Lecture Notes for Quantum Field Theory III

Javier Fuentes Martín and Matthias König
Physics Institute, University of Zürich

Contents

Part I

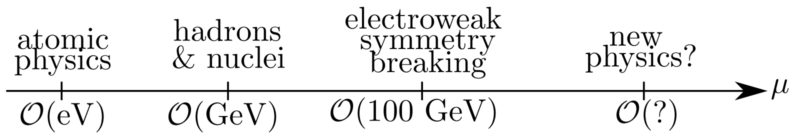
Effective Field Theories

CHAPTER 1 Introduction to Effective Field Theories

> The story so far: In the beginning the Universe was created. This has made a lot of people very angry and been widely regarded as a bad move.
>
> — *Douglas Adams*, The Hitchhiker's Guide to the Galaxy

1.1 Why Effective Field Theories?

The Standard Model (SM) of particle physics [1] is a powerful theory, capable of describing phenomena across a wide range of scales:



This makes it a powerful and versatile tool. But are all of these scales always important? The answer is clearly no: When we are describing low-energy physics, like for example the spectrum of hydrogen, we typically ignore effects of particles of weak-scale masses completely and expand around the limit of the proton being infinitely heavy. Similarly, when describing processes at very high energy, masses of light particles can typically be ignored. Of course, these statements depend on the level of precision with which we want to describe physics. At high precision, we should include also the small corrections from subleading effects.

The core concept goes beyond just quantum field theories: When asked to compute the velocity of an apple hitting the ground after falling from a tree, we would simply compute

$$mgh = \frac{mv^2}{2}\,, \qquad\qquad \Rightarrow v = \sqrt{2gh}\,, \tag{1.1}$$

even though we know that the gravitational potential is not linear in h. However, since the height of the tree is small compared to the scale over which the force of gravity changes (the radius of the earth R), the above result is accurate. Corrections to it would arise with a parametrical suppression of the relative order of $\mathcal{O}(h/R)$, which is roughly $\sim 10^{-6}$ for a typical apple tree on earth. The linear gravitational potential can be thought of an effective theory for the more complete Newtonian theory of gravity (which in turn, can be thought of as an effective theory for General Relativity).

Effective Field Theories (EFTs) are quantum field theories (QFTs) that are less general by construction. They focus on an isolated region compared to a more complete QFT (for example the SM), for which they are designed and treat effects from other regions as perturbations in a well-defined and systematic way. As an example, consider a theory of two real scalars, ϕ and φ with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \frac{g}{3!}\varphi^3\Phi\,. \tag{1.2}$$

Let Φ be much heavier than φ, meaning $M \gg m$ and let us consider a process at very low energy $E \ll M$. Processes with intermediate Φ particles will then be suppressed by the propagator

$$\langle 0|T\{\Phi(0)\Phi(x)\}|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{k^2 - M^2}\,, \tag{1.3}$$

where $k^2 \sim \mathcal{O}(E^2) \ll M^2$. We can see immediately, that neglecting k^2 makes the expressions we are dealing with structurally simpler while still being a good approximation up to corrections of order $\mathcal{O}(k^2/M^2)$.

The next important point is that at low energies the heavy scalar Φ cannot be produced as a real particle. We should therefore be able to describe physics with a Lagrangian that contains only φ:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \Delta\mathcal{L}\,. \tag{1.4}$$

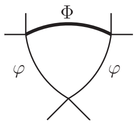
Here $\Delta\mathcal{L}$ is a new ingredient with interactions of φ that were previously not part of the Lagrangian (1.2). While in the full theory we had processes of the form $\varphi^3 \to \varphi^3$ through virtual Φ particles, the interaction terms generating these amplitudes are missing from the effective Lagrangian since it does not contain Φ. Therefore, we must include an interaction of the form

$$\Delta\mathcal{L} \supset \frac{\mathcal{C}_6}{M^2}\varphi^6\,, \tag{1.5}$$

to describe this process. Note how this operator needs to have a prefactor with two inverse powers of mass. We have chosen the heavy mass as a prefactor $1/M^2$ with no further explanation other than the propagator of Φ being of this form in the low-energy limit, but we will justify this later on in more detail.

You might now ask, why we need an effective Lagrangian when we can simply compute amplitudes in the full theory and expand them in the relevant limits we are interested in. And in fact, most of the times we need to do just that anyway to determine the coupling coefficients in what we called $\Delta\mathcal{L}$ above. The answer seems technical at first, but it is an important one. The issue hides at the loop-level, when we are computing radiative corrections. As an example, take the Lagrangian (1.2) again. At one loop, the interactions in this theory generate a contribution to the

$\varphi^3 \to \varphi^3$ process of the form:



$$\begin{aligned} &= \frac{ig^2}{M^2}\left\{(i\lambda)M^2 \int \frac{d^d l}{(2\pi)^d}\frac{i}{l^2-M^2}\left(\frac{i}{l^2-m^2}\right)^2\right\} \\ &= \frac{ig^2}{M^2}\left\{-\frac{\lambda}{16\pi^2}\left[1+\log\frac{m^2}{M^2}+\mathcal{O}\left(\frac{m^2}{M^2}\right)\right]+\mathcal{O}(\lambda^2)\right\} , \end{aligned} \tag{1.6}$$

where we have factored out the tree-level expression ig^2/M^2. While this contribution is formally of higher order in the expansion in the coupling λ, it exhibits a logarithm of the two masses in the process. For extreme enough values of the mass ratio m^2/M^2, this logarithm can become large enough to overpower the suppression of the coupling constant and spoil the convergence of our perturbation expansion. In this case, we should count these *large logarithms* as inverse powers in the coupling and assign them to the leading-order contribution.

We can already anticipate that these offending terms are absent in the effective theory: There will not be propagators of the form $1/(l^2-M^2)$ so the matrix elements there cannot give rise to logarithms of this mass. We will later see how EFTs provide a way of reorganizing the perturbation expansion and correctly including these large logarithmic contributions to any order in $[\lambda \log(m^2/M^2)]^n$ through the means of the renormalization group. In the jargon of the field, this goes by the term of *resummation of large logarithms*.

1.2 Core concepts

Let us now cast our collection of vague ideas into a more systematic approach. What goes into developing an EFT? The three key ingredients are:

1. **Identify the limit in which we want to work**
   Once we decide on the kind of approximation we want to work in (for example a particle being very heavy, a particle moving a very low velocities, etc), we need to find a way to turn this idea into a parametric limit in the theory. This means that we need to indentify the parameters that we wish to treat as small or large so that we can systematically expand in it. In our example of a heavy particle at low energies, this expansion parameter would be E/M.

   In this sense we define the *power-counting* of the effective theory and our Lagrangian, along with the fields are defined as expansions in this parameter. The power-counting is a profound ingredient and being consistent in it is just as important as the counting in the coupling constants of the Lagrangian.

2. **Determine the relevant degrees of freedom**
   In this step, one defines the building blocks out of which the effective Lagrangian is pieced together. This step can be as simple as dropping heavy particles and keeping only the light ones. However, there are EFT constructions in which this step is considerably more involved. We will encounter an example later when we discuss Chiral Perturbation Theory (χPT), the

low-energy effective theory of QCD. Other examples of EFTs with non-trivial field content are the Heavy Quark Effective Theory (HQET) [2] and the Soft-Collinear Effective Theory (SCET) [3].

3. **Determine the symmetries of the theory**
As with any theory in physics, knowing its symmetries is crucial in order to constrain possible interactions, especially in cases in which we do not know the full theory and we simply write down all possible operators in agreement with the symmetries and the power-counting of the theory. In other cases, we might gain new symmetries from the expansion one employs in the first step. An example would be HQET, a theory build for the physics of B-mesons. These mesons consist of a heavy quark (in this case the b-quark) and a light flavor and one expands in the heavy b-quark mass. At leading order, this theory predicts equal masses for the B-meson and its excited spin-1 counterpart, the B^*-meson, because spin-effects are suppressed by the b-mass. Indeed, the mass difference between the B and the B^* is tiny compared to their mass. This *heavy quark spin symmetry* is a new symmetry of the effective theory at leading power in the expansion parameter.

Before formalizing our ideas more, we should also clarify some more terminology. First off, we will very often refer to the region of high energy as *ultraviolet (UV)* and analogously to the low-energy region as *infrared (IR)*. Alternative jargon used for UV and IR are *hard* and *soft*. Another set of phrases are *top-down* and *bottom-up*:

- Doing a **top-down** construction means that we know the full theory is known and we find it useful (or necessary) to neglect UV effects. Most of the ideas and examples discussed to this point followed this philosophy: Using the full theory makes computations unnecessarily complicated and can even spoil perturbation theory and we therefore go to an effective description. Typical examples are HQET, SCET, and the so-called *Fermi theory*.

- On the other hand, we could also construct an EFT by deciding on the degrees of freedom, power-counting and symmetries and then simply writing down all possible operators in accordance with the symmetries up to a desired order in power-counting, with undetermined couplings. This type of construction is referred to as **bottom-up**. A typical example of this is the Standard Model Effective Field Theory (SMEFT) and χPT, and we will discuss both later.

1.3 Construction

We now want to formalize our core concepts and create a step-by-step program.

1. First, we choose a cutoff, which we will call Λ, and divide the field content of the theory. Let ϕ be a generic placeholder for fields of the theory. Then:
$$\phi = \phi_H + \phi_S \,, \tag{1.7}$$
where the subscripts H and S correspond to *hard* and *soft*, respectively. In this way, ϕ_S contains all Fourier modes of fields with frequencies below the cutoff $\omega < \Lambda$, whereas ϕ_H denotes the modes living above the cutoff, $\omega > \Lambda$.

