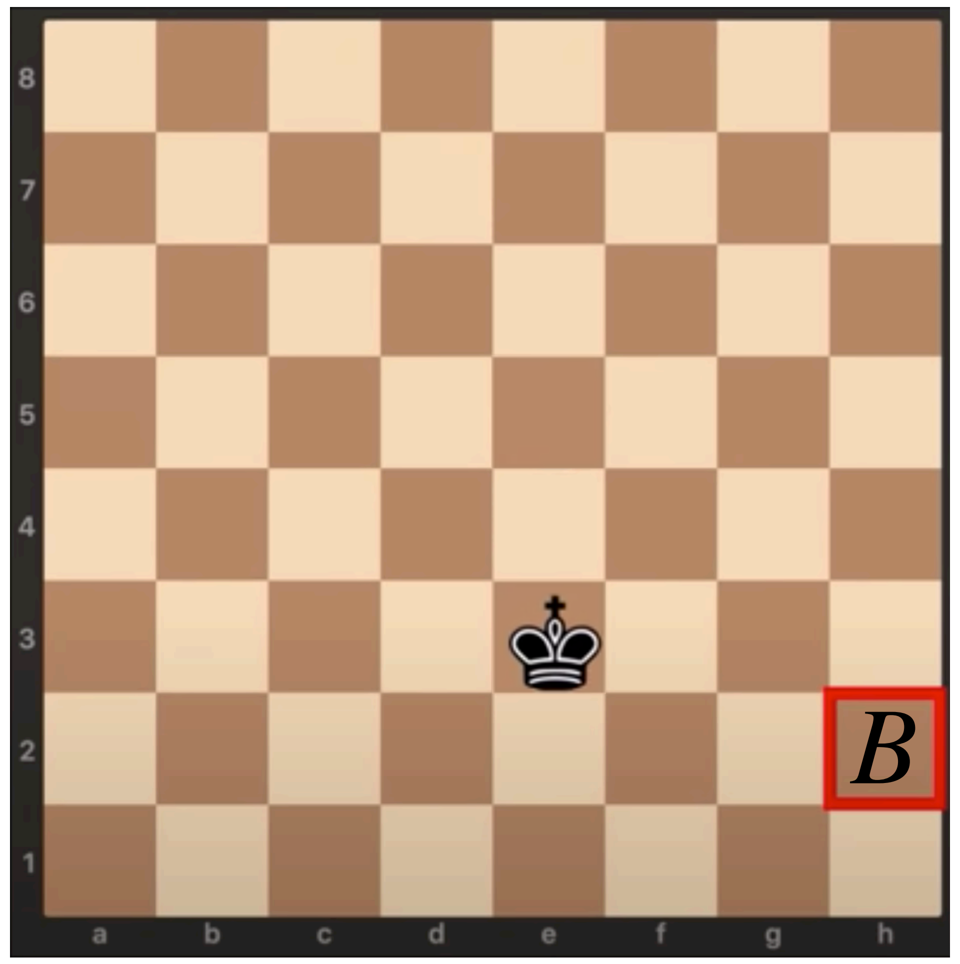


Manhattan distance

Chebyshev (or infinity) distance



Notion of distance applicable also between two functions, words, animals…

Metric spaces

Definition: A *metric space* is a pair (X, d) formed by a non-empty set, $X \neq \emptyset$, and application $d : X \times X \to \mathbb{R}$ (distance or metric) satisfying:

$$\begin{aligned} &i)\ d(x, y) \geq 0 \\ &ii)\ d(x, y) = 0 \Leftrightarrow x = y \\ &iii)\ d(x, y) = d(y, x) \\ &iv)\ d(x, z) \leq d(x, y) + d(y, z) \end{aligned} \qquad \forall\, x, y, z \in X$$

Properties:

$i)\ d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n)$

$ii)\ |\, d(x, z) - d(y, z)\,| \leq d(x, y)$

$iii)\ Y \subset X\,, d'(y_1, y_2) = d(y_1, y_2) \Rightarrow (Y, d')$ metric space with induced metric d'

Metric spaces

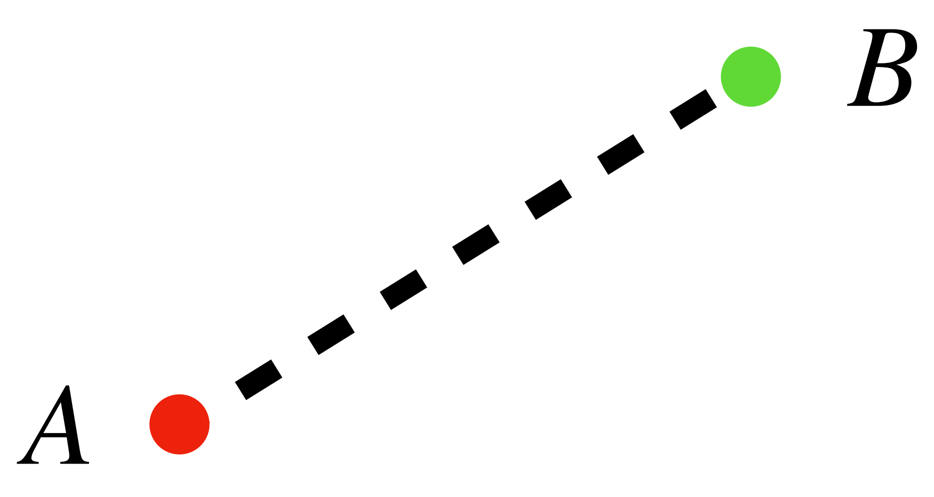
Example: d_p distance in $\mathbb{R}^2$

$$d_2(A,B) = \left(|x-x'|^2 + |y-y'|^2\right)^{1/2}$$

"Euclidean distance"

$A = (x, y)$

$B = (x', y')$



Metric spaces

Example: d_p distance in $\mathbb{R}^2$

$$d_{2}(A,B) = \left(|x - x'|^{2} + |y - y'|^{2}\right)^{1/2}$$

"Euclidean distance"

$$A = (x, y)$$

$$B = (x', y')$$

$$\downarrow$$

$$d_{p}(A,B) = \left(|x - x'|^{p} + |y - y'|^{p}\right)^{1/p}$$

$$p \in [1,\infty)$$

Metric spaces

Example: d_p distance in $\mathbb{R}^2$

$$d_2(A,B) = \left(|x-x'|^2 + |y-y'|^2 \right)^{1/2}$$

"Euclidean distance"

$A = (x, y)$

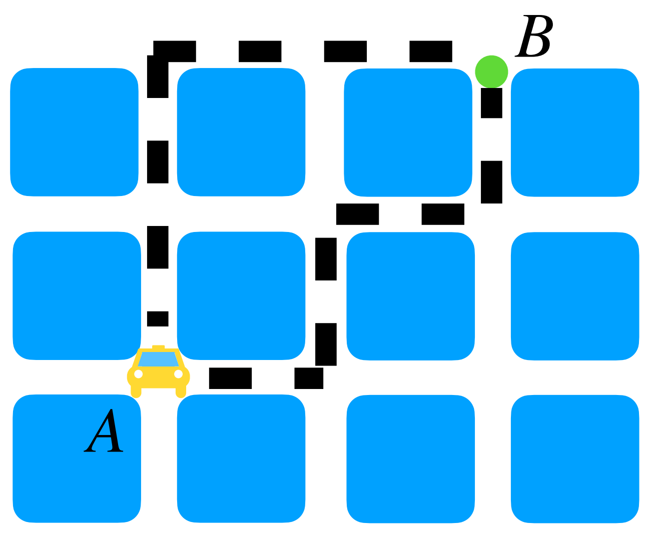
$B = (x', y')$

$$d_p(A,B) = \left(|x-x'|^p + |y-y'|^p \right)^{1/p}$$

$p \in [1, \infty)$

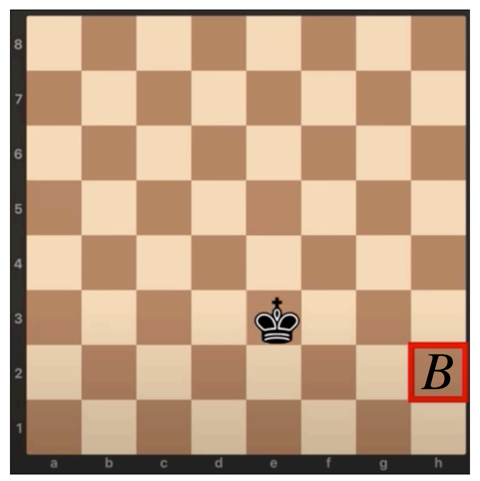
$$d_1(A,B) = |x-x'| + |y-y'|$$

Manhattan distance



$$d_\infty(A,B) = \max\{ |x-x'|, |y-y'| \}$$

Chebyshev (or infinity) distance



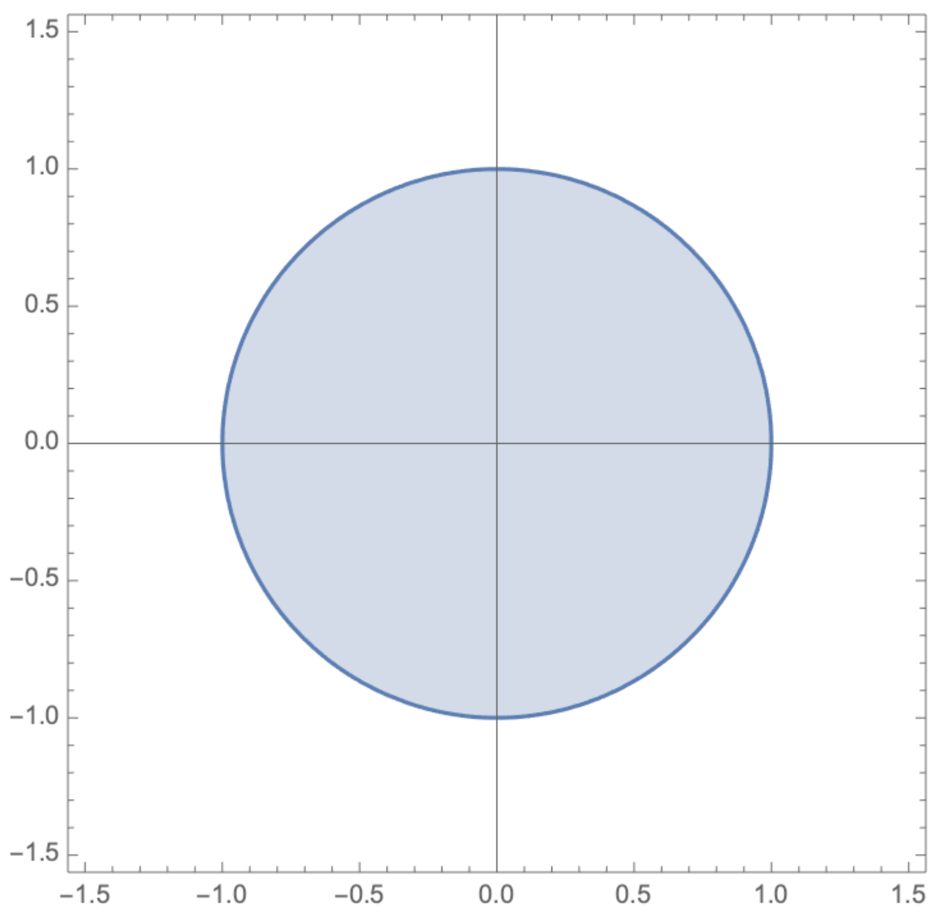
Metric spaces

Definition: For (X, d) metric space, we define the *open ball* of center $A \in X$ and radius r as:

$$B(A, r) = \{C \in X / d(A, C) < r\} \qquad \text{(}\textit{closed ball, } \bar{B}(A, r)\text{, for } d(A, C) \leq r\text{)}$$

Example: Open ball of center $A = (0,0)$ and radius $r = 1$ in d_p : $B(0,r) = \left\{(x, y) \in X / |x|^p + |y|^p < 1\right\}$

$p = 2$



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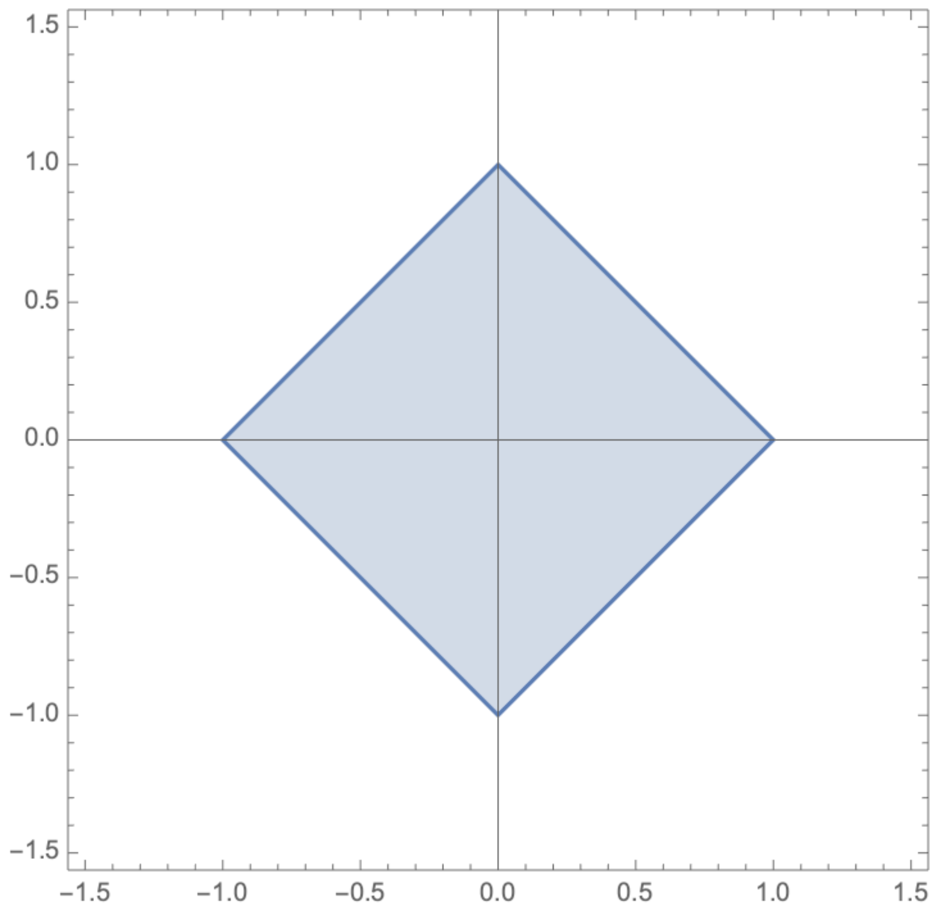
(*closed ball,* $\bar{B}(A, r)$, for $d(A, C) \leq r$)

Example: Open ball of center $A = (0,0)$ and radius $r = 1$ in d_p : $B(0,r) = \left\{(x, y) \in X \,/\, |x|^p + |y|^p < 1\right\}$