2. Physics at low energies are by construction now described exclusively by the modes denoted by ϕ_S. Matrix elements derived at low energies are then given by n-point vacuum correlators of the soft modes, which are of the form:

$$\langle 0|T\{\phi_S(x_1)\dots\phi_S(x_n)\}|0\rangle = \frac{1}{Z[0]}\left(-i\frac{\delta}{\delta J_S(x_1)}\right)\dots\left(-i\frac{\delta}{\delta J_S(x_n)}\right) Z[J_S]\Big|_{J_S=0}\,, \tag{1.8}$$

where the generating function of the theory is

$$Z[J_S] = \int \mathcal{D}\phi_S\,\mathcal{D}\phi_H \exp\left\{iS(\phi_S,\phi_H) + i\int d^dx J_S(x)\phi_S(x)\right\}\,, \tag{1.9}$$

where we did not include currents of the hard modes since they are not relevant for low-energy processes. From this formula, we can define the Wilsonian effective action:

$$\int \mathcal{D}\phi_H\, \exp\{iS(\phi_S,\phi_H)\} \equiv \exp\{iS_\Lambda(\phi_S)\}\,. \tag{1.10}$$

In computing this quantity, the path integral over the hard modes is performed and they are removed from the theory - they are "integrated out". The Wilsonian action encodes all the interactions at the hard scales. Once these are integrated out, the generating functional becomes:

$$Z[J_S] = \int \mathcal{D}\phi_S \exp\left\{iS_\Lambda(\phi_S) + i\int d^dx J_S(x)\phi_S(x)\right\}\,. \tag{1.11}$$

3. The object S_Λ encodes interactions at the hard scale Λ. In position space this means that it is non-local at scales $\Delta x \sim 1/\Lambda$ by definition. Since it no longer depends on the hard modes ϕ_H, it can be written as a series of interactions between fields ϕ_S with positions x_1 to x_n and $\Delta x^\mu_{ij} = (x_i - x_j)^\mu \sim \mathcal{O}(1/\Lambda)$. Since the wavelengths of the fields are large compared to Δx, we can expand around $\Delta x = 0$. In this limit, the product of operators is local, meaning they all depend on a single position x. This means, our next step is to write S_Λ as a local operator product expansion (OPE):

$$S_\Lambda(\phi_s) = \int d^dx\,\mathcal{L}_{\text{eff}}(x)\,, \qquad \text{with} \quad \mathcal{L}_{\text{eff}} = \sum_i g_i(\mu)\mathcal{O}_i\,, \tag{1.12}$$

where we introduced the effective Lagrangian $\mathcal{L}_{\text{eff}}$. This object is in principle an infinite series of operator products allowed by the symmetries of the theory. Once we have identified or decided on the power-counting parameter of the EFT, we can assign a power-counting to each operator $\mathcal{O}_i$ and truncate the sum at a given order.

Some comments are in order. An important point is the separation in step 1. As stated previously, this largely determines the construction of the effective theory. The common way to decide whether a mode is hard or soft is through its contribution to the matrix elements we are computing. In terms of Feynman diagrams, the contribution of a field to an amplitude is through propagators, so it is intuitive to determine the power-counting from the two-point correlators. At the example of a scalar field this would be:

$$\langle 0|T\{\varphi(0)\varphi(x)\}|0\rangle = \int \frac{d^dk}{(2\pi)^d} e^{-ik\cdot x} \frac{i}{k^2 - m_\varphi^2} \,. \tag{1.13}$$

Counting the powers of some expansion parameter λ in this expression tells you the order of two insertions of φ. Let us clarify this by a few examples: Consider again the theory with the Lagrangian (1.2). We want to build an effective theory for scattering processes of the light fields ϕ at very low energies E, much lower than the mass of the heavy fields Φ. This means, the momenta in the above formula are all $k^2 \sim \mathcal{O}(E^2)$ and we are power-counting in $\lambda = E/M$. Then we find the power-counting for the light fields by (setting $d = 4$ and counting $m^2 \sim E^2$):

$$\begin{aligned} \langle 0|T\{\phi(0)\phi(x)\}|0\rangle &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k^2 - m^2} \\ &\sim \quad \mathcal{O}(E^4) \quad \cdot \quad \mathcal{O}(E^{-2}) = \mathcal{O}(E^2) = \mathcal{O}(\lambda^2 M^2) \,. \end{aligned} \tag{1.14}$$

We thus see that two insertions of ϕ count as $\lambda^2 M^2$ and consequently a single field operator counts as λM. On the other hand for the heavy scalar we find:

$$\begin{aligned} \langle 0|T\{\Phi(0)\Phi(x)\}|0\rangle &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k^2 - M^2} \approx \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{-M^2} \\ &\sim \quad \mathcal{O}(E^4) \quad \cdot \quad \mathcal{O}(M^{-2}) = \mathcal{O}(\lambda^4 M^2) \,, \\ &\Rightarrow \; \Phi^2 \sim \lambda^4 M^2 \quad \Rightarrow \quad \Phi \sim \lambda^2 M \,. \end{aligned} \tag{1.15}$$

Therefore, if we take the hard scale M as reference, the heavy fields count as λ^2 and the light fields count as λ. The two take-away messages from this back-of-the-envelope calculation are:

1. We have seen directly that the heavy fields yield subleading contributions. This gives us a parametrical justification as to why we can remove them from the theory.

2. The power-counting of the modes is directly connected to their frequencies. The light fields have low invariant mass and thus count as soft. The heavy fields count as hard, since even for low momenta, their correlators are governed by the hard scale M.

Going back to our example, we now also know how to count the operators in the infinite sum in eq. 1.12: Each insertion of a soft field counts as one power of λ. Similarly, a derivative acting on the fields will contribute a power of the field momentum to the amplitude. Since we are counting $k^\mu \sim \lambda$, derivatives consequently also count as one power of λ. The result (and this hold for fermions and gauge fields as well), is that the order of an operator in our example is simply λ^{n-4} for n being the mass dimension of the operator. Therefore we can tweak our notation for $\mathcal{L}_{\text{eff}}$ in eq. 1.12

by turning the couplings g_i into dimensionless numbers and organize the OPE in our power-counting:

$$\mathcal{L}_{\text{eff}} = \sum_i g_i(\mu)\mathcal{O}_i = \sum_k^{\infty} \frac{1}{\Lambda^{k-4}} \sum_i^{n_k} \mathcal{C}_{(i,k)}(\mu)\mathcal{O}_{(i,k)} \,. \tag{1.16}$$

Here n_k denotes the number of operators at order k. The dimensionless coupling coefficients of the operators $\mathcal{C}_j$ are called *Wilson coefficients*. They encode the short-distance physics that we have integrated out, hence they are sometimes called *short-distance coefficients*. At a given order k, the number of operators n_k is finite. Once we truncate the sum over k at a finite value N, $\mathcal{L}_{\text{eff}}$ will contain a finite number of operators.

While the example of heavy and light particles in low-energy processes is by far the most common type of EFT construction, let us briefly digress into a different example to point out some points that are generic to the idea of EFTs and some points that are specific to the example at hand.

The light fields in our previous example counted as soft because of their low invariant mass. We immediately knew that their invariant mass would be small since we assumed every component of their momenta to be small, $k \sim E = \lambda M$. However, when looking at eq. (1.14), it is clear that it suffices to have k^2 to be small. This can be the case for massless, highly energetic fields with $k^\mu \sim (M, 0, 0, M)$. In this example, the field would still count as soft, since $k^2 \sim 0$ even though individual components of its momentum k^μ are large. Note that in contrast to our previous example, derivatives in the operators do not necessarily yield a power-suppression, since the derivative along the direction of k is actually of the high scale. Similarly, a completely massless field can count as hard, if it is highly off-shell. Its propagator would then be of the form $1/k^2$ with $k^2 \sim M^2$. This example leads to the Soft-Collinear Effective Theory, an effective theory developed for hard processes in QCD.

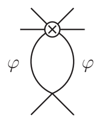
1.4 Matching

In the previous section we established that in order to construct an EFT, we split the degrees of freedom of the UV theory into hard and soft modes and remove hard modes, which we defined in connection with a cutoff Λ. The difference between the effective and the UV theory were the absence of the hard modes and the presence of new interactions of the soft modes in the EFT. These new interactions came with to-be-determined coupling constants, the Wilson coefficients. In a top-down EFT, i.e. when the UV theory is known, these constants can be determined from the following line of reasoning: The UV theory and the EFT must agree in the IR, but will generally differ in the UV.

The fact that they differ is almost too trivial to dwell upon: Processes in the deep UV can involve external fields that we classified as hard modes before, which can obviously not be generated in the EFT. But even scattering processes between the soft modes at hard-scale momentum transfers will in general produce different results. This is due to the fact that the effective interactions in the EFT are constructed in a local OPE, since we argued that the soft modes cannot probe the small non-locality in the short-distance effects generated by hard physics. In the language of Feynman diagrams, we expanded the propagator as

$$\frac{1}{k^2 - M^2} = -\frac{1}{M^2}\left[1 + \mathcal{O}\left(\frac{k^2}{M^2}\right)\right] \tag{1.17}$$

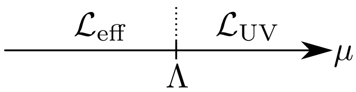
At high energies, this expansion is no longer valid and the effective description breaks down. This breakdown is particularly obvious when computing matrix elements at loop-order, where the EFT will produce matrix elements that are more divergent in the UV than their corresponding counterparts from the UV theory. This can be seen at the example in eq. (1.6). In the effective theory, this graph becomes:



$$= \frac{\lambda\,\mathcal{C}_6}{\Lambda^2}\int\frac{d^dl}{(2\pi)^d}\left(\frac{1}{l^2 - m^2}\right)^2 = \frac{i\lambda\,\mathcal{C}_6}{16\pi^2\Lambda^2}\left\{\frac{1}{\epsilon} + \log\frac{\mu^2}{m^2}\right\}\,, \tag{1.18}$$

where we worked in dimensional regularization with $d = 4 - 2\epsilon$. The quantity $\mathcal{C}_6$ is the Wilson coefficient of the effective ϕ^6 interaction. The ultraviolet divergence present in this graph was absent in eq. (1.6). It originates from regions in the integration over l where the virtual modes becomes hard enough to probe the non-locality in the effective vertex. In this region, the EFT breaks down and produces UV divergences. We can actually make use of that fact later to solve the problems of the aforementioned large logarithms.

We have yet to determine the Wilson coefficients of the EFT. To this end, let us return to the statement from which we started and focus on the part of it we have not discussed: The UV theory and the EFT must agree in the IR.



At the cutoff scale Λ at which we integrated out the hard modes, $\mathcal{L}_{\text{UV}}$ and $\mathcal{L}_{\text{eff}}$ must produce consistent matrix elements. Given that the full theory $\mathcal{L}_{\text{UV}}$ is known, this fixed the couplings of $\mathcal{L}_{\text{eff}}$ order by order in power-counting and in perturbation theory. This producedure is called *matching* and the cutoff scale, Λ is often also called *matching scale*.

Matching can be done through several methods. By far the most common one is diagrammatic. One computes matrix elements in both the effective and UV theories and equates them to detmermine the coupling constants of the effective Lagrangian. Note that this does not only apply to the Wilson coefficients of the "new" effective operators but also to the coupling constants of operators that both $\mathcal{L}_{\text{eff}}$ and $\mathcal{L}_{\text{UV}}$ share. For example, the coupling λ in our Lagrangian (1.4) will not be the same as the λ in the UV Lagrangian (1.2). Instead, it will receive corrections from virtual hard modes.

Another way of performing the matching is the background field method. In this method, fields are separated into the classical fields and quantum fluctuations. One can then integrate out the hard modes by solving the path integral for them explicitely. See section 1.6 for an introduction.

1.4.1 Matching at tree-level: Muon decay

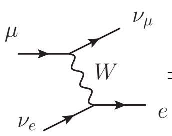
A classic example of an effective theory is the Fermi theory of muon decay. The decay of the muon $\mu \to e\bar{\nu}_e\nu_\mu$ proceeds through a virtual W boson in the SM. The momentum transfer is however much lower than the mass of the W boson, $m_W = 80$ GeV, since it is governed by the muon mass $m_\mu = 105$ MeV. Therefore, the interaction is well described by an effective four-fermion interaction of the form:

$$\mathcal{L}_{\text{eff}} \supset -\frac{4G_F}{\sqrt{2}}(\bar{\nu}_\mu\gamma^\alpha P_L\mu)(\bar{e}\gamma_\alpha P_L\nu_e)\,, \tag{1.19}$$

where G_F is Fermi's constant. By power-counting the fields we can see that it has mass dimension $G_F \sim \mathcal{O}(\text{GeV}^{-2})$. To connect it to our previous (and nowadays more standard) notation it corresponds to:

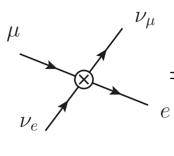
$$-\frac{4G_F}{\sqrt{2}} \equiv \frac{\mathcal{C}_{\mu e}}{\Lambda^2}\,. \tag{1.20}$$

Before the W boson was discovered, G_F would be a constant extracted from measuring the muon decay rate. Today, we know the UV theory to the Lagrangian (1.19) - the Standard Model. We can thus *match* the SM to the Fermi theory and compute G_F in terms of parameters of the SM. To this end, let us compute the matrix elements of muon decay in the full theory and the effective theory. In the SM we have:



$$\begin{aligned} &= \bar{u}(p_\nu)\left(\frac{ig}{\sqrt{2}}\gamma^\alpha P_L\right)u(p_\mu)\cdot\frac{-ig_{\alpha\beta}}{(k_e+k_\nu)^2-m_W^2}\cdot\bar{u}(k_e)\left(\frac{ig}{\sqrt{2}}\gamma_\alpha P_L\right)v(k_\nu) \\ &= \frac{ig^2}{2(q^2-m_W^2)}\left[\bar{u}(p_\nu)\gamma^\alpha P_L u(p_\mu)\right]\left[\bar{u}(k_e)\gamma_\alpha P_L v(k_\nu)\right] \\ &= -\frac{ig^2}{2m_W^2}\left[\bar{u}(p_\nu)\gamma^\alpha P_L u(p_\mu)\right]\left[\bar{u}(k_e)\gamma_\alpha P_L v(k_\nu)\right]+\mathcal{O}\left(\frac{q^2}{m_W^2}\right)\,. \end{aligned} \tag{1.21}$$

In the second line we have defined $q = k_e + k_\nu$ and in the last line we have expanded around $m_W \gg q^\mu$. The next step is to compute the matrix element for this process in the effective theory. It can be directly read off from the Lagrangian (1.19):



$$= -\frac{4iG_F}{\sqrt{2}}\left[\bar{u}(p_\nu)\gamma^\alpha P_L u(p_\mu)\right]\left[\bar{u}(k_e)\gamma_\alpha P_L v(k_\nu)\right]\,. \tag{1.22}$$

Equating (1.21) and (1.22), we find:

$$G_F = \frac{g^2}{4\sqrt{2}m_W^2}\,, \quad \text{or alternatively} \quad \mathcal{C}_{\mu e} = -\frac{g^2}{2}\,. \tag{1.23}$$

Here we set $\Lambda = m_W$ is the cutoff scale of the Fermi theory: It looses validity at momentum transfers close to the W-boson mass. We have computed the leading-power matching of the muon decay to the Fermi theory.

1.4.2 Matching at one-loop and the method of regions

Let us now carry out an example at loop-level. As a toy theory, we will go back to our model of two real scalars. However, we will modify the way in which they interact:

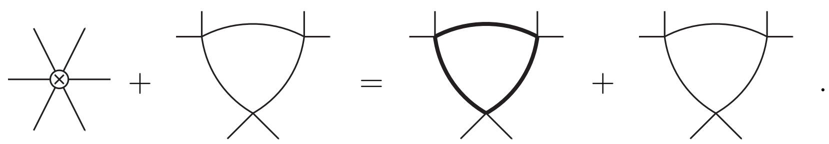
$$\mathcal{L}_{\text{int}} = \frac{\lambda_1}{4!}\varphi^4 + \frac{\lambda_2}{4}\varphi^2\Phi^2\,. \tag{1.24}$$

Now after removing the heavy scalar Φ, the effective Lagrangian has an operator to generate the six-point function for φ:

$$\mathcal{L}_{\text{eff}} \supset \frac{\mathcal{C}_6}{6!}\varphi^6\,. \tag{1.25}$$

The one-loop matching to this operator gives us the following diagrammatic matching equation:

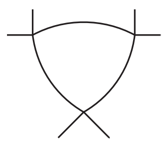
$$i\mathcal{A}_{EFT} \quad = \quad i\mathcal{A}_{\text{full}}$$



(1.26)

Before doing even a single line of computation, we seem to see that the second diagram on each side of the equation will give an identical contribution and will cancel out of the matching. The Wilson coefficient $\mathcal{C}_6$ would therefore be purely determined from the loop diagram with a heavy scalar. This seems reasonable, since we have stated earlier that the Wilson coefficients encode the hard-scale physics that we have integrated out from the theory. On the other hand, our previous comment about the origin of UV divergences in the EFT matrix elements should remind you of the fact that there are also regions of the loop integration in which $l^2 \sim \Lambda^2$ and thus the light fields become highly virtual - i.e. hard - modes. To see that these do not give matching corrections, we need to look at the loop integral a little more carefully.

The contribution from light fields in the full theory yields an amplitude of the following form:



$$= -\lambda_1^3 N \int \frac{d^d l}{(2\pi)^d}\left(\frac{1}{l^2 - m^2}\right)^3\,, \tag{1.27}$$

where N is a symmetry factor from interchanging the extern legs. To continue, we split the loop integral into the regions where the virtual fields are hard and soft:

$$I_{\text{full}} = I_H + I_S$$

$$\int \frac{d^d l}{(2\pi)^d}\left(\frac{1}{l^2 - m^2}\right)^3 = \left[\int \frac{d^d l}{(2\pi)^d}\left(\frac{1}{l^2 - m^2}\right)^3\right]_H + \left[\int_S \frac{d^d l}{(2\pi)^d}\left(\frac{1}{l^2 - m^2}\right)^3\right]_S\,. \tag{1.28}$$

In the effective theory, this diagram looks identical, but should not have the hard modes propagating in the loop, so it is given purely by I_S. The full theory on the other hand has both I_S and I_H. Consequently, the contribution from I_S cancels in the matching and the contribution to the Wilson coefficient is given by the piece I_H only. This, again, is intuitive: The Wilson coefficients are supposed to encode the hard dynamics only. Soft physics are described by the dynamic degrees of freedom in the effective theory. We have already arrived at an important conclusion:

The Wilson coefficients are fully determined by the hard region of the full theory amplitude.

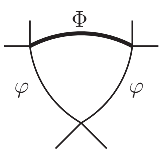
The last thing we need to work out is the computation of I_H. In dimensional regularization, the computation is done by simply power-counting l^2 of the hard scale, meaning $l^2 \sim \Lambda^2 \gg m^2$ and expanding the integrand around this limit. At leading power we obtain:

$$\left[\int \frac{d^d l}{(2\pi)^d}\left(\frac{1}{l^2-m^2}\right)^3\right]_H = \int \frac{d^d l}{(2\pi)^d}\left(\frac{1}{l^2}\right)^3 + \mathcal{O}\left(\frac{m^2}{\Lambda^2}\right) . \tag{1.29}$$

This integral is completely scaleless and thus vanishes in dimensional regularization. There is no matching correction from the hard region of the light fields.

1.4.3 Closing remarks on the method of regions

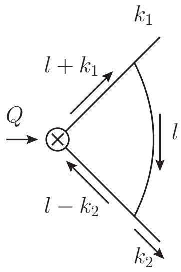
Before we move on to renormalization, we should go through a list of important remarks. If we look at tree-level matching from a diagrammatic perspective, it might seem that the corresponding diagrams in the effective theory are the same as the ones in the full theory, but with the hard propagators contracted to a point. This is of course not true: Loop diagrams with both hard and soft modes propagating in the loop have hard regions. For example, the loop diagram we saw earlier in the hard region is:



$$\begin{aligned} &= -\lambda g^2 N\left[\int \frac{d^d l}{(2\pi)^d}\frac{1}{l^2-M^2}\left(\frac{1}{l^2-m^2}\right)^2\right]_H \\ &= -\lambda g^2 N\int \frac{d^d l}{(2\pi)^d}\frac{1}{l^2-M^2}\left(\frac{1}{l^2}\right)^2 . \end{aligned} \tag{1.30}$$

This integral is not scaleless and thus gives a contribution to the Wilson coefficient of the effective six-point operator.

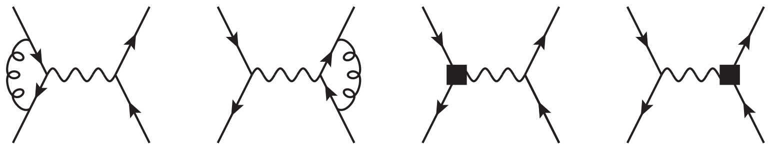
Another remark is that there are can in general be more than just hard and soft regions. For example, for processes at high energies, where the external momenta have large components, there are regions in which only individual light fields become hard. Consider a process of a heavy particle decaying to two very light particles, with momenta $k_1^2 \sim k_2^2 \sim 0$, but $(k_1+k_2)^2 = Q^2$. Then at one-loop we have an integral of the form:



$$\sim \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2} \frac{1}{(l+k_1)^2} \frac{1}{(l-k_2)^2} \,. \tag{1.31}$$

This integral has regions in which $l^\mu \propto k_i^\mu$, and consequently $l^2 \sim 0$ without every component of l^μ being small. At the example of $l^\mu \propto k_1^\mu$, this means that the propagators with l^2 and $(l+k_1)^2$ count as soft while $(l-k_2)^2 \sim (k_1-k_2)^2 \sim Q^2$ count as hard. These regions are called *collinear regions* and are relevant regions for SCET, an effective theory we mentioned a few times before.