$p = 1$



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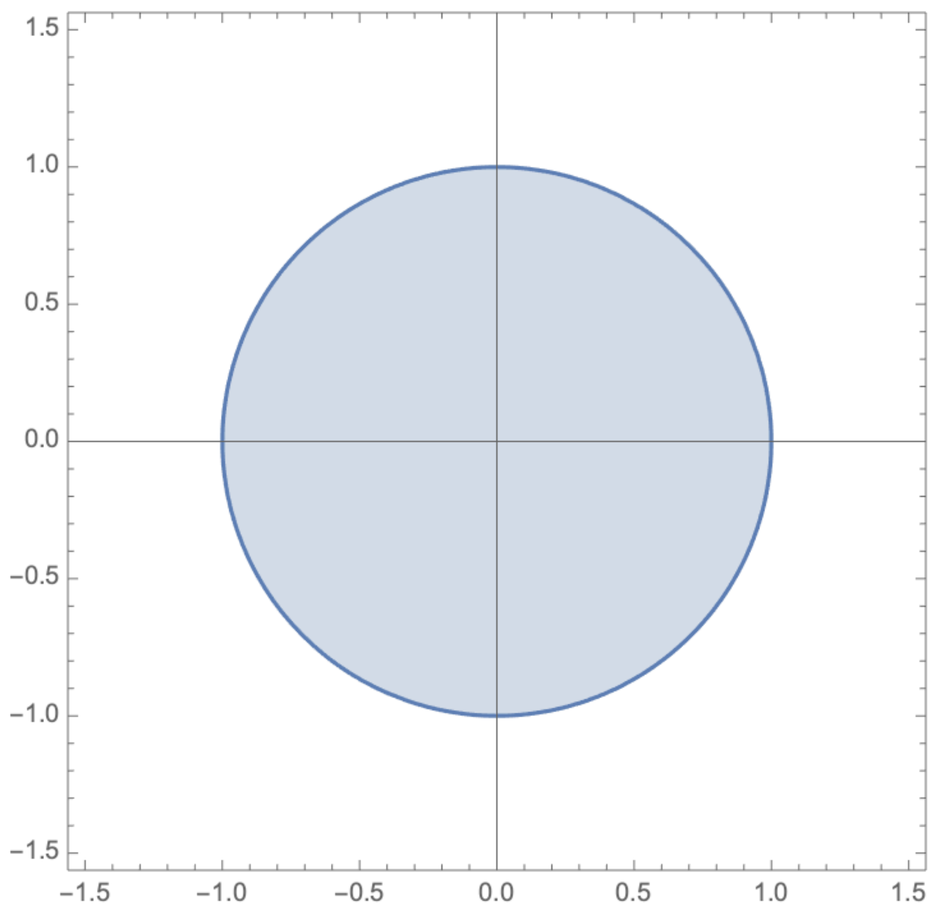

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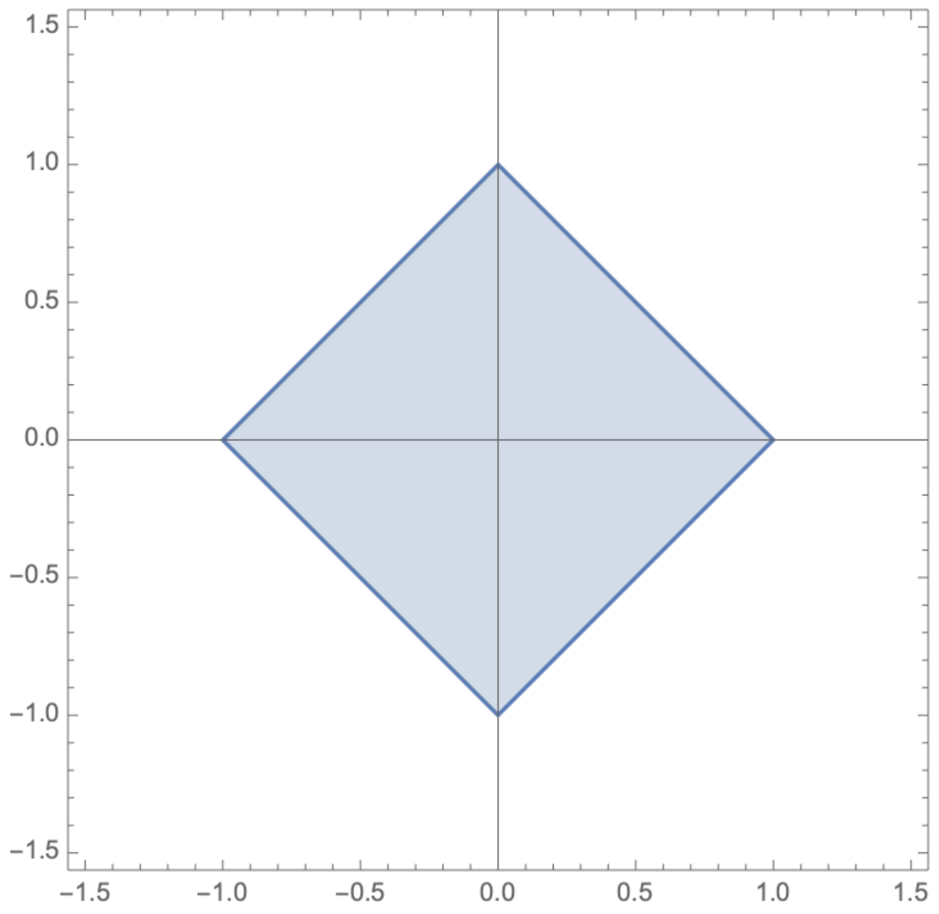
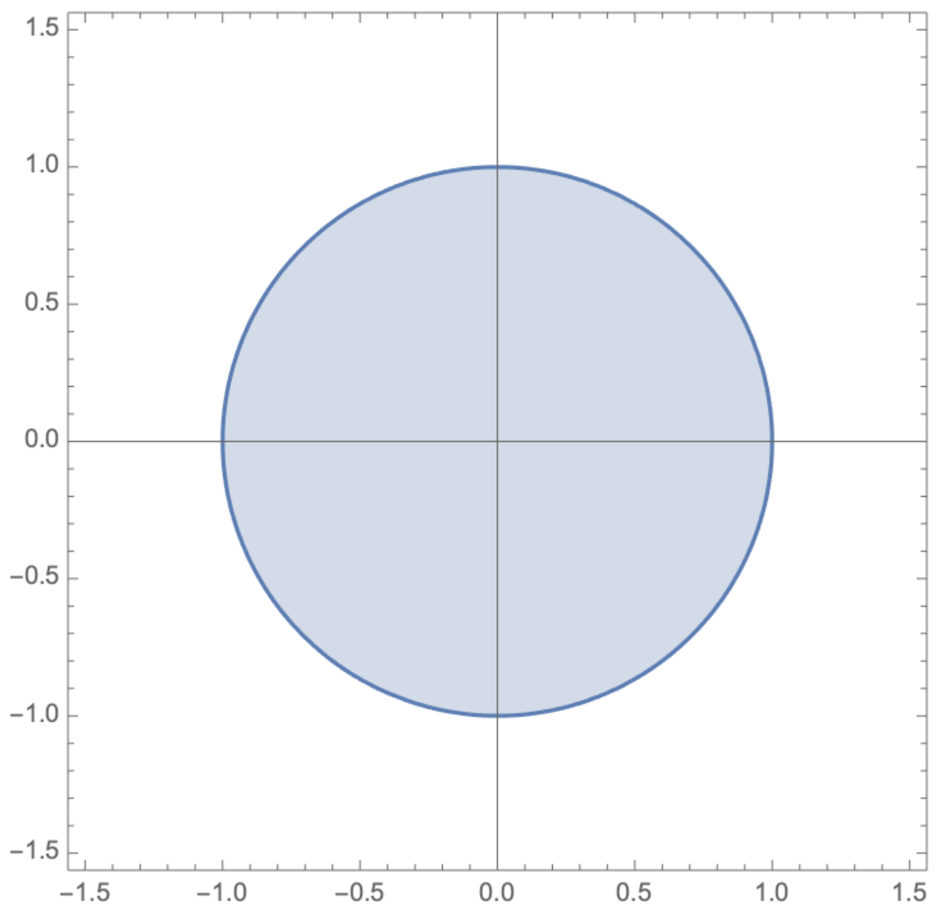
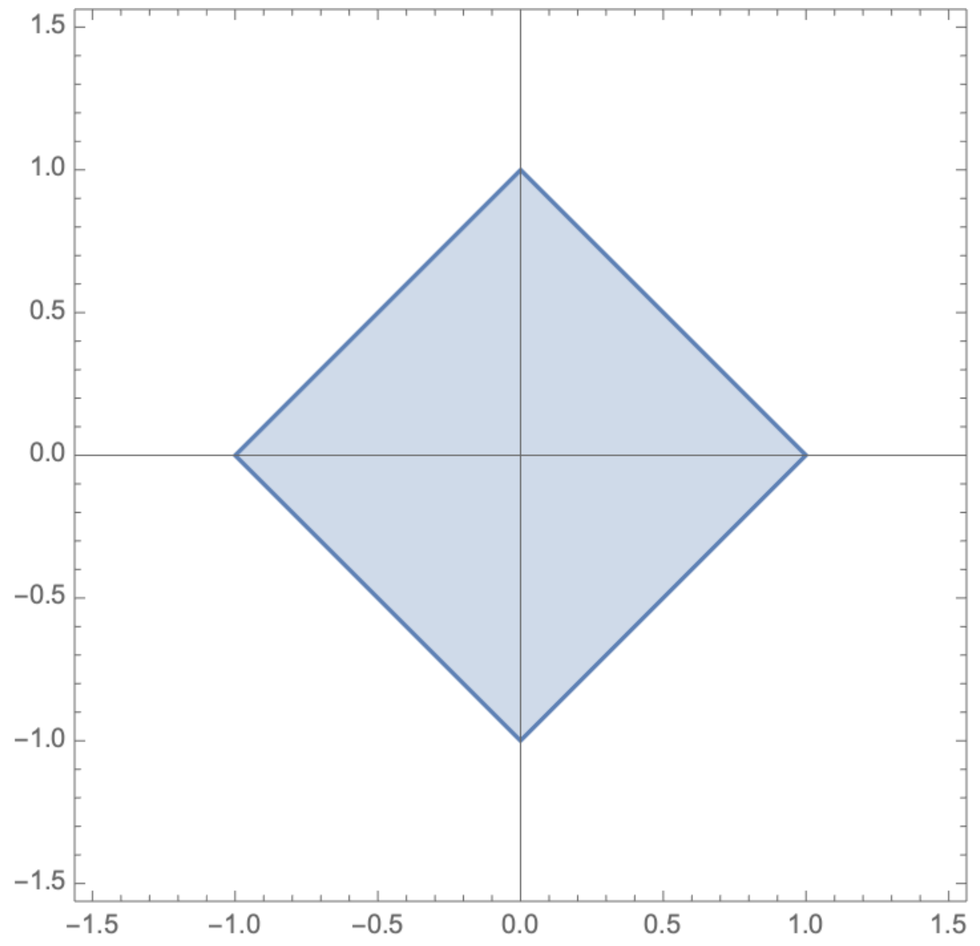


$p = 2$



$p \in [1,5]$

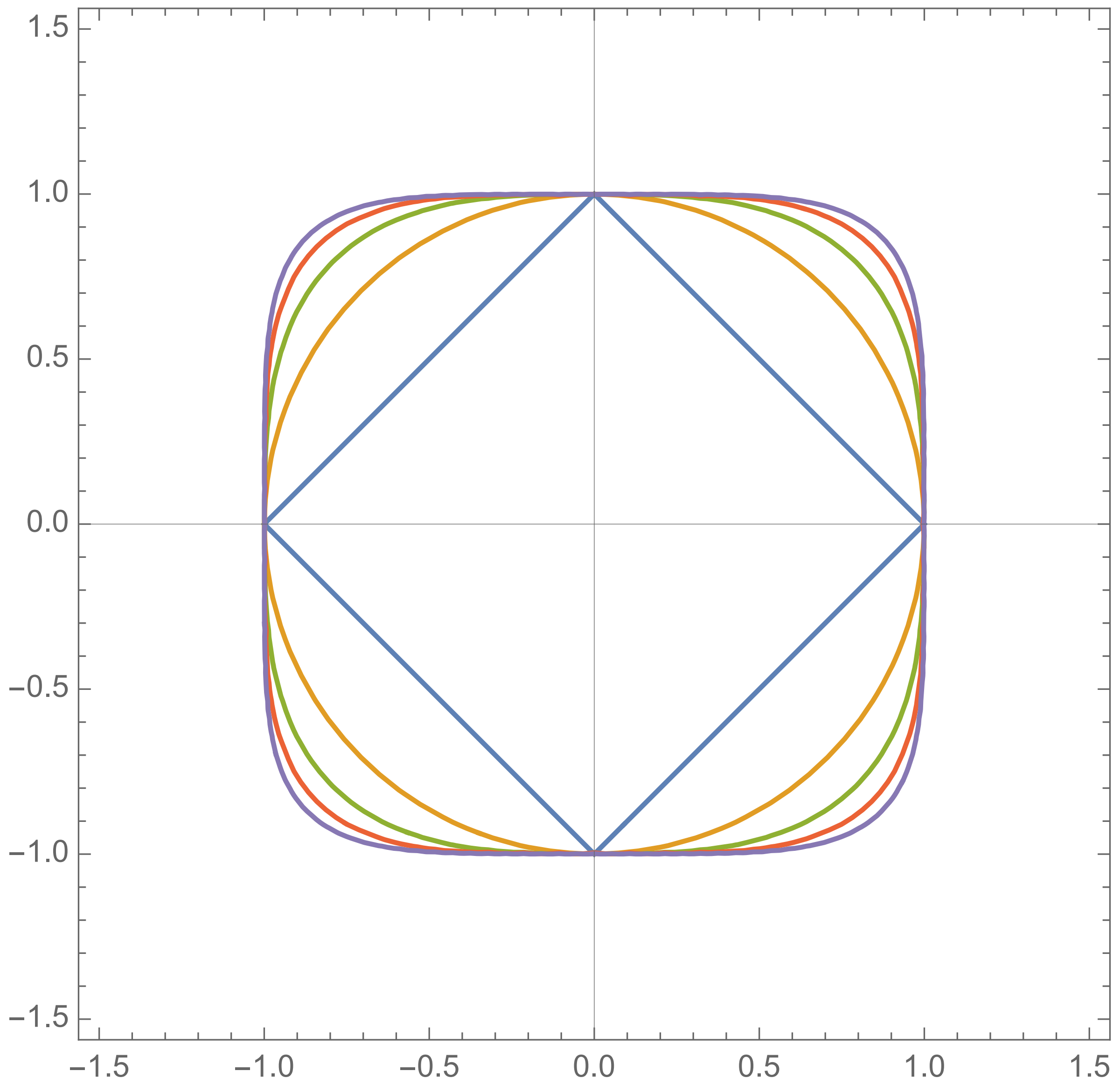


Metric spaces

Definition: For (X, d) metric space, we define the *open ball* of center $A \in X$ and radius r as:

$$B(A, r) = \{C \in X \,/\, d(A, C) < r\} \qquad \text{(}\textit{closed ball, } \bar{B}(A, r)\text{, for } d(A, C) \leq r\text{)}$$