Finally, we should address a subtlety about the nature of the divergences we encounter in these computations. We have established that the matching coefficients are determined by the hard region. In our computation, we have not distinguished between IR and UV divergences. Consider the case of the weak decay of a quark, mediated through a W boson, similar to the muon decay discussed above. The difference is that a four-quark transition receives important corrections from QCD. When computing these diagrams in the full theory, some diagrams produce UV divergences that are cancelled by the counterterms. In the example at hand, the diagrams in question are:



Defining the tree-level amplitude as $i\mathcal{A}_0$, each counterterm graph gives

$$i\mathcal{A}_{\delta_g} = i\mathcal{A}_0 \left(-\frac{\alpha_s C_F}{4\pi} \frac{1}{\epsilon_{\rm UV}} \right) \,, \tag{1.32}$$

irrespective of the region we are considering. The corresponding diagram producing the divergence that this term is supposed to cancel does not exist in the matching computation - it is scaleless and thus vanishes. We seem to have a leftover UV divergence. The resolution to this puzzle is that we were not careful enough about the scaleless integrals. They vanish in dimensional regularization only if we do not distinguish between IR and UV divergences. Treating them more carefully, each of the two vertex-correction diagrams gives a contribution

$$i\mathcal{A}_V = i\mathcal{A}_0 \left(\frac{\alpha_s C_F}{4\pi} \left[\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}} \right] + \mathcal{O}(\epsilon^0) \right) \,. \tag{1.33}$$

Thus, these graphs cancel the UV divergences from the counterterms, leaving an IR divergence. Equally, there will be matrix elements in the effective theory yielding scaleless integrals, whose UV divergences will be cancelled by counterterms for the effective operators. In the end, the matching coefficients will only contain IR divergences, as they should - see the next section.

1.5 Renormalization and Resummation

The concepts of effective field theories and renormalization are closely connected to each other. First, we will find that through the renormalization of the effective theory, we will be able to solve to problem of the large logarithms that we briefly touched upon earlier. In the end, the idea of EFTs will give a modern perspective on renormalization in general.

1.5.1 Renormalizability

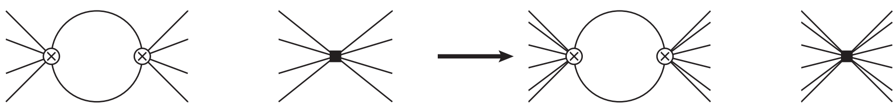
While we have discussed loops and even seen divergences in the matching. In order to remove these divergences, the theory needs to be renormalized. However, our infinite sum of operator products in the OPE will contain operators with mass dimension $k > 4$ and they are thus not renormalizable in the classical sense. First, let us get some more terminology in place. In our OPE, each operator's contribution to the action is (for a process with energies $E \ll \Lambda$):

$$\delta S_{(i,k)} \sim \mathcal{C}_{(i,k)} \left(\frac{E}{\Lambda}\right)^{k-4} , \tag{1.34}$$

with k being the operator dimension. Operators are then classified by their importance at low energies ($E \to 0$):

k	low-energy behavior	classical renormalizability	name
< 4	grows	super-renormalizable	*relevant*
$= 4$	constant	renormalizable	*marginal*
> 4	decreases	non-renormalizable	*irrelevant*

In our OPE, we allowed for operators of dimension $k > 4$, which are not renormalizable in the classical sense, meaning that we would need an infinite number of counterterms to cancel the divergences. This is easily seen by inserting a non-renormalizable operator twice in a loop-graph:



For example taking an operator of the form $\Delta\mathcal{L}_{\text{eff}} = g\varphi^6/\Lambda^2$, we can draw a diagram with two insertions of this operator and connecting four legs to form a loop-graph. This graph will have $2\times 6-4 = 8$ external legs and will be divergent. This divergence needs to be absorbed into a counterterm of a φ^8 operator. This in turn can be used to generate a divergent 10-point amplitude which introduces higher operators and so on.

This apparent problem with our effective operators is cured once we implement power-counting: The divergence from inserting the φ^6 twice gives an amplitude of

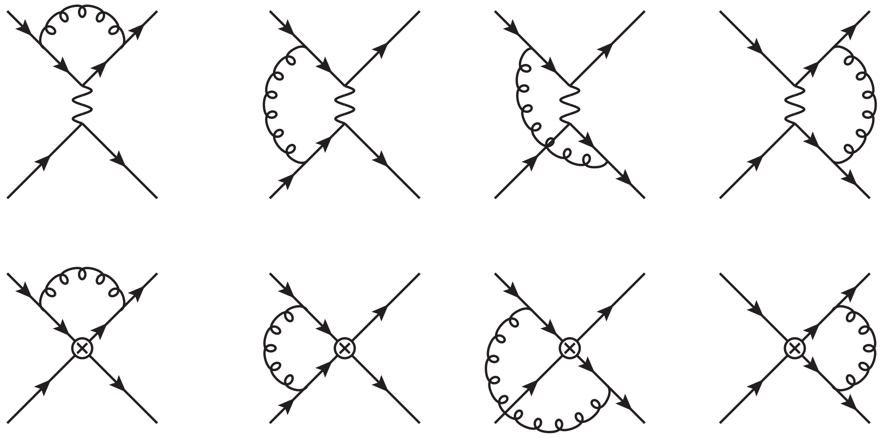
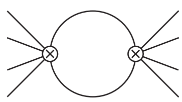


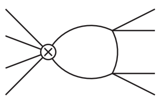
Figure 1.1: Example diagrams for the one-loop matching of the Fermi-theory. The top row depicts diagrams in the full SM whereas the bottom row shows corresponding diagrams in the EFT.

the form:



$$\sim \frac{g^2}{\Lambda^4} \int \frac{d^d l}{(2\pi)^d} \left(\frac{1}{l^2}\right)^2 = \frac{g^2}{\Lambda^4} \cdot \left(\frac{i}{16\pi^2\epsilon} + \mathcal{O}(\epsilon^0)\right) , \tag{1.35}$$

so it is of higher order in power-counting than the operators we inserted. Thus, if we decide to stop our OPE at dimension-six, then this correction is dropped. The only graph that does indeed contribute at this order in power-counting is the one where instead of the second operator insertion we insert a (renormalizable) quartic coupling twice. But this graph has more propagators, regulating it in the UV and rendering it finite:



$$\sim \frac{g\lambda^2}{\Lambda^2} \int \frac{d^d l}{(2\pi)^d} \left(\frac{1}{l^2}\right)^3 = \mathcal{O}(\epsilon^0) . \tag{1.36}$$

Therefore, once we stick consequently to our power-counting, our theory contains only a finite amount of counterterms and is thus renormalizable, in the modern sense.

1.5.2 Resummation of large logarithms

Now that we know the effective theory can be renormalized, we should discuss the renormalization in detail. We will do this at a realistic example, the quark transition $b \to c\bar{u}d$. In the full SM, this transition occurs through the weak decay of the b-quark through a virtual W-boson. Again, the momentum flow through the W-boson propagator is small compared to the mass $q^2 \sim m_b^2 \ll m_W^2$, so we can match it onto a set of effective operators obtained after integrating out the W. The tree-level contribution receives important corrections at one-loop order in QCD. Evaluating graphs like the ones in the top row of Fig. 1.1, we find the full $\mathcal{O}(\alpha_s)$-result:

$$\begin{aligned} i\mathcal{A}_{\text{full}} = -iG \Bigg\{ &(\bar{u}(p_c)\gamma^\mu P_L u(p_b))(\bar{u}(p_d)\gamma_\mu P_L v(p_u)) \left[1 - \frac{\alpha_s}{4\pi}\left(\log\frac{q^2}{m_W^2} + \frac{3}{2}\right)\right] \\ &+(\bar{u}(p_c)\gamma^\mu P_L v(p_u))(\bar{u}(p_d)\gamma_\mu P_L u(p_b)) \left[\frac{3\alpha_s}{4\pi}\left(\log\frac{q^2}{m_W^2} + \frac{3}{2}\right)\right]\Bigg\}, \end{aligned} \tag{1.37}$$

with $G = \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^*$. Note the large logarithm of the scale ratio q^2/m_W^2, rendering our perturbation series poorly convergent. Higher-order corrections will introduce terms of the form $\alpha_s^k \log^k(q^2/m_W^2)$, which will be large as well. We would need to go to very high loop-order to get a result with small perturbative uncertainty.

It would be desirable if we could introduce a new counting scheme: Instead of doing perturbation theory in α_s, we could introduce a parameter η and assign the counting

$$\alpha_s \sim \eta\,, \qquad\qquad \log\frac{q^2}{m_W^2} \sim \frac{1}{\eta}\,. \tag{1.38}$$

If we could now find a way to rearrange the perturbation series from an expansion in α_s into an expansion in η, then the resulting series would be more convergent. To resum the perturbation series in η, we would need the logarithmic terms to all orders in perturbation theory, meaning we would have to compute an infinite number of loop graphs.

The way out is to utilize the large separation between q^2 and m_W^2, integrate out the W and match it to a basis of operators. By the argument of regions, the Wilson coefficients can only be functions of the hard scale. Defining the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \mathcal{C}_1(\mu)\mathcal{O}_1 + \mathcal{C}_2(\mu)\mathcal{O}_2\,, \tag{1.39}$$

with the effective operators

$$\mathcal{O}_1 = (\bar{c}_L\gamma^\mu b_L)(\bar{d}_L\gamma_\mu u_L)\,, \qquad\qquad \mathcal{O}_2 = (\bar{c}_L\gamma^\mu u_L)(\bar{d}_L\gamma_\mu b_L)\,, \tag{1.40}$$

we can indeed evaluate the full theory diagrams once more in the hard region. In this case, we have:

$$\begin{aligned} i\mathcal{A}_{\text{full}}^{\text{hard}} = -iG \Bigg\{ &(\bar{u}(p_c)\gamma^\mu P_L u(p_b))(\bar{u}(p_d)\gamma_\mu P_L v(p_u)) \\ &\times \left[1 - \frac{\alpha_s}{4\pi}\left(\frac{11}{3}\frac{1}{\epsilon_{\text{IR}}} + \log\frac{\mu_{\text{IR}}^2}{m_W^2} + \frac{8}{3}\log\frac{\mu_{\text{IR}}^2}{\mu_{\text{UV}}^2} + \frac{3}{2}\right)\right] \\ &+(\bar{u}(p_c)\gamma^\mu P_L v(p_u))(\bar{u}(p_d)\gamma_\mu P_L u(p_b)) \left[\frac{3\alpha_s}{4\pi}\left(\frac{1}{\epsilon_{\text{IR}}} + \log\frac{\mu_{\text{IR}}^2}{m_W^2} + \frac{3}{2}\right)\right]\Bigg\}, \end{aligned} \tag{1.41}$$

which we can directly match onto the Lagrangian (1.39). Here we have assigned indices IR and UV not only to the dimensional regulators ϵ_i, but also to the corresponding renormalization scale μ_i. The $1/\epsilon_{\text{UV}}$ terms are cancelled by the QCD counterterms, as discussed earlier. The matching yields:

$$\begin{aligned}\mathcal{C}_1(\mu) &= -G\left\{1-\frac{\alpha_s(\mu)}{4\pi}\left(\frac{11}{3}\frac{1}{\epsilon_{\mathrm{IR}}}+\log\frac{\mu_{\mathrm{IR}}^2}{m_W^2}+\frac{8}{3}\log\frac{\mu_{\mathrm{IR}}^2}{\mu_{\mathrm{UV}}^2}+\frac{3}{2}\right)\right\}\,,\\ \mathcal{C}_2(\mu) &= -G\left\{\frac{3\alpha_s(\mu)}{4\pi}\left(\frac{1}{\epsilon_{\mathrm{IR}}}+\log\frac{\mu_{\mathrm{IR}}^2}{m_W^2}+\frac{3}{2}\right)\right\}\,.\end{aligned} \tag{1.42}$$

We are still keeping the renormalization scales μ_{IR} and μ_{UV} separate. When comparing the amplitudes (1.37) and (1.41), we see that the logarithmic dependence on q^2 has been turned into an IR divergence, $\log q^2 \to 1/\epsilon + \log\mu^2$. This makes sense, since the loop diagrams in the hard region lack the IR regulators that rendered the full theory result finite. The Wilson coefficients therefore now contain new divergences, associated to the fact that we separated the theory into a UV and an IR piece.

The logarithm that the coefficients now contain can either be large or small, depending on the scale μ at which the Wilson coefficients are evaluated. And Evaluating the Wilson coefficients at the natural scale of the process $\mu^2 = q^2$ restores the full theory result (up to the $1/\epsilon$-term, which can be absorbed into a counterterm). However, we can evaluate the Wilson coefficients at a different scale $\mu^2 \neq q^2$ and use the renormalization group (RG) to take it to the scale $\mu^2 = q^2$. This is the key to solving the problem of large logarithms: At the scale $\mu^2 = m_W^2$, the large logarithms are exactly zero. We then set up and solve the differential equations obtained by imposing invariance under the change of μ - the RG equation (RGEs) - and obtain the correct result at the low scale. Since at $\mu^2 = m_W^2$, the expression trivially contains all orders of $\alpha_s^k \log^k(m_W^2/m_W^2)$, solving the RGEs to get to the low scale will give us a result that also contains the large logarithms $\alpha_s^k \log^k(q^2/m_W^2)$. We say that the RG running resums the large logarithms.

Let us now return to our example and demonstrate the resummation explicitely. First, we renormalize our Lagrangian in the usual way, rescaling the fields:

$$\psi \to \sqrt{Z_\psi}\psi\,, \tag{1.43}$$

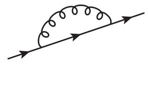
then we have:

$$\mathcal{L}_{\mathrm{eff}} = Z_\psi^2 \mathcal{C}_1^{(0)}\mathcal{O}_1 + Z_\psi^2 \mathcal{C}_2^{(0)}\mathcal{O}_2\,. \tag{1.44}$$

Now we assign

$$\mathcal{C}_i(1+\delta_{\mathcal{C}_i}) = Z_\psi^2 \mathcal{C}_i^{(0)}\,, \tag{1.45}$$

to find our counterterms. First, we need the wavefunction renormalization of the quarks to $\mathcal{O}(\alpha_s)$. It is a straightforward computation and yields:



$$\begin{aligned} &= (-4\pi\alpha_s C_F)\int\frac{d^d l}{(2\pi)^d}\frac{\gamma^\mu(\not{l}+\not{p})\gamma_\mu}{l^2(l+p)^2} = i\not{p}\,\frac{\alpha_s C_F}{4\pi\epsilon}\,,\\ &\Rightarrow Z_\psi = 1-\frac{\alpha_s(\mu)}{3\pi\epsilon}\,.\end{aligned} \tag{1.46}$$

Next, we need the counterterms of the effective couplings $\mathcal{C}_i$. To this end, we need to compute the divergences of the matrix elements of the EFT, corresponding to graphs like the ones in the bottom row of Fig. 1.1. We find:

$$\begin{aligned} i\mathcal{A}_{\mathrm{EFT}}^{(UV)} = i\Bigg\{&(\bar{u}(p_c)\gamma^\mu P_L u(p_b))(\bar{u}(p_d)\gamma_\mu P_L v(p_u))\mathcal{C}_1(\mu)\left(1+\frac{\alpha_s(\mu)}{4\pi\epsilon_{\mathrm{UV}}}\left[\frac{11}{3}-3\frac{\mathcal{C}_2(\mu)}{\mathcal{C}_1(\mu)}\right]\right) \\ &+(\bar{u}(p_c)\gamma^\mu P_L v(p_u))(\bar{u}(p_d)\gamma_\mu P_L u(p_b))\mathcal{C}_2(\mu)\left(1+\frac{\alpha_s(\mu)}{4\pi\epsilon_{\mathrm{UV}}}\left[\frac{11}{3}-3\frac{\mathcal{C}_1(\mu)}{\mathcal{C}_2(\mu)}\right]\right)\Bigg\}. \end{aligned} \tag{1.47}$$

Notice that once we insert (1.42) into this expression, the IR divergences of the bare Wilson coefficients cancel the UV divergences of the matrix element. This is because the product of both is an object that contains both the hard and soft region, which is the full result and thus finite (after the UV theory is renormalized).

We can read off the counterterms for the Wilson coefficients from eq. (1.47), they are

$$\begin{aligned} \delta_{\mathcal{C}_1} &= -\frac{\alpha_s(\mu)}{\pi\epsilon}\left(\frac{11}{12}-\frac{3\mathcal{C}_2(\mu)}{4\mathcal{C}_1(\mu)}\right)+\mathcal{O}(\alpha_s^2)\,, \\ \delta_{\mathcal{C}_2} &= -\frac{\alpha_s(\mu)}{\pi\epsilon}\left(\frac{11}{12}-\frac{3\mathcal{C}_1(\mu)}{4\mathcal{C}_2(\mu)}\right)+\mathcal{O}(\alpha_s^2)\,. \end{aligned} \tag{1.48}$$

Note that the divergence of $\mathcal{C}_2$ is proportional to $\mathcal{C}_1$ and vice versa. So even when there is no tree-level contribution to $\mathcal{C}_2$, there is a loop-induced contribution. We say $\mathcal{C}_2$ is *radiatively generated*.

With all this in place, we can write down the RG equations for the coefficients and solve them. First, demand that the bare coefficients $\mathcal{C}_i^{(0)}$ are independent of μ:

$$\frac{d\mathcal{C}_i^{(0)}}{d\log\mu} = \mu\frac{d}{d\mu}\left(\mathcal{C}_i\frac{Z_{\mathcal{C}_i}}{Z_\psi^2}\right) \overset{!}{=} 0\,. \tag{1.49}$$

By using the fact that the counterterms depend on the scale only through α_s, the equation can be rearranged into

$$\begin{aligned} \mu\frac{d}{d\mu}\mathcal{C}_i &= -\frac{Z_\psi^2}{Z_{\mathcal{C}_i}}\mathcal{C}_i\left(\mu\frac{d\alpha_s}{d\mu}\right)\frac{d}{d\alpha_s}\left(\frac{Z_{\mathcal{C}_i}}{Z_\psi^2}\right) \\ \mu\frac{d}{d\mu}\mathcal{C}_i &= -\mathcal{C}_i\left(-2\alpha_s\epsilon\right)\frac{d}{d\alpha_s}\left(\delta Z_{\mathcal{C}_i}-2\delta Z_\psi\right) \end{aligned} \tag{1.50}$$

In the second line we have inserted the leading term in the QCD beta function and dropped the prefactors since they differ from unity only at $\mathcal{O}(\alpha_s)$. Now we can insert the counterterms and get the evolution equations. It makes sense to define a vector $\vec{\mathcal{C}}(\mu) = (\mathcal{C}_1(\mu), \mathcal{C}_2(\mu))$ and write the equation in the compact form:

$$\mu\frac{d}{d\mu}\vec{\mathcal{C}}(\mu) = \frac{\alpha_s(\mu)}{2\pi}\begin{pmatrix}-1 & 3\\ 3 & -1\end{pmatrix}\cdot\vec{\mathcal{C}}(\mu) \tag{1.51}$$

With the QCD beta function $d\alpha_s(\mu)/d\log\mu = -\alpha_s^2(\mu)\beta_0/(2\pi)$, this equation becomes

$$\frac{d}{d\alpha_s}\vec{\mathcal{C}}(\alpha_s) = -\frac{1}{\alpha_s\beta_0}\begin{pmatrix}-1 & 3\\ 3 & -1\end{pmatrix}\cdot\vec{\mathcal{C}}(\alpha_s)\,. \tag{1.52}$$

The solution is given by:

$$\vec{\mathcal{C}}(\mu_l) = V \begin{pmatrix} R^{-4/\beta_0} & 0 \\ 0 & R^{2/\beta_0} \end{pmatrix} V^{-1} \vec{\mathcal{C}}(\mu_h) \tag{1.53}$$

with $R = \alpha_s(\mu_h)/\alpha_s(\mu_l)$ and

$$V = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} . \tag{1.54}$$

We now know the Wilson coefficients at the low scale μ_l as a function of the coefficients at the high scale μ_h. To see that this solution does indeed resum an infinite number of the logarithms $[\alpha_s(\mu_l)\log(q^2/m_W^2)]^k \equiv L^k$, we solve the QCD beta function at one-loop order:

$$\mu \frac{d\alpha_s(\mu)}{d\mu} = -\frac{\alpha_s^2}{2\pi}\beta_0 \qquad \Rightarrow \qquad \alpha(\mu_h) = \frac{\alpha(\mu_l)}{1 - \frac{\alpha(\mu_l)}{4\pi\beta_0}\log\frac{\mu_l^2}{\mu_h^2}} . \tag{1.55}$$

Then we can express our variable R in terms of $\alpha_s(\mu_l)$ to find:

$$R = \frac{1}{1 - \frac{\beta_0}{4\pi}\alpha_s(\mu_l)\log\frac{\mu_l^2}{\mu_h^2}} \tag{1.56}$$

Now seting $\mu_h^2 = m_W^2$ and $\mu_l^2 = q^2$, we see that the logarithms in R are exactly the logarithms L, that we want to resum and that the solution (1.53) indeed is an infinite tower of these logarithms:

$$R^b = \sum_{k=0}^{\infty} L^k \left(\frac{\beta_0}{4\pi}\right)^k \frac{\Gamma(b+k)}{\Gamma(b)\Gamma(1+k)} . \tag{1.57}$$

We say *the renormalization group evolution has resummed the large logarithms.* To see that these really correspond to the large logarithms we saw in eq. (1.37), we can simply stop the sum at $k = 1$. Then we find:

$$\begin{aligned} \mathcal{C}_1(q) &= \mathcal{C}_1(m_W)\left(1 - \frac{\alpha_s(q)}{4\pi}\log\frac{q^2}{m_W^2}\right) , \\ \mathcal{C}_2(q) &= \mathcal{C}_1(m_W)\left(\frac{3\alpha_s(q)}{4\pi}\log\frac{q^2}{m_W^2}\right) , \end{aligned} \tag{1.58}$$

which indeed reproduces the logarithmic terms.