Example: Open ball of center $A = (0,0)$ and radius $r = 1$ in d_p : $B(0,r) = \left\{(x, y) \in X \,/\, |x|^p + |y|^p < 1\right\}$



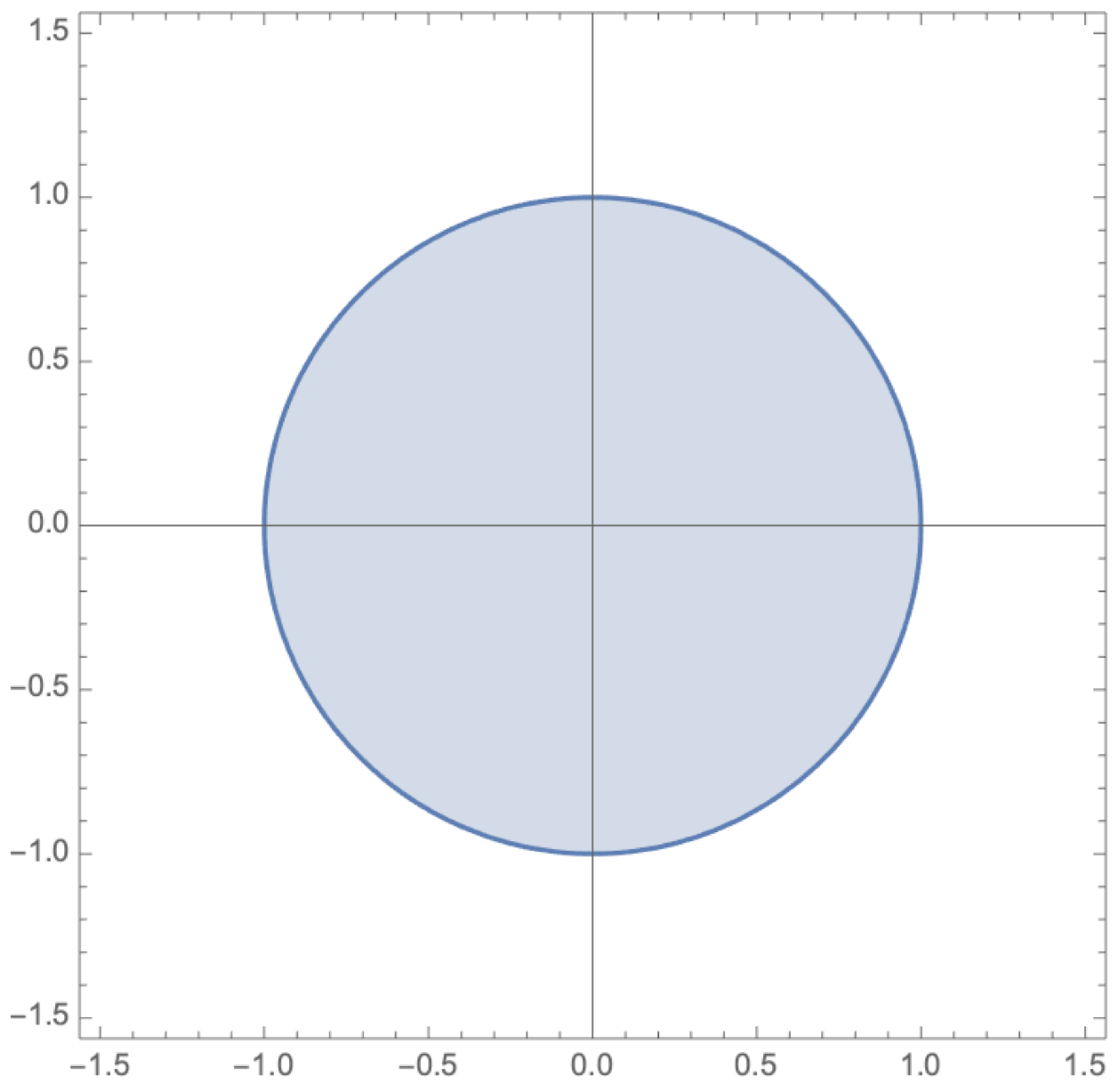
$$\left\{(x, y) \in X \,/\, |x|^p + |y|^p = 1\right\}$$

for $p = 1,2,3,4,5$

Metric spaces

Example: The different values of $\pi_p = \dfrac{\text{Length of the boundary of a ball}}{\text{Diameter of the ball}}$

$p = 2$

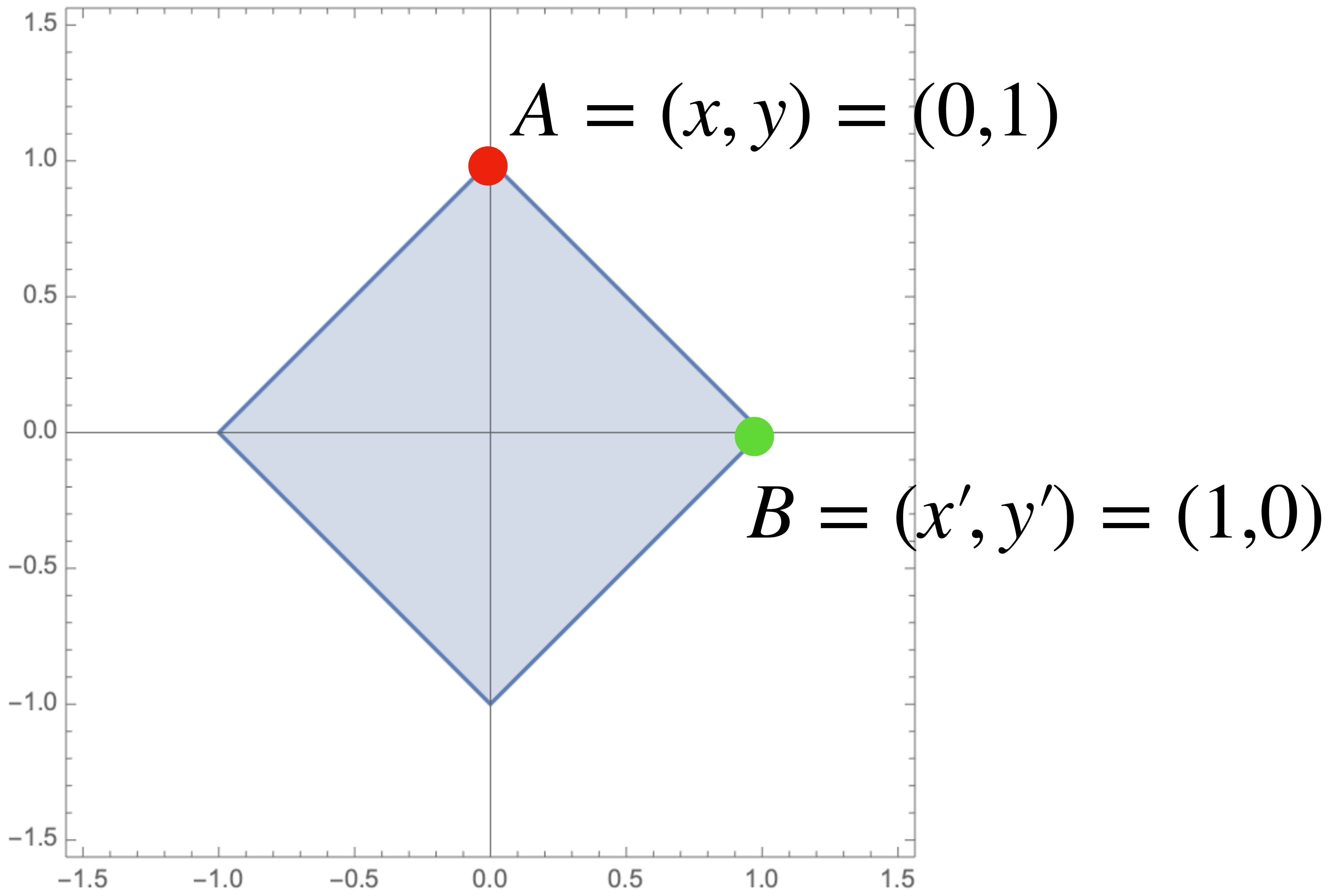


$$\pi_2 = \frac{2\pi r}{2r} = \pi = 3.1415\ldots$$

Metric spaces

Example: The different values of $\pi_p = \dfrac{\text{Length of the boundary of a ball}}{\text{Diameter of the ball}}$

$p = 1$

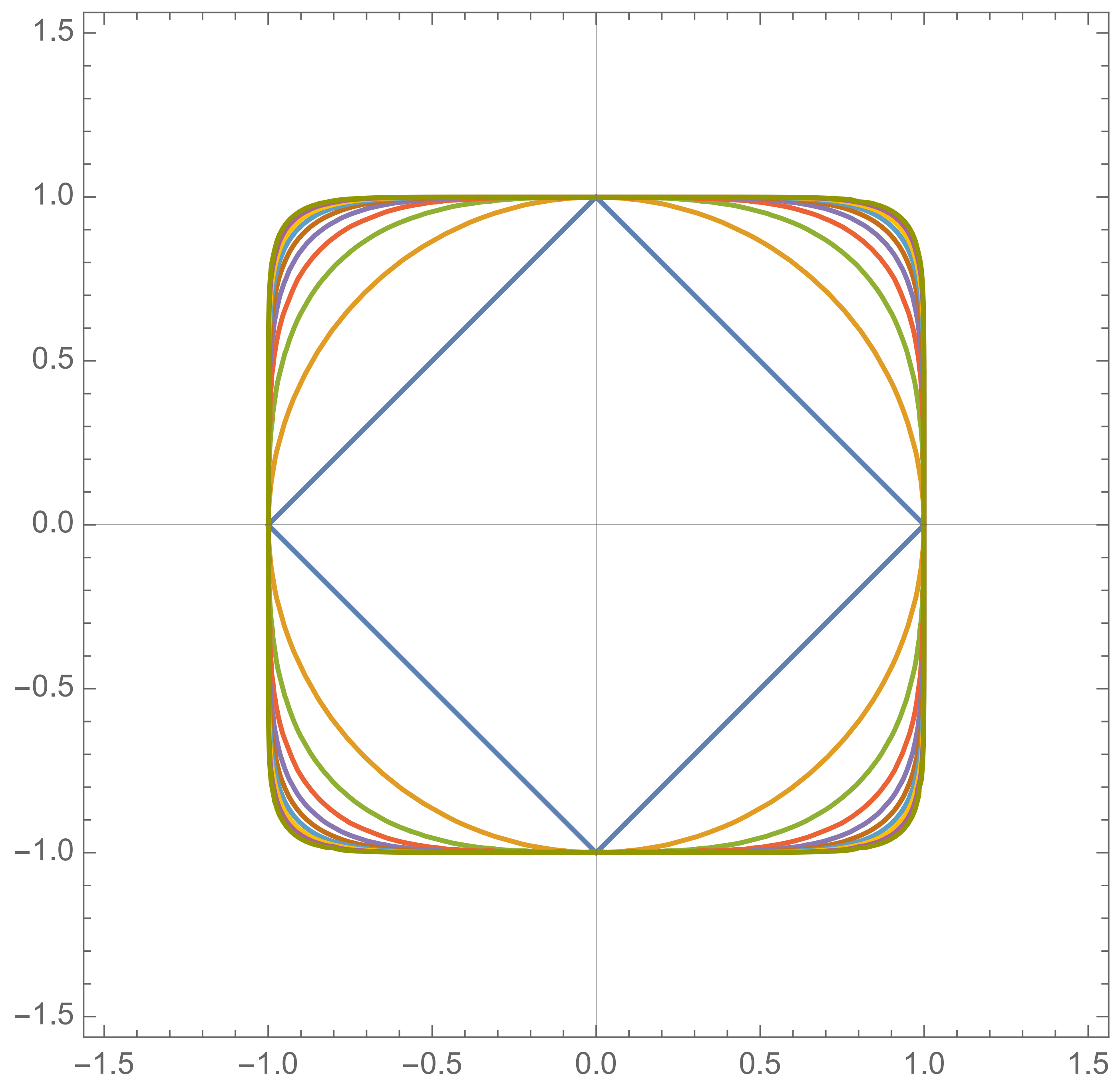


$$d(A, B) = |x - x'| + |y - y'| = 2$$

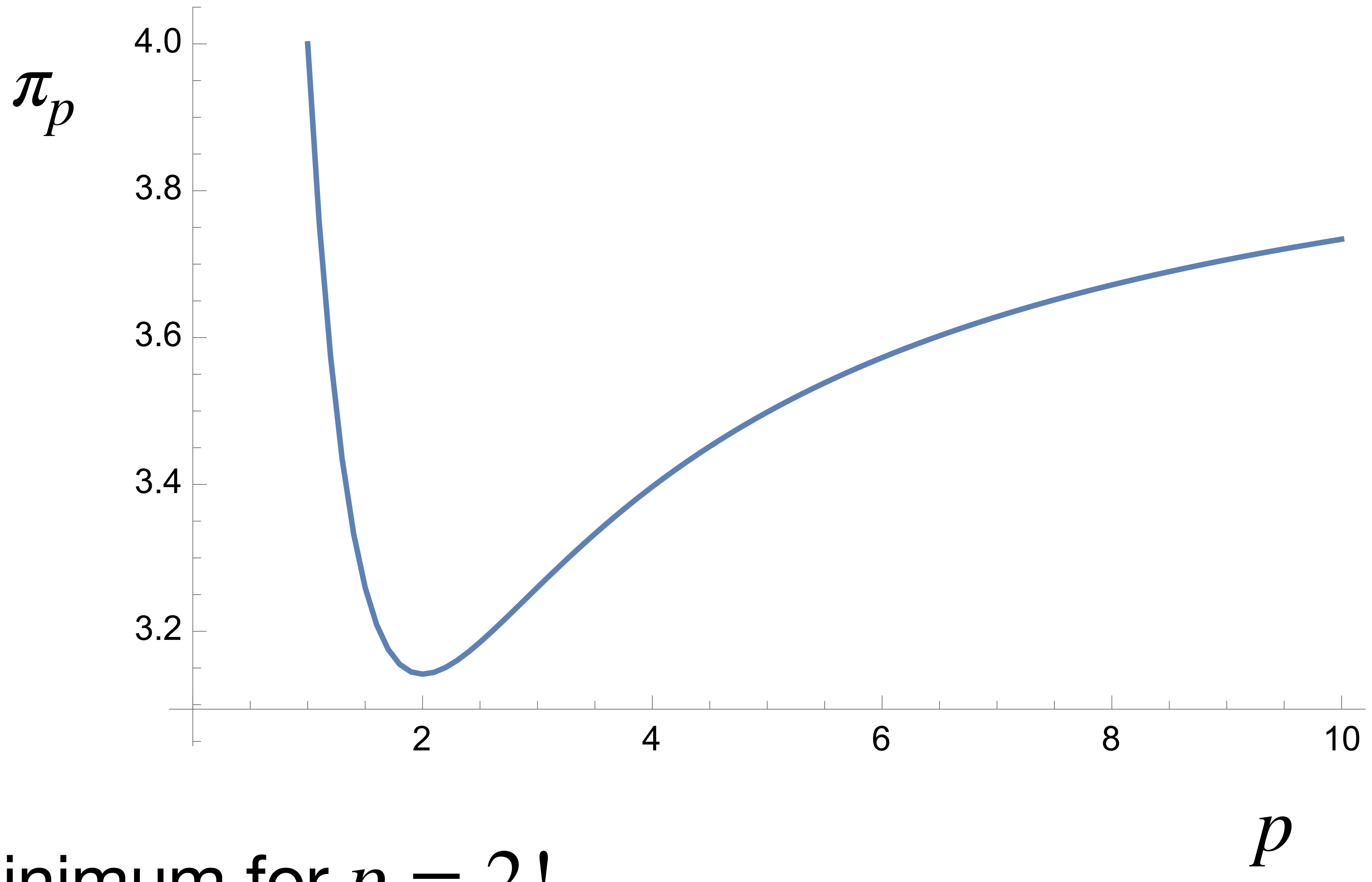
$$\implies \pi_1 = \frac{4 \cdot 2}{2} = 4$$

Metric spaces

Example: The different values of $\pi_p = \dfrac{\text{Length of the boundary of a ball}}{\text{Diameter of the ball}}$



$\Longrightarrow$



Minimum for $p = 2!$

Metric spaces

Definitions: Let (X, d) be a metric space

- Let $M \subset X$, $x \in M$ is an interior point if $\exists r > 0 \,/\, B(x, r) \subset M$
- Interior of M : $\operatorname{int} M = \{x \in X \,/\, x$ is interior point of $M\}$
- Open subspace: $M \subset X$ is open if $\operatorname{int} M = M$

- Given $M \subset X$, $x \in X$ is adherent point if $\forall r > 0$, $B(x, r) \cap M \neq \emptyset$
- Closure of M : $\bar{M} = \{x \in X \,/\, x$ is adherent point of $M\}$
- Closed subspace: $M \subset X$ is closed if $M = \bar{M}$

- $M \subset X$ is dense in X if $\bar{M} = X$

Metric spaces

Properties of open and closed sets: Let (X, d) be a metric space and $M \subset X$

$\emptyset\,, X$ are open and closed

M open $\Leftrightarrow M^c$ closed

$\cap_{i\in I} M_i$ closed if M_i are closed

$\cup_{i=1}^{n} M_i$ closed if M_i are closed

Any proposition above remains valid if we interchange *open* $\leftrightarrow$ *closed* and *union* $\leftrightarrow$ *intersection*

Metric spaces

Definition: Let X be a set and τ be a family of subsets of X. Then, τ is called a *topology* on X if:

$i)$ $\emptyset, X \in \tau$

$ii)$ For any $V_\alpha \in \tau$, $\cup_\alpha V_\alpha \in \tau$

$iii)$ For a finite family of $V_i \in \tau$, $\cap_{i=1}^{n} V_i \in \tau$

If τ is a topology on X, then the pair (X, τ) is called a topological space

Property: A metric space (X, d) is also a topological space (X, τ), and the topology defined by the metric is called induced topology

Metric spaces

Definition: Convergent sequence

$\{x_n\}_1^\infty \subset X$ converges to x in X, $x_n \to x$, if $\forall r > 0$, $\exists N / x_n \in B(x,r)$, $\forall n > N$

(equivalently, the real numbers sequence $\{d(x_n, x)\}$ converges to 0)

Definition: Cauchy sequence

$\{x_n\}_1^\infty \subset X$ is Cauchy if $\forall r > 0$, $\exists N / d(x_n, x_m) < r$, $\forall n, m > N$

Property: Every convergent sequence is also Cauchy $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) \to 0$

Definition: A metric space is complete if every Cauchy sequence is convergent

A subspace $S \subset X$ is complete if every Cauchy sequence in S converges in S

Property: Let $S \subset X$, $x \in X$ $\begin{cases} x \in \bar{S} \Leftrightarrow \exists \{x_n\}_1^\infty \subset S / x_n \to x \\ \text{Let } X \text{ be complete: } S \text{ complete} \Leftrightarrow S \text{ closed} \end{cases}$

Metric spaces

Summary of results:

- Metric (sub)spaces: (X, d)
- Open and closed balls. Induced topology
- Interior points, interior of a set. Open sets
- Adherent points, closure of a set. Closed sets. Dense subspaces
- Convergent sequences. Cauchy sequences
- In a complete metric space: Cauchy $\Rightarrow$ convergent

Other properties (applications, continuity, boundedness, …) could be introduced now but we will discuss them directly when we introduce Hilbert spaces

Normed spaces

Definition: A normed (vector) space is a pair $(X, \|\cdot\|)$ formed by a linear space X and an application $\|\cdot\| : X \to \mathbb{R}$ (norm) that satisfies:

$$i)\ \|x\| \geq 0$$

$$ii)\ \|x\| = 0 \Leftrightarrow x = 0$$

$$iii)\ \|\alpha x\| = |\alpha|\, \|x\|$$

$$iv)\ \|x + y\| \leq \|x\| + \|y\|$$

$\forall\, x, y \in X$

$\forall\, \alpha \in \Lambda$

Properties:

$i)$ Every $M < X$ with X being a normed space is normed subspace with the norm of X

$ii)$ Every normed space is a metric space with distance $d(x, y) = \|x - y\|$ satisfying:

$$d(x+z, y+z) = d(x, y)\,,\ d(\alpha x, \alpha y) = |\alpha|\, d(x, y)$$

$iii)$ Every linear metric space with these properties is also a normed space with $\|x\| = d(x, 0)$

Normed spaces

Properties: $(X, \|\cdot\|)$ normed space

$i)\ \Big| \|x\| - \|y\| \Big| \leq \|x - y\|\,, \forall\, x, y \in X$

$ii)\ B(x_0, r) = \{x_0\} + B(0,r)\,, \forall\, x_0 \in X\,,\ r > 0$

$iii)\ \overline{B}(x, r) = \overline{B(x, r)}\,, \forall\, x \in X\,,\ r > 0$

Definition: A Banach space is a normed space that is complete

Properties:

$i)\ X$ Banach $\Leftrightarrow \{a_n\}_1^\infty \in X\,,\ \sum_n \|a_n\| < \infty \Rightarrow \sum_n a_n$ convergent in X

$ii)$ Let X be Banach, a subspace Y is complete $\Leftrightarrow Y$ is closed in X

Completion theorem: Every normed space $L = (L, \|\cdot\|)$ admits a completion, $\tilde{L}$, which is unique except for norm isomorphisms, such that L is dense in $\tilde{L}$ and $\|x\|_{\tilde{L}} = \|x\|_L$

Normed spaces

Defintion:

Let $v_n \in X$, $v = \sum_{n=1}^{\infty} v_n$ if $\exists\, v \in X /$ $\left\| \sum_{n=1}^{k} v_n - v \right\| \xrightarrow{k \to \infty} 0$

Properties:

Hölder inequality (for sums)

$$i)\ \sum_{j=1}^{\infty} |a_j b_j| \leq \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |b_j|^q \right)^{1/q} \qquad p, q > 1\,,\ \frac{1}{p} + \frac{1}{q} = 1 \qquad \{a_j\}_1^{\infty} \in l_{\Lambda}^p \quad \{b_j\}_1^{\infty} \in l_{\Lambda}^q$$

Minkowski inequality (para sums)

$$ii)\ \left(\sum_{j=1}^{\infty} |a_j + b_j|^p \right)^{1/p} \leq \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{1/p} + \left(\sum_{j=1}^{\infty} |b_j|^p \right)^{1/p} \qquad p \geq 1 \qquad \{a_j\}_1^{\infty}, \{b_j\}_1^{\infty} \in l_{\Lambda}^p$$

Normed spaces

Summary of results:

- Normed (sub)space: $(X, \|\cdot\|)$
- Relation norm $\underset{\not\leftarrow}{\rightarrow}$ distance
- Banach space (complete normed space)
- Absolute convergence $\Rightarrow$ convergence in a Banach space
- A Banach subspace is Banach $\Leftrightarrow$ it is closed
- Completion theorem: every Banach space can be completed in a unique way
- Infinite sum converges in $(X, \|\cdot\|)$ to v if the sequence of partial sums converges to v
- Hölder and Minkowski inequalities

Hilbert spaces

Definition: A pre-Hilbert space is a pair $(X, \langle \cdot \rangle)$ formed by a linear space X and an application $\langle \cdot \rangle : X \times X \to \Lambda$ (scalar product) satisfying:

$$i)\ \langle v, v\rangle \geq 0\,,\ \langle v, v\rangle = 0 \Leftrightarrow v = \mathbf{0}$$

$$ii)\ \langle v, v_1 + v_2\rangle = \langle v, v_1\rangle + \langle v, v_2\rangle \qquad \forall\, v, v_1, v_2 \in X$$

$$iii)\ \langle v, \lambda w\rangle = \lambda\langle v, w\rangle \qquad \forall\, \lambda \in \Lambda$$

$$iv)\ \langle v, w\rangle = \overline{\langle w, v\rangle}$$

Properties:

$$i)\ \langle \lambda_1 v_1 + \lambda_2 v_2, v\rangle = \overline{\lambda_1}\langle v_1, v\rangle + \overline{\lambda_2}\langle v_2, v\rangle \ \ \forall\, \lambda_{1,2} \in \Lambda\,,\ v_{1,2} \in X$$

$$ii)\ \langle v, w\rangle = 0 \ \ \forall w \in X \Rightarrow v = \mathbf{0}$$

$$iii)\ \langle v_1, w\rangle = \langle v_2, w\rangle \ \ \forall w \in X \Rightarrow v_1 = v_2$$

Hilbert spaces

Property: Every pre-Hilbert space is a normed space with norm $\|v\| = +\sqrt{\langle v, v\rangle}$

Defintion: A Hilbert space is a pre-Hilbert space that is complete with the norm associated to the scalar product (more precisely, the distance asociated to this norm)

Properties: Let $(X, \langle \cdot \rangle)$ be a pre-Hilbert space and $\|\cdot\|$ the associated norm:

$i)$ $\|v+w\|^2 + \|v-w\|^2 = 2(\|v\|^2 + \|w\|^2)$ [Parallelogram identity]

$ii)$ $\mathrm{Re}\big[\langle v, w\rangle\big] = \frac{1}{4}\big[\|v+w\|^2 - \|v-w\|^2\big]$

$iii)$ $\mathrm{Im}\big[\langle v, w\rangle\big] = -\frac{1}{4}\big[\|v+iw\|^2 - \|v-iw\|^2\big]$ (si $\Lambda = \mathbb{C}$)

[Polarization identity]

Scalar product-norm relation: A normed space $(X, \|\cdot\|)$ that fulfills the parallelogram identity is a pre-Hilbert space with the scalar product defined by the polarization identity

Hilbert spaces

Properties: Let $(X, \langle \cdot \rangle)$ be a pre-Hilbert space and $\|\cdot\|$ the associated norm:

$i)$ Schwarz-Cauchy-Buniakowski inequality

$$|\langle v, w\rangle| \leq \|v\| \, \|w\| \quad \forall v, w \in X, \quad \big(\text{"="} \Leftrightarrow v, w \text{ lin. dep.}\big)$$

$ii)$ Triangular inequality

$$\|v + w\| \leq \|v\| + \|w\| \quad \forall v, w \in X, \quad \big(\text{"="} \Leftrightarrow w = \mathbf{0} \text{ ó } v = \lambda\, w, \lambda \geq 0\big)$$

$iii)$ Continuity of the scalar product

$$v_n \to v\,,\, w_n \to w \Rightarrow \langle v_n\,, w_n\rangle \to \langle v, w\rangle$$

$$\{v_n\}_1^\infty\,,\,\{w_n\}_1^\infty \text{ Cauchy in } X \Rightarrow \big\{\langle v_n\,, w_n\rangle\big\}_1^\infty \text{ Cauchy in } \Lambda$$

Hilbert spaces

Properties: Let $(X, \langle \cdot \rangle)$ be a pre-Hilbert space and $\|\cdot\|$ the associated norm:

$i)$ $v, w \in X$ are orthogonal if $\langle v, w \rangle = 0$ (we denote $v \perp w$)

$ii)$ $S = \{v_\alpha\}_{\alpha \in A} \subset X$ is an orthogonal set if $\langle v_\alpha, v_\beta \rangle = 0 \ \ \forall \alpha \neq \beta$

$iii)$ $S = \{v_\alpha\}_{\alpha \in A} \subset X$ is an orthonormal set if $\langle v_\alpha, v_\beta \rangle = \delta_{\alpha\beta}$

$iv)$ Every orthogonal set of non-zero vectors is l.i.