1.5.3 Concluding remarks

Let us summarize what we have learned in a sentences without getting lost in lengthy computations.

Resummation: We have explicitely seen that multi-scale loop-amplitudes will generically contain logarithms of ratios of the various scales. For strong scale hierarchies, these logarithms can be problematic for the convergence of our perturbation expansion. Instead of counting them as suppressed by the coupling α, we count them as inverse powers of the coupling such the combination of both is of the same order as the leading term. Schematically, we rearrange our series as follows:

$$\begin{aligned} &1+\frac{\alpha}{4\pi}\left(a_1\log\lambda+c_1\right)+\frac{\alpha^2}{16\pi^2}\big(a_2\log^2\lambda+b_2\log\lambda+c_2\big)+\dots \\ &=\left(1+a_1\frac{\alpha}{4\pi}\log\lambda+a_2\frac{\alpha^2}{16\pi^2}\log^2\lambda\right)+\left(\frac{\alpha}{4\pi}c_1+b_2\frac{\alpha^2}{16\pi^2}\log\lambda\right)+\frac{\alpha^2}{16\pi^2}c_2\,. \end{aligned} \tag{1.59}$$

In the first line, terms are sorted by their order in α, whereas in the second line, they are ordered by our new counting scheme. The two schemes are called *fixed-order perturbation theory* (first line) and *RG-improved perturbation theory* (second line). The different orders in the fixed-order scheme are referred to as leading order (LO), next-to-leading order (NLO), et cetera, for fixed-order perturbation theory. When computations are done in RG-improved perturbation theory, one refers to the different orders as leading log (LL), next-to-leading log (NLL), et cetera. Thus when results are being quoted as a certain order in logarithmic counting, it is done so to indicate that logarithms have been resummed.

Using the RG we have shown that we can indeed obtain the logarithmic terms to all orders in perturbation theory: We have removed one of the scales by splitting the theory into several theories that each only contain a single scale. Matching and the method of region immediately shows how the full result *factorizes* into a product of something that contains only the hard scale (the Wilson coefficients) and something that contains only the soft scale (the EFT matrix element). Since we are free to do this splitting at any scale, we can choose one at which the logarithms are small or even vanish so the result contains all powers of the logarithms to a good approximation. We then use the renormalization group to move this scale to the physical value. Since the RG evolution is the solution to imposing independence of this scale, we get a result at a different scale that still contains all powers of these logarithms, even if they are no longer small.

There is a range of effective theory constructions that are completely driven by resummation in the sense that some result contains large logarithms, that need to be resummed. Then one identifies the relevant physics contributing the different scales and build an EFT that separates them. By renormalizing the theory, one resums the large logarithms. Many factorization theorems, especially in QCD, can be derived this way.

Renormalization: The fact that the theory continuously changes with the variation of the scale μ goes back to our original method of splitting the theory into regions, $\phi = \phi_H + \phi_S$. By lowering the scale, we continuously shuffle Fourier modes of the fields from ϕ_S to ϕ_H, even though we might not cross thresholds of heavy particles being integrated out. With lower and lower scales, the remaining modes of the theory are of longer and longer wavelengths. The shortest distance these fields can resolve thus becomes longer and longer. Physics at shorter distances then appear as local effects. This is reflected in the running of the short-distance coefficients, the couplings of the theory.

The above sentiment is not limited to EFTs, it holds for the renormalization group in general. It is then safe to say that every renormalized theory is an effective theory. We can even interpret counterterms in a way less ad-hoc than they appeared before: We know that "physics is finite". When our loop computations give divergent result, we are simply extrapolating our theory into a region in which it probes short-distance effects that we are approximating to be local. In the full theory, some new effect - we do not know what - becomes relevant at high scales, preventing the amplitude from diverging. If these new high-scale effects are integrated out, they appear as an effective interaction cancelling the divergence in the effective theory, just as our Wilson coefficients cancel the divergences of the EFT matrix elements.

1.6 Addendum: Matching with the background field method

In section 1.3, we introduced the idea of integrating out the hard modes from the theory to define the effective theory. Then, in sections 1.4 and 1.5, we implemented our ideas by matching matrix elements of the effective theory to matrix elements of the full theory. The decoupling of hard modes happened in an ad-hoc way, simply by matching up a Lagrangian that did not contain any hard modes to the full theory. In this section, we perform the one-loop matching by actually carrying out the function integral over the hard modes. This can be done by employing the background field method.

1.6.1 Derivation of the formalism

Remember that we want to compute the Wilsonian effective action:

$$\int \mathcal{D}\phi_H \exp\left\{iS(\phi_S, \phi_H)\right\} = \exp\left\{iS_\Lambda(\phi_S)\right\} . \tag{1.60}$$

To this end, we want to find a way of evaluating the functional integral on the left-hand side of explicitely. We can do this even at one-loop order. In the background field method, one separates fields into their classical parts (the background field) and quantum fluctuations around these:

$$\phi = \phi_{\rm cl} + \eta \,, \tag{1.61}$$

where $\phi_{\rm cl}$ here denotes the classical field and η denotes the quantum fluctuations. Do not confuse this with the separation of modes we discussed earlier, this step comes in at a later stage. The classical fields are defined by the part of the fields satisfying the classical equations of motion:

$$\left.\frac{\delta\mathcal{L}(\phi)}{\delta\phi}\right|_{\phi=\phi_{\rm cl}} = 0\,. \tag{1.62}$$

To compute the effective action, we start with the action $iS = i\int d^4x\mathcal{L}(\phi_{\rm cl}+\eta)$ and expand it around the classical fields:

$$\begin{aligned} iS = i\int d^4x\,\mathcal{L}(\phi) =& i\int d^4x\,\mathcal{L}(\phi_{\rm cl}) + i\int d^4x\,\left(\left.\frac{\delta\mathcal{L}(\phi)}{\delta\phi(x)}\right|_{\phi=\phi_{\rm cl}}\right)\eta(x) \\ &+\frac{i}{2}\int d^4x\,d^4y\,\eta(x)\left(\frac{\delta^2\mathcal{L}(\phi)}{\delta\phi(x)\,\delta\phi(y)}\right)\eta(y)+\dots \end{aligned} \tag{1.63}$$

The first term in the expansion is the Lagrangian evaluated for classical fields. The linear term in the fluctuations vanishes by definition, since its coefficient is zero by the equations of motion (1.62). The second-order term contains the leading quantum corrections to the classical component. In the Language of Feynman diagrams, this term contains the one-loop corrections.

The next step is to split the fields into hard and soft modes. Next, we want to remove the hard modes from each term in the Lagrangian (1.63), order by order in η. In the leading term, it suffices to substitute hard modes by their classical values. This means, we simply solve the equations of motion for them

$$\frac{\delta L}{\delta\phi_{{\rm cl},H}} = \frac{\partial\mathcal{L}}{\partial\phi_{{\rm cl},H}} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_{{\rm cl},H})} \stackrel{!}{=} 0\,. \tag{1.64}$$

and solve for $\phi_{{\rm cl},H}$. The evaluation of the quadratic term is slightly more complicated, but still feasible. First, let us write the modes in a vector notation, $\phi_i = (\phi_H, \phi_S)$. Then the quadratic term in (1.63) becomes:

$$\frac{1}{2}\eta\left(\frac{\delta^2\mathcal{L}(\phi)}{\delta\phi\,\delta\phi}\right)\eta = \frac{1}{2}\eta_i\left(\frac{\delta^2\mathcal{L}(\phi)}{\delta\phi_i\,\delta\phi_j}\right)\eta_j \equiv \frac{1}{2}\eta_i\,\mathcal{Q}_{ij}\,\eta_j\,. \tag{1.65}$$

Here we have defined the fluctuation operator:

$$\mathcal{Q} = \begin{pmatrix}\Delta_H & X_{SH}^\dagger \\ X_{SH} & \Delta_S\end{pmatrix}\,. \tag{1.66}$$

The off-diagonal blocks X_{SH} mix soft and hard modes. They should in principle not exist but are a residue of being too simplistic in the way we assigned fields to regions. In our familiar example of a theory of heavy and light scalars, we will have these mixed terms if we identify all light fields with soft modes and all heavy fields with hard modes. We know however that we can have light fields with large virtualities, which we should count as part of the hard field content. To correctly account for this, we simply perform a field redefinition $\eta_i \to \tilde{\eta}_i = V_{ij}\eta_j$ that puts $\mathcal{Q}$ in a block-diagonal form. Then the fields $\tilde{\eta}_H$ contain light fields with hard invariant masses k^2. The transformation matrix V is given by:

$$V = \begin{pmatrix}\mathbf{1} & 0 \\ -\Delta_S^{-1}X_{SH} & \mathbf{1}\,,\end{pmatrix} \tag{1.67}$$

leading to the diagonal operator:

$$\tilde{\mathcal{Q}} = V^\dagger\mathcal{Q}V = \begin{pmatrix}\tilde{\Delta}_H & 0 \\ 0 & \Delta_S\end{pmatrix}\,, \tag{1.68}$$

with

$$\tilde{\Delta}_H = \Delta_H - X_{SH}^{\dagger}\Delta_S^{-1}X_{SH}\,. \tag{1.69}$$

Now we can perform the integration over the hard modes:

$$e^{iS_\Lambda} = \int \mathcal{D}\tilde{\eta}_H \,\exp\left\{\frac{i}{2}\int d^4x\, d^4y\, \tilde{\eta}_H\cdot\tilde{\mathcal{Q}}_H\cdot\tilde{\eta}_H\right\} = \left(\det\left[-\frac{i}{2}\tilde{\mathcal{Q}}_H\right]\right)^{-c}\,, \tag{1.70}$$

where $c = 1/2$ for bosons and $c = -1$ for fermions. In the case of mixed operators, the operator must be further decomposed and diagonalized. Now the action can be written as:

$$S_\Lambda = ic\,\log\det\left(\tilde{\Delta}_H\right)\,. \tag{1.71}$$

Our remaining task is now to evaluate the determinant in the above expression. First note that a determinant can be written as the product of eigenvalues:

$$\det A = \prod_i a_i = \exp\left\{\sum_i \log a_i\right\} = \exp\{\mathrm{Tr}\log A\}\,. \tag{1.72}$$