Generalized Pythagoras theorem: Let $\{v_i\}_1^n$ be orthonormal in X

$$\|v\|^2 = \sum_{i=1}^n |\langle v_i, v \rangle|^2 + \left\| v - \sum_{i=1}^n \langle v_i, v \rangle v_i \right\|^2, \quad \forall v \in X$$

Pythagoras theorem:

$$\left\| \sum_{i=1}^n v_i \right\|^2 = \sum_{i=1}^n \|v_i\|^2, \text{ si } \langle v_i, v_j \rangle = 0 \ \ (i \neq j)$$

Hilbert spaces

Properties:

$i)$ Bessel inequality: let $\{v_\alpha\}_{\alpha\in A}$ be an arbitrary orthonormal set

$$\|v\|^2 \geq \sum_{\alpha\in A} |\langle v_i , v\rangle|^2, \quad \forall v \in X$$

$ii)$ Let $\{v_\alpha\}_{\alpha\in A}$ be an arbitrary orthonormal set

$A^{(v)} \equiv \{\alpha \in A/\langle v_\alpha , v\rangle \neq 0\}$ is finite or countable infinite

Completion theorem:

Given a pre-Hilbert space $(X, \langle \cdot , \cdot \rangle)$, there exists a Hilbert space H (unique except for isomorphisms) and an isomorphism $A : X \to W$ with W dense in H

Hilbert spaces

Definition (orthogonal complement): Let H Hilbert and $M \subset H$, $M \neq \emptyset$

$$M^{\perp} \equiv \{v \in H \,/\, v \perp w \quad \forall\, w \in M\} \quad \text{(also denoted as } M^{\perp} = H \ominus M)$$

Propiedades:

$i)$ $M^{\perp}$ is a closed linear subspace $\forall\, M \subset H$, H Hilbert

$ii)$ $M \cap M^{\perp} = \{\mathbf{0}\}$ or $M \cap M^{\perp} = \emptyset$

$iii)$ $M^{\perp\perp} \equiv (M^{\perp})^{\perp} \supset M$

$iv)$ $M^{\perp} = (\overline{M})^{\perp} = [M]^{\perp} = (\overline{[M]})^{\perp}$

$v)$ $\{\mathbf{0}\}^{\perp} = H$, $\;H^{\perp} = \{\mathbf{0}\}$

Hilbert spaces

Orthogonal projetion theorem:

Let M be a closed linear subspace of the Hilbert space H, i.e $M \lhd H$, then
$\forall\, v \in H \;\; \exists!\, v_1 \in M\,, \exists!\, v_2 \in M^{\perp} / \, v = v_1 + v_2$ (v_1: orthogonal projection of v in M)

equivalent statement:

Let $M \lhd H$ with H Hilbert, then
$$\forall\, v \in H \;\; \exists!\, v_1 \in M \,/\, \|v - v_1\| = \inf_{y \in M} \|v - y\|\,, v - v_1 \in M^{\perp}$$

Hilbert spaces

Definition (orthogonal direct sum): Let $M, N \lhd H$ with H Hilbert

$$H = M \oplus N \text{ if } H = M \overrightarrow{\oplus} N \text{ y } M \perp N$$

Properties:

$i)\ H = M \oplus M^{\perp}, \forall M < H, M = \overline{M}$

$ii)\ S^{\perp\perp} = \overline{[S]} \quad \forall S \subset H,\ S \neq \emptyset$ (S closed subspace $\Rightarrow S^{\perp\perp} = S$)

$iii)\ S < H$ is dense in $H \Leftrightarrow S^{\perp} = \{\mathbf{0}\}$

Definition: Orthogonal projector onto M, $P_M : H \to M\,, \quad P_{M^{\perp}} : H \to M^{\perp}$

$$P_M v = v_1\,,\ v = v_1 + v_2 \text{ with } v_1 \in M, v_2 \in M^{\perp}$$

$$P_M + P_{M^{\perp}} = 1_H\,, \quad P_M P_{M^{\perp}} = P_{M^{\perp}} P_M = 0\,, \quad P_M^2 = P_M\,, \quad P_{M^{\perp}}^2 = P_{M^{\perp}}$$

Hilbert spaces

Theorem: Let $\{x_n\}_1^\infty$ be an orthonormal set in H (Hilbert) and $\{\lambda_n\}_1^\infty \subset \Lambda$, then:

$$\sum_{n=1}^{\infty} \lambda_n x_n \text{ converges } \Leftrightarrow \sum_{n=1}^{\infty} |\lambda_n|^2 \text{ converges}$$

Theorem: Let $S = \{x_\alpha\}_{\alpha\in A}$ be an orthonormal set in H (Hilbert) and $M = \overline{[S]}$

$i)\ x_M \equiv \sum_{\alpha\in A} \langle x_\alpha, x\rangle\, x_\alpha \in M$

$ii)\ x_M$ is the only vector that satisfices $x - x_M \perp M$

$iii)\ x \in M \Rightarrow x = x_M$

$iv)\ d(x, M) \equiv \inf_{y\in M} \|x - y\| = d(x, x_M)$

$$\boxed{\text{The optimal approximation of a vector } x \text{ by elements of } M = \overline{[\{x_\alpha\}_{\alpha\in A}]} \text{ is given by } P_M x = x_M}$$

Hilbert spaces

Gram-Schmidt orthogonalization theorem:

Let $\{v_j\}_{j\in J} \subset H$ be a l.i. set, with J finite or countable infinite, $\exists\,\{u_j\}_{j\in J}$ orthonormal such that:

$$i)\; u_i \in [\{v_j\}_{j\in J}]\,,\; v_i \in [\{u_j\}_{j\in J}] \qquad\qquad ii)\; \overline{[\{u_j\}_{j\in J}]} = \overline{[\{v_j\}_{j\in J}]}$$

Solution: $u_m \equiv \dfrac{w_m}{\|w_m\|}$, with $w_m \equiv v_m - \sum_{k=1}^{m-1} \langle u_k, v_m \rangle\, u_k$

Definition (orthonormal basis): Orthonormal set $\{v_\alpha\}_{\alpha\in A} \subset H$ that is maximal

Theorem: Every Hilbert space $\neq \{\mathbf{0}\}$ has an orthonormal basis

Hilbert spaces

Theorem (charaterization of orthonormal bases): Let $S = \{v_\alpha\}_{\alpha \in A} \subset H \neq \{\mathbf{0}\}$ be an orthonormal set. The following statements are equivalent:

$i)$ S is an orthonormal basis of H

$ii)$ $\overline{[S]} = H$

$iii)$ $v \perp v_\alpha \, , \forall\, \alpha \in A \Rightarrow v = \mathbf{0}$. That is, $S^\perp = \{\mathbf{0}\}$

$iv)$ $\forall v \in H \Rightarrow v = \sum_\alpha \langle v_\alpha \, , v \rangle \, v_\alpha$ (Fourier series expansion)

$v)$ $\forall v, w \in H \Rightarrow \langle v \, , w \rangle = \sum_\alpha \langle v \, , v_\alpha \rangle \, \langle v_\alpha \, , w \rangle$ (Parseval identity)

$vi)$ $\forall v \in H \Rightarrow \|v\|^2 = \sum_\alpha | \, \langle v_\alpha \, , v \rangle \, |^2$ (Parseval identity)

Hilbert spaces

Definition: Separable topological (and metric) spaces

- A topological space X is said to be *separable* when it contains a countable subset that is dense in X
- A metric space M is separable iff it has a numerable basis of open sets

Separability criterium in Hilbert spaces

A Hilbert space $H \neq \{\mathbf{0}\}$ is separable	$\Longleftrightarrow$	It admits a countable orthonormal basis (finite o countable infinite)

Proposition: Every orthonormal basis in a Hilbert space H has the same cardinality (Hilbert dimension of H)

Hilbert spaces

Definition: Two Hilbert spaces, H_1 y H_2, over Λ are said to be isomorphic iff

$\exists\, U : H_1 \to H_2\,, U$ linear isomorphism $/ \langle Ux\,, Uy\rangle_{H_2} = \langle x, y\rangle_{H_1}\,, \forall\, x\,, y \in H_1$

Isomorphism theorem:

Every Hilbert space $H \neq \{\mathbf{0}\}$ is isomorphic to $l^2_\Lambda(A)$ with $\operatorname{card} A =$ Hilbert dimension of H

Corollaries:

- Every Hilbert space of Hilbert dimension n (finite) is isomorphic to Λ^n
- Every separable Hilbert space of infinite Hilbert dimension is isomorphic to $l^2_\Lambda(\mathbb{N})$
- Let H be a separable Hilbert space of Hilbert dimension h and linear dimension l
  - $h < \infty \Rightarrow l = h$ and every orthogonal basis is a Hamel basis
  - $h = \infty \Rightarrow l > h$ and no orthogonal basis is a Hamel basis

Hilbert spaces

Summary of results:

- (Pre-)Hilbert space: Linear space with a scalar product and complete
- Hilbert $\underset{\not\longleftarrow}{\longrightarrow}$ Normed
- Parallelogram and polarization identities
- Schwarz and triangular inequalities. Continuity of the scalar product
- Orthonormality. Pythagoras theorem and Bessel inequality
- Completion theorem
- Orthogonal complement and projectors. Optimal approximation to a vector
- Gram-Schmidt orthonormalization
- Orthonormal bases. Separable spaces
- Isomorphism theorem

Function spaces

Examples of function spaces:

$i)\ (C_\Lambda[a,b], \|\cdot\|_\infty)$ complete, not Hilbert

$ii)\ (C_\Lambda[a,b], \|\cdot\|_p)\,, p \geq 1$ not complete ($p = 2$ pre-Hilbert)

$iii)\ (B(\mathbb{R})\,, \|\cdot\|_\infty)$ complete, not Hilbert

$iv)\ (R^p(\mathbb{R})\,, \|\cdot\|_p)\,, p \geq 1$ not complete ($p = 2$ pre-Hilbert)

Example of incompleteness of $(\boldsymbol{C_\Lambda[a,b]}, \|\cdot\|_2)$

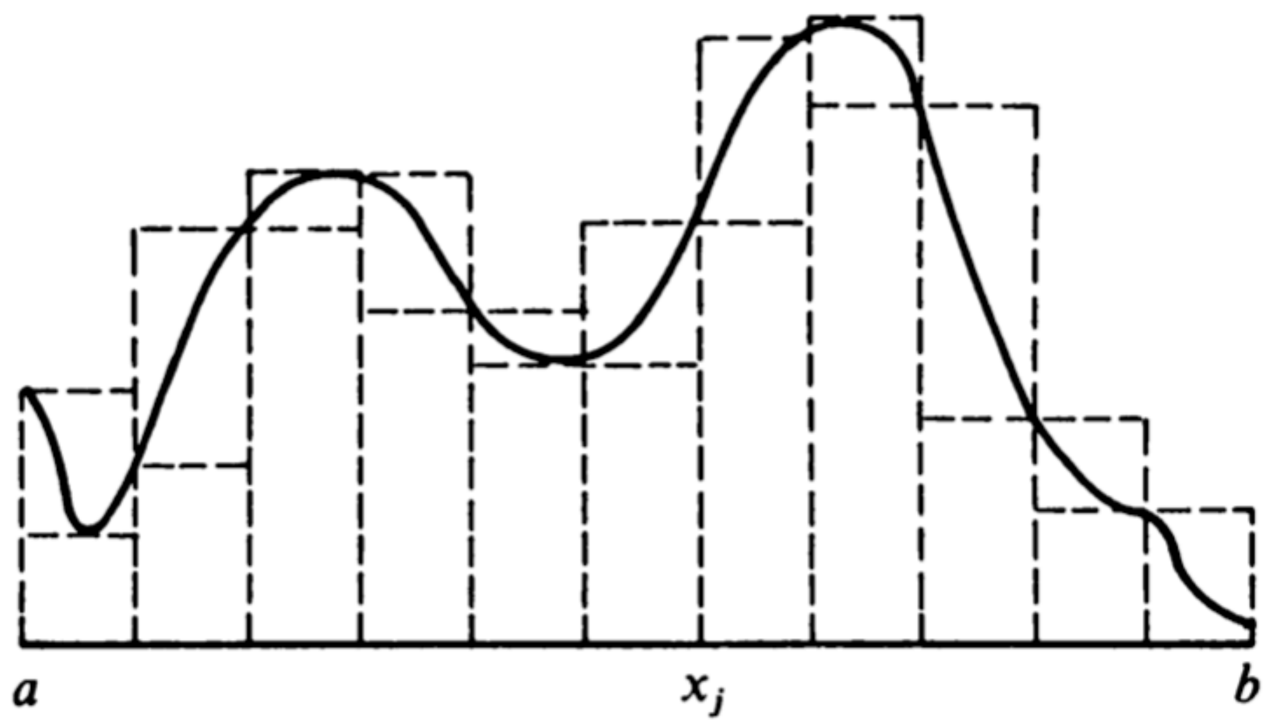
$$f_n(x) = \begin{cases} 0\,, & x \leq \frac{1}{2} - \frac{1}{n}\,, \\ nx - \frac{n}{2} + 1\,, & \frac{1}{2} - \frac{1}{n} < x \leq \frac{1}{2}\,, \\ 1\,, & \frac{1}{2} \leq x\,, \end{cases}$$

Is Cauchy but not convergent in $(C_\mathbb{R}[0,1]\,, \|\cdot\|_2)$

We can complete it by adding to the space the limit of every Cauchy sequence but we will need to extend the notion of integral for that

Function spaces

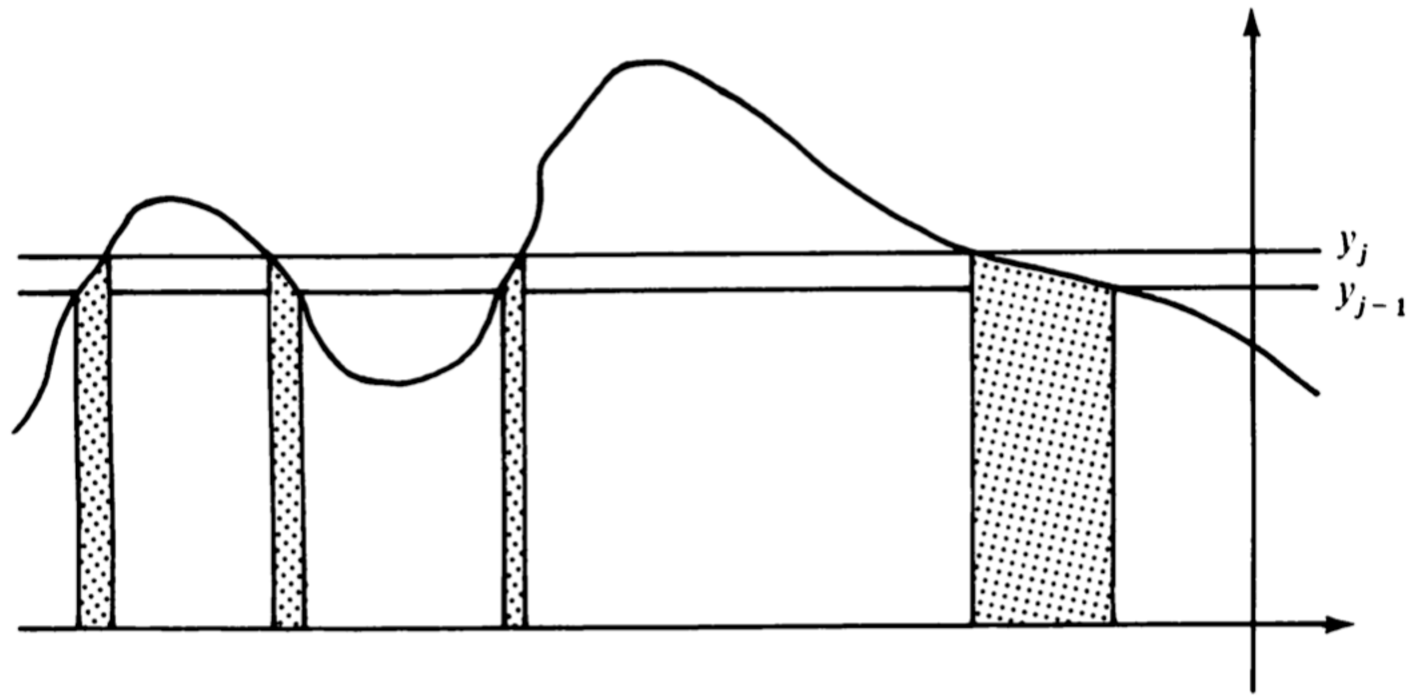
Riemann integral: Partition of “x axis” and equal convergence upper and lower integrals



$$\int_a^b f(x)\,dx = I$$

$$\text{if } I = \lim_{|\pi|\to 0} \sum_{k=1}^{n} R_k^{\inf} = \lim_{|\pi|\to 0} \sum_{k=1}^{n} R_k^{\sup} < \infty \qquad \pi \equiv \sup_j |x_j - x_{j-1}|$$

Lebesgue integral: Partition of “y axis” and measure of subsets in the “x axis”



$$\int_{\mathbb{R}} f(x)\,dx = \lim_{|\pi|\to 0} \Sigma_\pi(f) \qquad \pi \equiv \sup_j |y_j - y_{j-1}|$$

$$\Sigma_\pi(f) \equiv \sum_{j=1}^{n} y_{j-1}\,\mu\{f^{-1}([y_{j-1}, y_j])\}$$

Function spaces

We need to develop a new concept of *measure*

Borelian ($\boldsymbol{B}$): Element of $\mathscr{B}$, minimal family of subsets of $\mathbb{R}$ that contains every open interval (a, b) and satisfies

$$i)\ \{B_j\}_1^\infty \subset \mathscr{B} \Rightarrow \cup_{j=1}^\infty B_j \in \mathscr{B} \qquad\qquad ii)\ B \in \mathscr{B} \Rightarrow \mathbb{R} - B \in \mathscr{B}$$

Borel-Lebesgue measure (of a borelian $\boldsymbol{B \in \mathscr{B}}$): $\mu(B) \equiv \inf\limits_{I \supset B} l(I) \qquad l(I) \equiv \sum\limits_{j=1}^\infty |b_j - a_j|$

[I open interval or union of disjoint open intervals, i.e. $I = \cup_{j=1}^\infty (a_j, b_j)$]

Properties:

$$i)\ B \in \mathscr{B} \Rightarrow \mu(B) = \inf\{\mu(A), A\,\text{open} \supset B\} = \sup\{\mu(C), C\,\text{compact} \subset B\}$$

$ii)\ B_n \in \mathscr{B}, n \geq 1$ disjoint to each other $\Rightarrow \mu(\cup_1^\infty B_n) = \sum_1^\infty \mu(B_n)$

Function spaces

Borel measurable function: $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable iff $f^{-1}(B) \in \mathscr{B}$, $\forall\, B \in \mathscr{B}$

- A complex function is Borel measurable iff its real and imaginary parts are Borel
- Given f, g real: $f + g$, $\lambda f\,(\lambda \in \mathbb{R})$, fg , $|f|$, $f \circ g$ are Borel
- Characterization of Borel measurable functions

$i)\, f : \mathbb{R} \to \mathbb{R}$ is Borel $\Leftrightarrow f^{-1}\{(a,b)\} \in \mathscr{B}\ \forall a, b$

$ii)\, f_n(x) \to f(x)$, $\forall x$, f_n Borel $\Rightarrow f$ Borel

$iii)\, f : \mathbb{R} \to \mathbb{R}$ is Borel $\Leftrightarrow \{x\,/\,f(x) < b\} \in \mathscr{B}$, $\forall\, b$

Lebesgue integral: Let $f \geq 0$, bounded and Borel measurable. We define its integral as

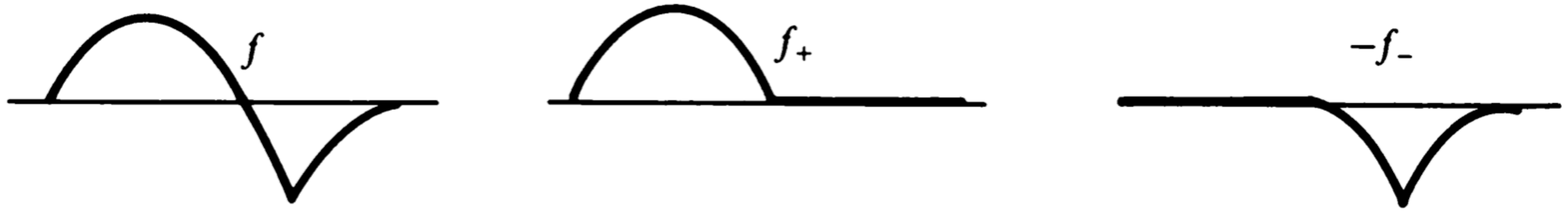
$$\int_{\mathbb{R}} f(x)\, dx = \lim_{|\pi| \to 0} \Sigma_\pi(f)$$

$\pi : 0 = y_0 < y_1 < \ldots < y_n = \sup f$ (partition of the image of f)

$$\Sigma_\pi(f) \equiv \sum_{j=1}^{n} y_{j-1}\, \mu\{f^{-1}([y_{j-1}, y_j])\}$$

Function spaces

Let f be real (not necessarily ≥ 0) measurable Borel



$$f_+ \equiv \max\{f(x)\,,0\} \geq 0$$

$$f_- \equiv \max\{-f(x)\,,0\} \geq 0$$

$$|f| = f_+ + f_- \geq 0 \qquad (f = f_+ - f_-)$$

We say that $f \in \mathscr{L}^1_{\mathbb{R}}(\mathbb{R})$ if $\int_{\mathbb{R}} |f|\,dx < +\infty \Rightarrow \int_{\mathbb{R}} f\,dx \equiv \int_{\mathbb{R}} f_+\,dx - \int_{\mathbb{R}} f_-\,dx$

We say that $f \in \mathscr{L}^1_{\mathbb{C}}(\mathbb{R})$ if $\int_{\mathbb{R}} |f|\,dx < +\infty \Rightarrow \int_{\mathbb{R}} f\,dx \equiv \int_{\mathbb{R}} \mathrm{Re}(f)\ dx + i\int_{\mathbb{R}} \mathrm{Im}(f)\ dx$

Let f be real defined on $[a,b]$, we say that $f \in \mathscr{L}^1_{\mathbb{R}}([a,b])$ iff

$$F(x) = \begin{cases} f(x) & x \in [a,b] \\ 0 & x \neq [a,b] \end{cases} \in \mathscr{L}^1_{\mathbb{R}}(\mathbb{R}) \Rightarrow \int_a^b f(x)\,dx = \int_{\mathbb{R}} F(x)\,dx$$

Function spaces

Properties *almost everywhere* (a.e.):

A property $P(x)$, $x \in \mathbb{R}$, is said to be satisfied almost everywhere (a.e.) if the set $\{x \,/\, P(x) \text{ false}\}$ has null measure. In particular, $f_1 = f_2$ a.e. $\Leftrightarrow \int_{\mathbb{R}} |f_1 - f_2| \, dx = 0$

L^1 spaces:

$L^1(\mathbb{R})$ is the set of equivalence classes of functions of $\mathscr{L}^1(\mathbb{R})$ under the equivalence relation: $f_1 = f_2$ a.e.

L^p spaces:

$$f \in \mathscr{L}^p(X) \text{ iff } \|f\|_p \equiv \left| \int |f|^p \, dx \right|^{1/p} < +\infty \,, \quad \forall\, 1 \leq p \leq +\infty$$

Definition: $L^p(X)$ is the set of equivalence classes of functions in $\mathscr{L}^p(X)$ with the equivalence relation $f_1 = f_2$ a.e.

Function spaces

Properties of L^p:

$i)$ $\left(L^p(\mathbb{R})\,,\|\cdot\|_p\right)\,,\left(L^p(B)\,,\|\cdot\|_p\right)$ are Banach

$ii)$ $C[a,b]$ is dense in $\left(L^p([a,b])\,,\|\cdot\|_p\right)$

$iii)$ $\left(L^p([a,b])\,,\|\cdot\|_p\right)$ is the completion of $\left(C([a,b])\,,\|\cdot\|_p\right)$ (idem for $[a,b]\to\mathbb{R}$)

$iv)$ $L^2(\mathbb{R})$ is Hilbert with the scalar product

$$\langle f\,,g\rangle \equiv \int_{\mathbb{R}} \bar{f}(x)\,g(x)\,dx\,, \qquad \text{(analogously for } [a,b])$$

Function spaces

Hölder and Minkowski inequalities (for integrals)

Let $f, g \in L^p(X)\,, g \in L^q(X)\,, \quad 1 < p < \infty\,, \quad \dfrac{1}{p} + \dfrac{1}{q} = 1$

Hölder inequality

$$\int_X |f(x)g(x)|\,dx \leq \left(\int_X |f(x)|^p\,dx \right)^{1/p} \left(\int_X |g(x)|^q\,dx \right)^{1/q}$$

Minkowski inequality

$$\left(\int_X |f(x) + h(x)|^p\,dx \right)^{1/p} \leq \left(\int_X |f(x)|^p\,dx \right)^{1/p} + \left(\int_X |h(x)|^p\,dx \right)^{1/p}$$

Function spaces

Some important orthonormal bases in $L^2(X)$

<u>Legendre basis</u>

$$P_n(x) \equiv \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \quad \text{(Legendre polynomial)}$$

$\left\{\sqrt{n+1/2}\, P_n\right\}_0^\infty$ is orthonormal basis of $L^2([-1,1])$

$$(1-x^2)\, P_n'' - 2x\, P_n' + n(n+1)\, P_n = 0\,, \quad n = 0, 1, \ldots \quad \text{(Legendre equation)}$$

<u>Hermite basis</u>

$$H_n(x) \equiv (-1)^n\, e^{x^2}\frac{d^n}{dx^n}\, e^{-x^2} \quad \text{(Hermite polynomials)}$$

$\left\{\left(\sqrt{\pi}\, 2^n n!\right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n\right\}_0^\infty$ is orthonormal basis of $L^2(\mathbb{R})$

$$H_n'' - 2x\, H_n' + 2n\, H_n = 0\,, \quad n = 0, 1, \ldots \quad \text{(Hermite equation)}$$

Function spaces

Some important orthonormal bases in $L^2(X)$

Laguerre basis

$$L_n(x) \equiv \frac{1}{n!}\, e^x \frac{d^n}{dx^n}\left(e^{-x}\, x^n\right) \quad \text{(Laguerre polynomials)}$$

$\left\{e^{-\frac{x}{2}}\, L_n\right\}_0^\infty$ is orthonormal basis of $L^2([0,\infty])$

$$x\, L_n'' + (1-x)\, L_n' + n\, L_n = 0\,, \quad n = 0, 1, \ldots \quad \text{(Laguerre equation)}$$

Fourier basis

$\left\{e^{2\pi i\, nx/L}/\sqrt{L}\right\}_{-\infty}^{+\infty}$ is orthonormal basis of $L^2([a, a+L])$

$\left\{\frac{1}{\sqrt{L}}\,, \sqrt{\frac{2}{L}}\, \cos\left(\frac{2\pi nx}{L}\right)\,, \sqrt{\frac{2}{L}}\, \sin\left(\frac{2\pi nx}{L}\right)\right\}$ is orthonormal basis of $L^2([a, a+L])$

Function spaces

Series expansions in eigenfunctions

Given the differential operator

$$\mathcal{O} \equiv -\frac{d^2}{dx^2}$$

every function $f \in L^2[a, a+L]$ can be written in terms of eigenfunctions of $\mathcal{O}$ as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n f_n(x)$$

with

$$f_n(x) = e^{i\frac{2\pi n x}{L}} \qquad\qquad \mathcal{O} f_n = \left(\frac{2\pi n}{L}\right)^2 f_n$$

Function spaces

Orthonormal basis of polynomials associated to a weight function

Let $0 \neq \rho \in L^1(\mathbb{R})$, non-negative $/\, \exists\, \alpha > 0$ such that $\int_{\mathbb{R}} e^{|\alpha| t} \rho(t)\, dt < \infty$

If $\{p_n(t)\}_0^\infty$ are the orthonormal polynomials in terms of the scalar product $\langle f, g \rangle_\rho \equiv \int_{\mathbb{R}} \bar{f}\, g\, \rho$, obtained from $\{t^n\}_0^\infty$ via the Gram-Schmidt orthonormalization process, then $\{p_n(t)\, \rho^{1/2}(t)\}_0^\infty$ is an orthonormal basis of $L^2(\operatorname{supp} \rho)$

Theorem: Given two borelians B_1, B_2 and their orthonormal bases $\{f_i\}_1^\infty, \{g_i\}_1^\infty$ of $L^2(B_1), L^2(B_2)$, the set $\{f_i\, g_j\}_{i,j=1}^\infty$ constitutes an orthonormal basis of $L^2(B)$ where $B \equiv B_1 \times B_2$

Function spaces

Summary of results:

- Borelians. Borel-Lebesgue measure. Borel measurable functions
- Lebesgue integral. Relation to Riemann integral
- Lebesgue integrable functions. $\mathscr{L}^p(X)$ spaces
- Properties almost everywhere. $L^p(X)$ spaces
- $L^2(X)$ is a Hilbert space (completion of $C(X)$)
- Hölder and Minkowski inequalities
- Basis of orthonormal polynomials in $L^2(X)$
- Series expansions in eigenfunctions

Linear operators

Definition: Given two linear spaces L_1 and L_2 and the map $T: L_1 \to L_2$ (univalued) with domain $D(T) < L_1$ and image $I(T) \subset L_2$, we say that T is a *linear operator* if

$$T(x+y) = Tx + Ty \qquad T(\lambda x) = \lambda \cdot Tx \qquad \begin{array}{l} \forall x, y \in D(T) \\ \forall \lambda \in \Lambda \end{array}$$

Conversely, if $\Lambda = \mathbb{C}$ and $T(\lambda x) = \bar{\lambda} \cdot Tx$ we say that T is an *antilinear operator*

Properties: If T is a linear operator, then

$i)\ I(T) < L_2$

$ii)\ T\,\mathbf{0}_{\boldsymbol{D}} = \mathbf{0}_{\boldsymbol{I}}$

$iii)\ T(-x) = -Tx$

$iv)\ \ker(T) = \{x \in D(T)\,/\,Tx = \mathbf{0}_{\boldsymbol{I}}\} < L_1$

$v)$ If $D(T) = L_1$ and $\dim L_1$ is finite,

$$\dim D(T) = \dim \ker(T) + \dim I(T)$$

Linear operators

Definition: The set of all linear operators admits a structure of linear space with field Λ:

$$(T_1 + T_2)x \equiv T_1 x + T_2 x \qquad\qquad (\lambda T)x \equiv \lambda \cdot Tx$$

We denote this linear space as $\mathscr{L}(L_1, L_2)$ and if $L_1 = L_2$ as $\mathscr{L}(L)$

Some important lemmas:

$i)$ $\forall T \in \mathscr{L}(D(T) < H)\,,\ \overline{D(T)} = H$ (Hilbert) $\Rightarrow \big(\langle x\,, y\rangle = 0 \ \ \forall\, y \in D(T) \Rightarrow x = \mathbf{0}\big)$

$ii)$ Let $T \in \mathscr{L}(L)$ with L pre-Hilbert $\Rightarrow \big(T = \mathbf{0} \Leftrightarrow \langle x\,, Tx\rangle = 0 \quad \forall x \in L\big)$

$iii)$ Let $T_i \in \mathscr{L}(L)\,,\ i = 1{,}2$ with L pre-Hilbert $\Rightarrow \big(T_1 = T_2 \Leftrightarrow \langle x\,, T_1 x\rangle = \langle x\,, T_2 x\rangle \quad \forall\, x \in L\big)$