With this we can expand the trace in momentum eigenstates of the operator:

$$\begin{aligned} S_\Lambda &= ic\,\mathrm{tr}\int\frac{d^dk}{(2\pi)^k}\,\langle k|\log\tilde{\Delta}_H|k\rangle \\ &= ic\,\mathrm{tr}\int d^dx\int\frac{d^dk}{(2\pi)^d}e^{-ikx}\,\log(\tilde{\Delta}_H)e^{ikx}\,. \end{aligned} \tag{1.73}$$

The symbol $\mathrm{tr}(A)$ indicates that part of the original trace $\mathrm{Tr}(A)$ has been carried out by the sum over momentum eigenstates. The operator $\tilde{\Delta}_H$ is an operator product depending on the coordinate x and partial derivatives ∂:

$$\tilde{\Delta}_H \equiv \tilde{\Delta}_H(x,\partial)\,. \tag{1.74}$$

These derivatives can act on the exponentials and one can show that:

$$\begin{aligned} S_\Lambda &= ic\,\mathrm{tr}\int d^dx\int\frac{d^dk}{(2\pi)^d}e^{-ikx}\,\log\left(\tilde{\Delta}_H(x,\partial)\right)e^{ikx} \\ &= ic\,\mathrm{tr}\int d^dx\int\frac{d^dk}{(2\pi)^d}\log\left(\tilde{\Delta}_H(x,\partial+ik)\right)\,. \end{aligned} \tag{1.75}$$

1.6.2 A scalar example

It is instructive to evaluate this expression explicitely for a scalar theory. Let us use the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}m^2\varphi^2 + \frac{\lambda_1}{4!}\varphi^4 + \frac{\lambda_2}{4}\varphi^2\Phi^2\,. \tag{1.76}$$

As a first step, let us solve the classical equations of motion to remove Φ at tree-level. We find:

$$\begin{aligned}\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_{\text{cl}})} &= \frac{\partial \mathcal{L}}{\partial \Phi_{\text{cl}}} \\ -\partial^2 \Phi_{\text{cl}} &= -M^2 \Phi_{\text{cl}} + \frac{\lambda_2}{2}\varphi_{\text{cl}}^2 \Phi_{\text{cl}}\,.\end{aligned} \tag{1.77}$$

Since we are only interested in the the low-energy theory, the momentum term can be neglected compared to the mass term. Thus the left-hand side of the above equation vanishes and consequently the classical field Φ_{cl} satisfies

$$\Phi_{\text{cl}} = 0\,. \tag{1.78}$$

There is no tree-level matching in this theory. We can see this diagrammatically when looking at the Lagrangian: There is simply no tree-level diagram with only external φ mediated by a virtual Φ.

Now, let us compute the one-loop effective action. First, we compute the blocks Δ_H, Δ_S and X_{SH}. We obtain:

$$\begin{aligned}\Delta_H(x,\partial) &= -\partial^2 - M^2 + \frac{\lambda_2}{2}\varphi_{\text{cl}}^2\,, \\ \Delta_S(x,\partial) &= -\partial^2 - m^2 + \frac{\lambda_1}{2}\varphi_{\text{cl}}^2 + \frac{\lambda_2}{2}\Phi_{\text{cl}}^2\,, \\ X_{SH} &= 0\,.\end{aligned} \tag{1.79}$$

The last term in the second line vanishes by the equations of motion. Since $X_{SH} = 0$, the matrix V defined in eq. (1.67) is simply identity and $\tilde{\Delta}_H = \Delta_H$. So now we want to compute:

$$\begin{aligned}S_\Lambda &= \frac{i}{2}\int d^dx \int \frac{d^dk}{(2\pi)^d} \log\left(\tilde{\Delta}_H(x,\partial + ik)\right) \\ &= \frac{i}{2}\int d^dx \int \frac{d^dk}{(2\pi)^d} \log\left(k^2 - M^2 - \left(2ik\cdot\partial + \partial^2 - \frac{\lambda_2}{2}\varphi_{\text{cl}}^2\right)\right) \\ &= \frac{i}{2}\int d^dx \int \frac{d^dk}{(2\pi)^d} \left\{\log\left(k^2 - M^2\right) + \log\left(1 - \frac{2ik\cdot\partial + \partial^2 - \frac{\lambda_2}{2}\varphi_{\text{cl}}^2}{k^2 - M^2}\right)\right\}\end{aligned} \tag{1.80}$$

The first term in the curly brackets does not generate any interactions between the fields φ_{cl}, so we can neglect it. Focussing on the second term and rewriting the logarithm through its series representation,

$$\log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}\,, \tag{1.81}$$

we find

$$S_\Lambda = \frac{i}{2}\int d^dx \int \frac{d^dk}{(2\pi)^d} \log\left(1 - \frac{2ik\cdot\partial + \partial^2 - \frac{\lambda_2}{2}\varphi_{\text{cl}}^2}{k^2 - M^2}\right) \tag{1.82}$$

$$= -\frac{i}{2}\int d^dx \sum_{n=1} \frac{1}{n} \int \frac{d^dk}{(2\pi)^d} \left(\frac{2ik\cdot\partial + \partial^2 - \frac{\lambda_2}{2}\varphi_{\text{cl}}^2}{k^2 - M^2}\right)^n\,. \tag{1.83}$$

Remember that for the matching, we need to evaluate the integrals in the hard region, meaning $k^2 \sim M^2$. This means that terms of higher order in n are suppressed by the hard scale and that we can truncate the sum at a given order. To obtain the operators up to order $\mathcal{O}(1/\Lambda^2)$, we truncate the sum at $n = 3$. Let us work out the terms, order by order in this expansion. The first order simply gives us:

$$S_\Lambda^{(n=1)} = -\frac{i}{2} \int d^d x \int \frac{d^d k}{(2\pi)^d} \frac{2ik \cdot \partial + \partial^2 - \frac{\lambda_2}{2}\varphi_{\text{cl}}^2}{k^2 - M^2} \,. \tag{1.84}$$

The derivatives act on the vacuum, so they vanish. The only contribution is thus

$$S_\Lambda^{(n=1)} = -\frac{i\lambda_2}{4} \int d^d x \, \varphi_{\text{cl}}^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - M^2} \,. \tag{1.85}$$

To evaluate this and following integrals, let us introduce a handy master forumla: In the hard region, all loop integrals will be of the form:

$$I_{\alpha\beta}(M^2) = \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2}\right)^\alpha \left(\frac{1}{k^2 - M^2}\right)^\beta \,, \tag{1.86}$$

which can be evaluated in closed form to find:

$$I_{\alpha\beta}(M^2) = \frac{i}{16\pi^2} \left(-\frac{1}{M^2}\right)^{-2+\alpha+\beta} \left(\frac{4\pi\bar{\mu}^2}{M^2}\right)^\epsilon \frac{\Gamma(2-\alpha-\epsilon)\Gamma(\alpha+\beta+\epsilon-2)}{\Gamma(\beta)\Gamma(2-\epsilon)} \,, \tag{1.87}$$

with $\bar{\mu}$ being the renormalization scale. In the $\overline{\text{MS}}$ scheme, we have

$$\bar{\mu}^2 = \frac{e^{\gamma_E}}{4\pi} \mu^2 \,. \tag{1.88}$$

Evaluating the expression (1.85) at the renormalization scale $\mu^2 = M^2$ and absorbing the UV divergence into a counterterm, we find the contribution to the Lagrangian

$$\Delta\mathcal{L}_{\text{eff}}^{(n=1)} = \frac{M^2}{2} \frac{\lambda_2}{32\pi^2} \varphi_{\text{cl}}^2 \,. \tag{1.89}$$

This is a correction to the mass term of the classical field. In the Language of Feynman graphs, this term corresponds to the loop diagram:

$$\rightarrow \qquad \frac{M^2}{2} \frac{\lambda_2}{32\pi^2} \varphi_{\text{cl}}^2 \,. \tag{1.90}$$

Moving on to the second term in the expansion we have:

$$S_\Lambda^{(n=2)} = -\frac{i}{4} \int d^d x \int \frac{d^d k}{(2\pi)^d} \frac{(2ik \cdot \partial + \partial^2 - \frac{\lambda_2^2}{2}\varphi_{\text{cl}}^2)(2ik \cdot \partial + \partial^2 - \frac{\lambda_2^2}{2}\varphi_{\text{cl}}^2)}{(k^2 - M^2)^2} \,. \tag{1.91}$$

In the second bracket, we can drop derivatives, as they act on the vacuum again. Thus:

$$\begin{aligned} S_\Lambda^{(n=2)} &= \frac{i\lambda_2}{8} \int d^d x \int \frac{d^d k}{(2\pi)^d} \frac{(2ik \cdot \partial + \partial^2 - \frac{\lambda_2^2}{2}\varphi_{\text{cl}}^2)}{(k^2 - M^2)^2} \varphi_{\text{cl}}^2 \,. \\ &= \int d^d x \int \frac{d^d k}{(2\pi)^d} \left[-\frac{\lambda_2}{4} \frac{(k \cdot \partial)\varphi_{\text{cl}}^2}{(k^2 - M^2)^2} + \frac{i\lambda_2}{8} \frac{\partial^2 \varphi_{\text{cl}}^2}{(k^2 - M^2)^2} - \frac{i\lambda_2^2}{16} \frac{\varphi_{\text{cl}}^4}{(k^2 - M^2)^2} \right] \,. \end{aligned} \tag{1.92}$$

The first and second term in the brackets do not contribute since they represent total derivatives. Only the last term contributes a matching correction to the quartic coupling of the light scalar:

$$\Delta\mathcal{L}_{\text{eff}}^{(n=2)} = -\frac{i\lambda_2^2}{16}\varphi_{\text{cl}}^4\, I_{0\,2}(M^2)\,. \tag{1.93}$$

Again absorbing the divergence into a counterterm choosing $\mu^2 = M^2$, we find that this integral has no finite term. There is no correction to the quartic coupling at this order in n.