Linear operators

Given a linear operator $T: D(T) < L_1 \to I(T) < L_2$, we define the following:

Definition: The inverse map is given by $T^{-1} : D(T^{-1}) = I(T) \to I(T^{-1}) = D(T)$ such that $\forall y \in D(T^{-1})$ one has $T^{-1}y = x$ with $x \in D(T)$ y $Tx = y$

Definition: T is non-singular $\Leftrightarrow \exists T^{-1}$ linear operator. Moreover, if T is non-singular

$$i)\ \exists T^{-1}T \equiv I_D : D(T) \to D(T) \,/\, T^{-1}T\,x = x \quad \forall\, x \in D(T)$$

$$ii)\ \exists TT^{-1} \equiv I_I : I(T) \to I(T) \,/\, TT^{-1}\,y = y \quad \forall\, y \in I(T)$$

Definition: T is invertible $\Leftrightarrow D(T) = I(T) = L$, T is non-singular and $TT^{-1} = T^{-1}T = I$

Linear operators

Definition: Given two linear operators $T_i : D(T_i) < L_1 \to I(T_i) < L_2$, with $D(T_1) \subset D(T_2)$ and $T_2 x = T_1 x \;\; \forall x \in D(T_1)$, we say that T_2 is an extension of T_1 to $D(T_2)$ and that T_1 is a restrictions of T_2 to $D(T_1)$, which we denote as $T_1 \supset T_2$

Definition: Given two normed spaces L_1 and L_2 over the same field Λ, a linear operator $A : D(A) < L_1 \to I(A) < L_2$ is continuous in $x \in D(A)$

$$\Leftrightarrow \forall \epsilon > 0 \; \exists \, \delta(x, \epsilon) / \;\; \|x - y\|_{L_1} < \delta \; \forall y \in D(A) \Rightarrow \|Ax - Ay\|_{L_2} < \epsilon$$

$\Leftrightarrow \forall \{x_n\}_1^\infty \subset D(A)$ we have that $x_n \to x \Rightarrow A(x_n) \to A(x)$

Definition: A linear operator A is said to be continuos if it is continuous $\forall x \in D(A)$

Linear operators

Definition: Let H_1 and H_2 be two Hilbert spaces over Λ and $T \in \mathscr{L}(H_1, H_2)$. We say that T is bounded iff

$$\|T\| \equiv \sup_{v \in H,\, v \neq 0} \frac{\|Tv\|}{\|v\|} < +\infty$$

and $\|T\|$ is called the norm of the operator T. Note that $\|T\|$ is the infimum of every possible bound for T of the type $\|Tv\| \leq M\,\|v\|\ \forall v \in H_1\ (M > 0)$

Definition: We denote $\mathscr{A}(H_1, H_2)$ as the set of bounded linear operators from H_1 to H_2. We further use the compact notation $\mathscr{A}(H)$ when $H_1 = H_2 \equiv H$

Linear operators

Theorem: Let $T \in \mathscr{L}(H_1, H_2)$ with H_1, H_2 Hilbert. The following statements are equivalent:

$i)$ $T \in \mathscr{A}(H_1, H_2)$ $\quad$ $ii)$ T is continuous $\quad$ $iii)$ T is continuous in $x \in H_1$

Definition: A linear operator $T : D(T) < H_1 \to H_2$ is bounded in its domain if

$$\sup_{v \in D(T),\, v \neq 0} \frac{\|Tv\|}{\|v\|} < +\infty$$

Theorem: If a linear operator, $T : D(T) < H_1 \to H_2$, is bounded in its domain with $\overline{D(T)} = H_1$ $\Rightarrow \exists!\, \tilde{T} \in \mathscr{A}(H_1, H_2)$ that extends T to H_1. Furthermore, $\|\tilde{T}\| = \|T\|$.

Theorem: Let $H_1, H_2 \neq \{\mathbf{0}\}$ be Hilbert spaces and $T \in \mathscr{A}(H_1, H_2)$ with $I(T) = H_2$, then

$$T^{-1} \in \mathscr{A}(H_2, H_1) \Leftrightarrow \exists k > 0 \;/\; \|Tv\| \geq k\|v\| \quad \forall\, v \in H_1$$

Linear operators

Theorem: H_1, H_2 Hilbert spaces $\Rightarrow \mathscr{A}(H_1, H_2)$ is a Banach space

Note: If $\dim H_2 > 1 \Rightarrow \mathscr{A}(H_1, H_2)$ does not admit a scalar product

Definition (Banach algebra): A normed linear space L over Λ, is a *unital Banach algebra* if it has a product rule $L \times L \to L$ denoted as $x, y \to xy,$ which is associate and such that

$i)\ x(y+z) = xy + xz$ $\qquad$ $iv)\ \|xy\| \leq \|x\|\,\|y\|$

$ii)\ (x+y)z = xz + yz$ $\qquad$ $v)\ \exists e \in L$ with $\|e\| = 1 \ /\ xe = ex = x$

$iii)\ (\lambda x)y = \lambda(xy) = x(\lambda y)$ $\qquad$ $vi)\ L$ is complete

Proposition: H Hilbert $\Rightarrow \mathscr{A}(H)$ is a unital Banach algebra

Definition (commutator): Given $T_{1,2} \in \mathscr{A}(H): [T_1, T_2] = T_1T_2 - T_2T_1$ (in general $\neq 0$)

Linear operators

Definition: We define the *graph of an operator* as

$$\Gamma(T) = \{(x, y) \in L_1 \times L_2 \,/\, x \in D(T)\,, y = Tx\} < L_1 \overrightarrow{\oplus} L_2$$

Definition: T is said to be closed iff $\Gamma(T) = \overline{\Gamma(T)}$ in the normed space defined as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$\alpha(x, y) = (\alpha x, \alpha y)$$

and the norm $\|(x, y)\| = \|x\|_{L_1} + \|y\|_{L_2}$ (assuming L_1 and L_2 are normed spaces)

Note: If X and Y are Banach, $X \times Y$ are also Banach

Linear operators

Theorem (of the closed graph): Let $T : D(T) < B_1 \to B_2$ be a linear operator with B_1 and B_2 Banach spaces with $D(T) = \overline{D(T)}$, then:

$$T \text{ bounded } \Leftrightarrow \Gamma(T) \text{ is closed}$$

Definition: A linear operator T is defined as closable iff it has a closed extension $\widetilde{T} \supseteq T$

Definition: Given a closable operator T, we define its closure, $\overline{T}$, as its minimal closed extension

Linear operators

Summary of results:

- Linear operators and inverse operators
- Continuos and bounded operators. Relationship between continuity and boundedness
- Criterium of inversion keeping boundedness
- Space of bounded operators with domain and image in a Hilbert are Banach
- Banach algebra. $\mathscr{A}(H)$ forms a unital Banach algebra
- Graph of an operator
- Theorem of the closed graph
- Closure of an operator

Some important types of bounded operators

Adjoint operator: Given $A \in \mathscr{A}(H)$ with H Hilbert, we define its adjoint operators as the <u>unique</u> operator $A^\dagger \in \mathscr{A}(H)$ that satisfies

$$\langle w, Av \rangle = \langle A^\dagger w, v \rangle \qquad \forall\, v, w \in H$$

Properties:

$i)$ The application $\mathscr{G} : \mathscr{A}(H) \to \mathscr{A}(H)$ / $\mathscr{G}(A) = A^\dagger$ is an antilinear and isometric bijection

$ii)$ $(AB)^\dagger = B^\dagger A^\dagger$ $\qquad [(\alpha A + \beta B)^\dagger = \bar{\alpha} A^\dagger + \bar{\beta} B^\dagger\,,\ \|A^\dagger\| = \|A\|]$

$iii)$ $\left(A^\dagger\right)^\dagger = A$

$iv)$ $A\,, A^{-1} \in \mathscr{A}(H) \Rightarrow \left(A^\dagger\right)^{-1} = \left(A^{-1}\right)^\dagger$

$v)$ $\|A^\dagger A\| = \|A\|^2$

$vi)$ If H is separable, $[A^\dagger]_{ij} = \overline{A_{ji}}$

Some important types of bounded operators

Equality of operators: $A = B \Leftrightarrow D(A) = D(B) = D\,, Ax = Bx\,, \quad \forall x \in D$

(equiv. if $D(A) = D(B) = D\,, \langle y, Ax \rangle = \langle y, Bx \rangle\,, \quad \forall x \in D)$
$\forall y \in H$

Operator types: Given $T : D(T)$ dense in $H \to H$

Symmetric or Hermitian operator: $T \subset T^\dagger \left[D(T) \subsetneq D(T^\dagger)\,, \langle x, Ty \rangle = \langle Tx\,, y \rangle\,, \quad \forall x\,, y \in D(T) \right]$

Self-adjoint operator: $T = T^\dagger \left[D(T) = D(T^\dagger)\,, \langle x, Ty \rangle = \langle Tx\,, y \rangle\,, \quad \forall x\,, y \in D(T) \right]$

Bounded self-adjoint operator: $A \in \mathscr{A}(H) \,/\, A = A^\dagger \qquad \left[A = A^\dagger \Leftrightarrow \langle x, Ax \rangle \in \mathbb{R}\ \forall x \in H \right]$

Some important types of bounded operators

Definition: Given $A \in \mathscr{A}(H)$, we say it is *positive,* $A \geq 0$, if $\langle x, Ax \rangle \geq 0 \quad \forall\, x \in H$

Definition: Given A, B self-adjoint in $\mathscr{A}(H)$, we say that A is *larger or equal* than B if $A - B \geq 0$ and we denote this as $A \geq B$

Theorem of the square root: Given $A \in \mathscr{A}(H)$, self-adjoint and positive, then $\exists!\, B,$ also positive, such that $B^2 = A$, and we denote $B \equiv A^{1/2}$. We further denote $|A| \equiv (A^\dagger A)^{1/2}$

Defintion (isometric operador): $T \in \mathscr{A}(H)$ / $\|Tx\| = \|x\|$, $\forall\, x \in H$

$$\boxed{T \text{ isometric}} \quad \begin{array}{c} \Longrightarrow \\ \cancel{\Longleftarrow} \end{array} \quad \boxed{T \text{ bounded in its domain with } \|T\| = 1}$$

Some important types of bounded operators

Definition (unitary operator): $U \in \mathscr{A}(H)$ / $U^\dagger = U^{-1}$

Note: $T \in \mathscr{A}(H)$ isometric $\Longleftrightarrow T^\dagger T = I$

$U \in \mathscr{A}(H)$ unitary $\Longleftrightarrow U^\dagger U = UU^\dagger = I$

Characterization of unitary operators: Given $U \in \mathscr{A}(H)$, we have that:

U unitary $\Longleftrightarrow U$ bijective and $\langle Ux, Uy \rangle = \langle x, y \rangle \quad \forall\, x, y \in H$

$\Longleftrightarrow U$ bijective and $\|Ux\| = \|x\| \quad \forall\, x \in H$

$\Longleftrightarrow \{e_\alpha\}_{\alpha \in A}$ orthonormal basis of $H \Rightarrow \{Ue_\alpha\}_{\alpha \in A}$ orthonormal basis of H

$\Longleftrightarrow U^\dagger$ unitary

Some important types of bounded operators

Definition (normal operator): $A \in \mathscr{A}(H)$ / $[A, A^\dagger] = 0$

Note: $A \in \mathscr{A}(H) \Longrightarrow A^\dagger \in \mathscr{A}(H) \Longrightarrow D(AA^\dagger) = D(A^\dagger A) = H$

$A \in \mathscr{A}(H)$ normal $\Longleftrightarrow \|Av\| = \|A^\dagger v\| \quad \forall v \in H$

Properties:

A self-adjoint $\Longrightarrow A$ normal $(AA^\dagger = A^\dagger A = A^2)$

A hermitian $\not\Longrightarrow A$ normal $(D(AA^\dagger) \neq D(A^\dagger A))$

A unitary $\Longrightarrow A$ normal $(AA^\dagger = A^\dagger A = I)$

A isometric $\not\Longrightarrow A$ normal $(I(AA^\dagger) \neq I(A^\dagger A))$

Definition (orthogonal projector): $P \in \mathscr{A}(H)$ is orthogonal projector iff $P^2 = P = P^\dagger$

Theorem: Given P orthogonal proj. $\Rightarrow \exists\, M \triangleleft H$ / P is orthogonal proj. in M

Some important types of bounded operators

Summary of results:

- Adjoint operators. Properties
- Hermitian and self-adjoint operators. Properties
- Positive operators. Ordering of operators. Square root of operators
- Isometric and unitary operators
- Normal operators
- Orthogonal projectors

Linear functionals

Definition: Given a linear space L over Λ ($\mathbb{R}$ ó $\mathbb{C}$), a *linear functional* (or *linear form*) over L is any linear operator of the from $F : L \to \Lambda$. That is, $F \in \mathscr{L}(L, \Lambda)$

Definition: If L is a normed space, we say that a *continuous linear functional* is any element in $\mathscr{A}(L, \Lambda)$ and we denote as *dual space, $\tilde{L}$,* the normed space

$$\tilde{L} \equiv (\mathscr{A}(L, \Lambda), \|\cdot\|_{\mathscr{A}})$$

Proposition: Let H be a Hilbert space of <u>finite dimension</u> (and thus isomorphic to Λ^n), we have that

$$i)\ \mathscr{A}(H, \Lambda) = \mathscr{L}(H, \Lambda) \qquad\qquad ii)\ \dim \tilde{H} = \dim H$$

Linear functionals

Riesz-Fréchet theorem: Let $(H, \langle \cdot , \cdot \rangle)$ be a Hilbert space (separable or not)

$$\forall\, F : H \to \Lambda \text{ linear and continuous}$$

$$\exists!\, f \in H \,/\, F(g) = \langle f, g \rangle \,, \forall\, g \in H$$

Corollaries:

$i)$ Let $F \in \tilde{H}$, we then have:

$$\ker F \subsetneq H \Rightarrow \dim (\ker F)^{\perp} = 1 \qquad\qquad \ker F = H \Rightarrow \dim (\ker F)^{\perp} = 0$$

$ii)$ $F \in \tilde{H} \Leftrightarrow \ker F \lhd H$

$iii)$ $\|F_f\|_{\mathscr{A}(H,\Lambda)} = \|f\|_H$

$iv)$ Let $\{e_j\}_1^n$ be orthonormal basis of $\Lambda^n \Rightarrow \forall\, \varphi : H \to \Lambda^n$ linear and continuous

$$\exists f_1, \ldots, f_n \in H \,/\, \varphi(g) = \sum_1^n \langle f_i , g \rangle \, e_i$$

Linear functionals

Using Riesz-Fréchet theorem, we can see that $\tilde{H}$ is a Hilbert space with

$$\langle \cdot\,, \cdot \rangle : \tilde{H} \times \tilde{H} \to \Lambda$$

$$F_f\,, F_g \to \langle F_f\,, F_g \rangle \equiv \langle g\,, f \rangle$$

Corollary: The application $\tau : f \in H \to F_f \in \mathscr{A}(H, \Lambda)$, with $F_f(g) \equiv \langle f, g \rangle$ is an isometric antilinear bijection (it follows that both spaces are isomorphic)

Proposition: Let $(L, ||\cdot||)$ be a normed space, $\tilde{L}$ and $\tilde{\tilde{L}}$ are Banach (and similarly for Hilbert)

Definition: Let $(L, ||\cdot||)$ be a normed space, we say it is reflexive iff L and $\tilde{\tilde{L}}$ are isomorphic

Linear functionals

Dirac notation: The Riesz-Fréchet theorem suggests the following notation

$$\text{Vector of } H \longrightarrow \text{"ket" } |\,\phi\rangle \in H \quad \left[\alpha\,|\,\phi\rangle = |\,\alpha\phi\rangle\right]$$

$$\text{Functional of } \tilde{H} \longrightarrow \text{"bra" } \langle F\,| \in \tilde{H} \quad \left[\alpha\,\langle\phi\,| = \langle\overline{\alpha}\phi\,|\right]$$

$$\text{Functional acting on a vector} \longrightarrow \text{"bracket" } \langle F\,|\,\phi\rangle \equiv F(\phi)$$

from the Riesz-Fréchet theorem we have

$$\begin{aligned} R : \tilde{H} &\to H \\ \langle F| &\to |\,\phi_F\rangle \end{aligned} \implies \langle F\,|\psi\rangle = \Big\langle\,|\,\phi_F\rangle\,, |\psi\rangle\Big\rangle$$

Normally, we simplify the notation and use the same name for F and ϕ_F, yielding

$$\langle\phi\,|\,\psi\rangle = \Big\langle\,|\,\phi\rangle\,, |\psi\rangle\Big\rangle$$

Linear functionals

Definition: Let H be a Hilbert space over Λ. We define a bilinear form (more precisely a sesquilinear form) as the application $\varphi : H \times H \to \Lambda$ such that

$$i)\ \varphi(\alpha w, \beta v) = \bar{\alpha}\beta\, \varphi(w, v)\,, \quad \forall\, \alpha, \beta \in \Lambda\,, \quad \forall\, v, w \in H$$

$$ii)\ \varphi(w_1 + w_2, v) = \varphi(w_1, v) + \varphi(w_2, v)$$

$$ii)\ \varphi(w, v_1 + v_2) = \varphi(w, v_1) + \varphi(w, v_2)$$

Linear functionals

Definition: φ is bounded if $\exists k \geq 0$ / $|\varphi(w,v)| \leq k\,\|w\|\,\|v\| \quad \forall\, w,v \in H$. If bounded, we define the norm of φ as

$$\|\varphi\| \equiv \sup_{w \neq 0 \neq v} \frac{|\varphi(w,v)|}{\|w\|\,\|v\|}$$

Theorem: Let $\varphi : H \times H \to \Lambda$ be a bounded bilinear form over H Hilbert, then $\exists!\, A \in \mathscr{A}(H)$ such that

$$\varphi(w,v) = \langle w\,, Av \rangle\,, \qquad \forall\, v, w \in H$$

Moreover $\|\varphi\| = \|A\|$.

Linear functionals

Strong convergence: $x_n \xrightarrow{s} x \Leftrightarrow \|x_n - x\| \to 0$ (usual topology induced by the norm)

Weak convergence: $x_n \xrightarrow{w} x \Leftrightarrow |F(x_n) - F(x)| \to 0 \qquad \forall F \in \tilde{H}$

Theorem: Let H be a Hilbert space and $\{x_n\}_1^\infty \subset H$, then

$i)$ If H has finite dimension, $x_n \xrightarrow{s} x \Leftrightarrow x_n \xrightarrow{w} x$

$ii)$ If H has infinite dimension, $x_n \xrightarrow{s} x \Rightarrow x_n \xrightarrow{w} x$

$iii)$

$$\left.\begin{array}{c} x_n \xrightarrow{w} x \\ \|x_n\| \to \|x\| \end{array}\right\} \Leftrightarrow x_n \xrightarrow{s} x$$

Linear functionals

Topologies in $\mathscr{A}(B_1, B_2)$ with $B_{1,2}$ Banach:

- Uniform or norm topology: $A_n \xrightarrow{u} A \Leftrightarrow \|A_n - A\| \xrightarrow{n\to\infty} 0$
- Strong topology: $A_n \xrightarrow{s} A \Leftrightarrow \|A_n x - Ax\| \xrightarrow{n\to\infty} 0 \qquad \forall\, x \in H$
- Weak topology: $A_n \xrightarrow{w} A \Leftrightarrow |\, F(A_n x) - F(Ax)\,| \xrightarrow{n\to\infty} 0 \qquad \forall\, F \in \tilde{H}$

[In $\mathscr{A}(H)$, $\Leftrightarrow |\,\langle f, A_n x\rangle - \langle f, Ax\rangle\,| \xrightarrow{n\to\infty} 0$]

In finite-dimensional spaces, all topologies are the same. In infinite-dimensional ones

$$\text{Uniform top.} \quad \underset{\neq}{>} \quad \text{Strong top.} \quad \underset{\neq}{>} \quad \text{Weak top.}$$

Linear functionals

Summary of results:

- Linear functionals. Dual space
- Riesz-Frechét theorem and corollaries
- Scalar product of linear functionals
- Dual spaces and reflexive normed spaces
- Dirac notation
- Bilinear forms
- Different convergence criteria

Theory of distributions

Space of test functions

Space of test functions of compact support

$$\mathscr{D}(\mathbb{R}) = \{f \in \mathscr{C}^{\infty}(\mathbb{R}) \,/\, \operatorname{supp} f \text{ compact of } \mathbb{R}\}$$

is a linear space and forms an algebra of functions

Convergence: $$f_n \xrightarrow{\mathscr{D}} f \text{ if } \begin{cases} i) \operatorname{supp} f_n \subset \mathbb{K} \text{ compact and independent of } n \\ ii) \|f_n^{(p)} - f^{(p)}\|_{\infty} \xrightarrow{n \to \infty} 0 \,,\ \forall p \geq 0 \end{cases}$$

Theory of distributions

Space of test functions

Space of rapidly decreasing test functions

$$\mathcal{S}(\mathbb{R}) = \{f \in \mathscr{C}^\infty(\mathbb{R}) / \sup_{k,m \in \mathbb{N}} \|x^k f^{(m)}\|_\infty < \infty\}$$

is a seminormed space with seminorm $\|f\|_{k,m} = \|x^k f^{(m)}\|_\infty$

Convergence: $f_n \xrightarrow{\mathcal{S}} f$ if $\|x^k f_n^{(m)} - x^k f^{(m)}\|_\infty \xrightarrow{n\to\infty} 0 \ \forall \, k, m \in \mathbb{N}$

Property: $f_n \xrightarrow{\mathscr{D}} f \Rightarrow f_n \xrightarrow{\mathcal{S}} f$, $\mathscr{D}$ is dense in $\mathcal{S}$. We have that $\mathscr{D} \subset \mathcal{S} \subset L^2(\mathbb{R})$

Theory of distributions

Definition: A distribution (or generalized function) defined over $\mathbb{R}$ is defined as any continuous linear functional with domain the space $\mathscr{D}$

$$T : \mathscr{D} \to \Lambda$$

i) Linear: $T[\alpha\phi_1 + \beta\phi_2] = \alpha\, T[\phi_1] + \beta\, T[\phi_2] \qquad \forall\, \alpha, \beta \in \Lambda \quad \forall\, \phi_{1,2} \in \mathscr{D}$

ii) Continuous: $\phi_n \to \phi \Rightarrow T[\phi_n] \to T[\phi]$

Space of distributions: $\widetilde{\mathscr{D}(\mathbb{R})} = \{T\,/\,T \text{ distribution}\}$ [Dual of $\mathscr{D}(\mathbb{R})$]

Sufficient condition for continuity:

$\exists\, M > 0 \,/\, |\,T[\phi]\,| \leq M\|\phi\|_\infty, \forall\, \phi \in \mathscr{D}(\mathbb{R}) \Rightarrow T$ continuous in $\mathscr{D}$

Theory of distributions

Definition: A tempered distribution defined over $\mathbb{R}$ is defined as any continuous linear functional with domain the space $\mathcal{S}$

$$T : \mathcal{S} \to \Lambda$$

Space of distributions: $\widetilde{\mathcal{S}(\mathbb{R})} = \{T \,/\, T \text{ tempered distribution}\}$ [Dual of $\mathcal{S}(\mathbb{R})$]

Sufficient condition for continuity:

$\exists\, M > 0 \,/\, |\, T[\phi]\, | \leq M \|\phi\|_\infty \,, \forall\, \phi \in \mathcal{S}(\mathbb{R}) \Rightarrow T$ continuous in $\mathcal{S}$

Property: $\widetilde{L^2(\mathbb{R})} \subset \widetilde{\mathcal{S}(\mathbb{R})} \subset \widetilde{\mathscr{D}(\mathbb{R})}$

Theory of distributions

3 possibilities for $T \in \widetilde{\mathscr{D}(\mathbb{R})}$ analogously for $\widetilde{\mathscr{S}(\mathbb{R})}$)

1) Normal distribution: $\exists f \in \mathscr{D}(\mathbb{R}) \,/\, F[\phi] = \int_{\mathbb{R}} \bar{f}(x)\, \phi(x)\, dx$

2) Regular distribution: $\exists f : \mathbb{R} \to \Lambda\,, f \notin \mathscr{D}(\mathbb{R}) \,/\, F[\phi] = \int_{\mathbb{R}} \bar{f}(x)\, \phi(x)\, dx$

3) Singular distribution: $\nexists f : \mathbb{R} \to \Lambda \,/\, F[\phi] = \int_{\mathbb{R}} \bar{f}(x)\, \phi(x)\, dx$

Theory of distributions

Examples of distributions:

1) Every integrable function over any compact in $\mathbb{R}$ defines a regular distribution

2) Characteristic distribution: Let $X \subset \mathbb{R}$, $\chi_X : \phi \to \chi_X[\phi] = \int_X \phi(x)\, dx$

which is normally represented by $\chi_X(x) = \begin{cases} 1\,, & x \in X \\ 0\,, & x \notin X \end{cases}$ such that $\chi_X[\phi] = \int_{\mathbb{R}} \chi_X(x)\, \phi(x)\, dx$

3) Dirac delta: $\delta_{x_0} : \phi \to \phi(x_0)$ (singular tempered distribution)

4) Principal value of $1/x$: $\mathrm{PV}_{\frac{1}{x}}[\phi] = \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x}\, dx$ (singular tempered distribution)

Theory of distributions

Dirac delta: $\delta_{x_0} : \phi \to \phi(x_0) \qquad (\delta \equiv \delta_0)$

It is often represented as a "function": $\delta_{x_0}[\phi] = \displaystyle\int \delta(x - x_0)\, \phi(x)\, dx$ with

$$\delta(x - x_0) = \begin{cases} \infty\,, & x = x_0 \\ 0\,, & x \neq x_0 \end{cases}$$

or as the limit of sequences of functions

$$\delta(x) = \lim_{\lambda \to \infty} \sqrt{\frac{\lambda}{\pi}}\, e^{-\lambda x^2} = \lim_{\lambda \to \infty} \frac{\sin \lambda x}{\pi x} = \lim_{\epsilon \to 0^+} (i\pi\epsilon)^{-1/2}\, e^{ix^2/\epsilon} = \lim_{\epsilon \to 0^+} \frac{\epsilon}{\pi\,(x^2 + \epsilon^2)}$$

Theory of distributions

Dirac delta: $\delta_{x_0} : \phi \to \phi(x_0)$

Theorem: Let φ_n be real-valued functions satisfying that $\forall\, n \in \mathbb{N}$:

$i)\ \varphi_n(x) > 0 \quad \forall\, x \in \mathbb{R}$

$ii)\ \int_{-\infty}^{+\infty} \varphi_n(x)\, dx = 1$

$iii)\ \int_{|x|\geq a} \varphi_n(x)\, dx \to 0 \quad \forall\, a > 0$

Then, $\varphi_n(x) \to \delta(x)$ in $\widetilde{\mathcal{S}(\mathbb{R})}$

Property: Let $f(x)$ with a finite number of simple zeroes, then

$$\delta\left(f(x)\right) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|} \qquad\qquad f(x_i) = 0$$

Theory of distributions

Principal value of $1/x$: $\mathrm{PV}_{\frac{1}{x}}[\phi] = \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x}\, dx$

It can be represented as a limiting “function”

$$\frac{1}{x \mp i0} \equiv \lim_{\epsilon \to 0^+} \frac{1}{x \mp i\epsilon} = \mathrm{PV}_{\frac{1}{x}} \pm i\pi\delta$$

Theory of distributions

Operations with distributions (also for tempered distributions):

- Multiplication with functions: $\rho T : \phi \to T[\bar{\rho}\phi] \in \widetilde{\mathcal{D}(\mathbb{R})}\,, \forall\, \rho \in \mathcal{C}^\infty(\mathbb{R})$

  $\in \widetilde{\mathcal{S}(\mathbb{R})}\,, \forall\, \rho \in \mathcal{S}(\mathbb{R})$

- Translation: $T_a : \phi \to T[\phi_{-a}]$ with $\phi_a(x) \equiv \phi(x + a)$

- Derivative of a distribution: $T^{(m)} : \phi \to T[(-1)^m \phi^{(m)}]$

These operations preserve the convergence in the sense of distributions (weak convergence)

$$T_n \to T \Leftrightarrow T_n[\phi] \to T[\phi]\,, \quad \forall\, \phi \in \widetilde{\mathcal{D}(\mathbb{R})}$$

Theory of distributions

Regularity theorem for distributions:

$\forall\, T \in \widetilde{\mathscr{D}(\mathbb{R})}\,, \exists f$ continuous in $\mathbb{R}\,, \exists\, n \in \mathbb{N}$ / $T = T_f^{(n)}$ where $T_f[\phi] \equiv \int_{\mathbb{R}} \bar{f}(x)\phi(x)\,dx$

Fourier transform:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} f(x)\,dx \qquad \text{(direct transform)}$$

$$\check{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} f(k)\,dk \qquad \text{(Inverse transform)}$$

we have that $\hat{\check{f}} = \check{\hat{f}} = f$

Fourier transform of distributions: $\hat{T}[\phi] : \phi \to T[\check{\phi}] \quad \forall\, T \in \widetilde{\mathscr{D}(\mathbb{R})}$

Theory of distributions

Theory of distributions

Summary of results

- Space of test functions (of bounded support and rapidly decreasing)
- (Tempered) distributions: continuous linear functional acting on the space of test functions
- Type of distributions: normal, regular and singular
- Examples of distributions: Dirac delta, characteristic distribution, principal value of $1/x$
- Operations over distributions: multiplication with functions, translation, derivative
- Regularity theorem for distributions
- Fourier transform