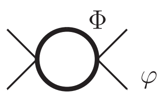
Finally, let us evaluate the third order in our sum. We now have three insertions of the operator $(2ik\cdot\partial + \partial^2 - \lambda_2\varphi_{\text{cl}}^2/2)$. In the third bracket, we can again drop derivatives as they are acting on nothing. In the first bracket, we can drop derivatives as well, since they only generate terms that are total derivatives. In the middle bracket, the term linear in k can be dropped since the integral over the momentum vanishes by symmetry. Therefore, we have:

$$\begin{aligned} S_\Lambda^{(n=3)} &= -\frac{i}{6}\int d^dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-M^2)^3}\left(-\frac{\lambda_2}{2}\varphi_{\text{cl}}^2\right)\left(\partial^2 - \frac{\lambda_2}{2}\varphi_{\text{cl}}^2\right)\left(-\frac{\lambda_2}{2}\varphi_{\text{cl}}^2\right) \\ &= -\frac{i\lambda_2^2}{24}\int d^dx \int \frac{d^dk}{(2\pi)^d}\frac{\varphi_{\text{cl}}^2\partial^2\varphi_{\text{cl}}^2}{(k^2-M^2)^3} + \frac{i\lambda_2^3}{48}\int d^dx\int \frac{d^dk}{(2\pi)^d}\frac{\varphi_{\text{cl}}^6}{(k^2-M^2)^3}\,. \end{aligned} \tag{1.94}$$

This generates a new quartic interaction as well as a six-point interaction:

$$\mathcal{L}_{\text{eff}}^{(n=3)} = -\frac{\lambda_2^2}{48(16\pi^2)}\varphi_{\text{cl}}^2\partial^2\varphi_{\text{cl}}^2 + \frac{\lambda_2^6}{96(16\pi^2)}\varphi_{\text{cl}}^6\,. \tag{1.95}$$

The quartic operator comes from the one-loop diagram with two heavy scalars in the loop:



$$\rightarrow \qquad -\frac{\lambda_2^2}{48(16\pi^2)}\varphi_{\text{cl}}^2\partial^2\varphi_{\text{cl}}^2\,. \tag{1.96}$$

One might now wonder how this diagram can give us an expression with three heavy propagators, as it can be seen from the last line in eq. (1.94). The way this comes about is that this is a subleading-power expression in the expansion of external momenta over the heavy mass:

$$\left[\frac{1}{(l+p)^2-m^2}\right] = \frac{1}{l^2-m^2}\left[1 - \frac{2l\cdot p + p^2}{l^2-M^2} + \dots\right]. \tag{1.97}$$

The leading piece ends up in the correction to the φ_{cl}^4 operator, whereas the next-to-leading piece introduces the term we find in our expansion at $n = 3$. The p^2 in the numerator of the expansion of the heavy propagator corresponds to the ∂^2 in the effective operator in (1.96).

We have computed the one-loop effective Lagrangian for the theory (1.76). While the expressions can be related to Feynman diagrams, we have done so without computing a single diagram, notably without having to worry about any kinds of symmetry and combinatorial factors. The example we have worked out was lucky enough to have $X_{SH} = 0$, meaning that all the loop diagrams contained only heavy scalars. There are no diagrams where we had both light scalars and heavy scalars in the loop. We should conclude this chapter by elaborating on the additional complication that arise in the more general scenario where this is not the case.

For $X_{SH} \neq 0$, the fluctuation operator $\tilde{\Delta}_H$ contains a term of the form

$$\tilde{\Delta}_H \supset X_{SH}^{\dagger} \Delta_S^{-1} X_{SH} \,. \tag{1.98}$$

In this case, we should further separate Δ_S into the soft kinetic term and interactions:

$$\Delta_S = \tilde{\Delta}_S + X_S \,. \tag{1.99}$$

Then we can write the inverse of Δ_S appearing in (1.98) using the Neumann expansion:

$$\Delta_S^{-1} = \sum_{n=0}^{\infty} (-1)^n \left(\tilde{\Delta}_S^{-1} X_S \right)^n \tilde{\Delta}_S^{-1} \,, \tag{1.100}$$

where $\tilde{\Delta}_S^{-1}$ simply gives the propagator of the soft field. One can then work out the full fluctuation operator to a given order in the expansion above. This will reproduce terms in S_H corresponding to diagrams with both heavy and light fields in the loop. These loop integrals can always be evaluated with our master formula (1.87).

CHAPTER 2

The Standard Model Effective Theory

"We demand rigidly defined areas of doubt and uncertainty!"

— *Douglas Adams*, The Hitchhiker's Guide to the Galaxy

2.1 Motivation

In the previous chapter we have learned how it is possible to build an effective quantum field theory, by expanding a given theory around a certain limit. We have done so by assigning a power-counting to the various fields in the theory and thereby classifying them by importance at low energy scales. We have then split up the field modes of the theory into the categories *hard* and *soft* and constructed the effective theory by removing the hard modes and absorbing their effects on the soft modes into local operators. These local operators, built out of soft modes exclusively, came with coupling constants that encoded the underlying hard physics that we have removed from the theory. The result was that matrix elements of the effective theory factorized into the soft matrix elements and the Wilson coefficients, which encoded the hard physics.

The consequence of this factorization is that the effective theory knows nothing about the ultraviolet dynamics of the full theory it was made from. The only remnants of the full theory, the Wilson coefficients, carry no deeper information about the UV picture - they only determine the interaction strength of the effective operators. For example, in the Fermi theory, the Wilson coefficients did not uniquely give away the presence of the W boson, it was merely a coupling constant that could be measured in beta decay. A W boson with double the coupling strength g and double the mass m_W would have given the same effective coupling.

What sounds like an apparent shortcoming can be turned into a benefit: Using effective theories, we can derive predictions for low-energy processes as functions of the Wilson coefficients without ever having to know about the UV theory: The UV theory is a detail that we need not care about and might very well not even know. We can be fairly certain that this applies to the SM itself: We know from its inability to answer a number of questions, that the SM can not be a complete theory. However, at scales probed by current experiments, it does provide an accurate description of the phenomena observed. So while it cannot be a full theory, it most certainly can be viewed as an EFT.

This is the underlying train of thought behind the Standard Model Effective Field Theory (SMEFT): It is a bottom-up EFT, where the fields of the SM take the roles of the soft modes. The symmetries of the theory are exactly those of the SM. Whatever UV theory we can imagine will result in a certain set of Wilson coefficients in the SMEFT, but at low energy we do not need to know about these. The power-counting is given by $\lambda^2 \sim q^2/\Lambda^2$, where Λ is the scale at which the hard modes have been removed.

You can already see a caveat: We do not actually know Λ, since we do not know what the hard modes were in the UV theory. Therefore, when computing processes in the SMEFT, one has to be careful about the validity of the underlying expansion. We have good reasons to believe that new physics (NP) should be situated at scales at least $\mathcal{O}(\text{TeV})$. If there are new particles at 1 TeV, then the local OPE cannot be trusted for processes at high momentum transfer, like the ones happening at the Large Hadron Collider, which has a maximum center-of-mass energy of 14 TeV [1].

In the following we will briefly review the SM itself and then construct the SMEFT from its building blocks.

2.2 Field content and gauge symmetries

Let us begin by specifying the field content of the SM. Let us begin with the gauge group of the SM. It is given by

$$\mathcal{G}_{\text{SM}} = \text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y \,, \tag{2.1}$$

where $\text{SU}(3)_C$ gives rise to the strong interactions and $\text{SU}(2)_L \times U(1)_Y$ governs the electroweak interactions. After electroweak symmetry breaking (EWSB), the electroweak symmetry is broken into its diagonal subgroub, $\text{SU}(2)_L \times \text{U}(1)_Y \to U(1)_{\text{em}}$, which is the symmetry group of quantum electrodynamics (QED).

The next step is to list the field content along with its transformation properties under this group. Let us begin with the fermions. There are five types of multiplets, transforming under the gauge group as follows:

$$Q_L^i \sim (3,2)_{1/6},\ u_R^i \sim (3,1)_{2/3},\ d_R^i \sim (3,1)_{-1/3},\ L_L^i \sim (1,2)_{-1/2},\ e_R^i \sim (1,1)_{-1}. \tag{2.2}$$

They correspond to the left-handed quarks, the right-handed up- and down-type quarks, the left-handed leptons and the right-handed charged leptons. There are no right-handed neutrinos in the SM. The first number in the bracket denotes the transformation under $\text{SU}(3)_C$, the second number the transformation under $\text{SU}(2)_L$ and the subscript denotes the hypercharge. For example, the fields Q_L^i transform as triplets under the strong group, are weak-isospin doublets and have a hypercharge

[1] This however does not mean that the momentum scale of the scattering processes are $q^2 = (14\ \text{TeV})^2$. Since the LHC is colliding protons, the scattering of indivdual partons receives only a fraction of the collision energy and a large part is carried away by the proton remnants not participating in the hard scattering.

of 1/6. The index $i = 1, 2, 3$ is the generation index. For example, the doublet $Q_L^3 = (t_L, b_L)$ contains the left-handed top and bottom quarks.

The SM also contains a scalar doublet, the Higgs doublet. It transforms as:

$$\Phi \sim (1, 2)_{1/2} \,. \tag{2.3}$$

After it acquires a vacuum expectation value, it generates the mass terms for the fermions and the weak bosons W and Z. The SMEFT however is a theory valid between the weak scale $v = 246$ GeV and the NP-scale Λ. In this region, we can treat electroweak symmetry as unbroken and the weak bosons as massless.

We are still missing the four vector bosons in our list: The gluons, the W and Z bosons and the photon. However, as they are gauge fields, they do not enter the Lagrangian as individual fields but instead automatically arise from derivatives on the fields: Since the fields transform differently at each point in space time, the derivatives on the fields contain an extra term that compares the difference in the gauge phase rather than the difference of the field value itself. One then introduces a vector field, defined to compensate this term and subtract it from the derivative:

$$\partial_\mu \to D_\mu = \partial_\mu - ig\tau^a A_\mu^a \,. \tag{2.4}$$

In short, the gauge fields appear whenever the gauge phase is compared along a path, either between two points - where it enters as a term in the derivative - or along a closed path - where it gives rise to the field strength tensors:

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] \,. \tag{2.5}$$

Now we have gathered all our building blocks: The fermions in (2.2), the Higgs doublet Φ, the covariant derivative D_μ and the field strength tensors.

2.3 The operator basis

2.3.1 Operator reduction

With the building blocks and the symmetries in place, we can proceed to build the operator basis. In principle, this sounds like a simple enough task to perform in a "brute-force" approach by simply piecing together all products of the building blocks in accordance with the symmetries of the theory.

While this is certainly an appropriate first step, care must be taken about the completeness of the operator basis: Not all operators written down this way will be independent of each other. There are three ways by which operators can be connected with each other:

1. **Integration-by-parts relations:** Terms that can be written as total derivatives cannot contribute to the action, since they correspond to a boundary term at infinity. So by imposing $D_\mu \mathcal{O}^\mu = 0$ and using the product rule, one can find relations among operators containing derivatives. An example:

$$\begin{aligned} 0 &= D_\mu(\phi^2 D^\mu \phi^2) = (D_\mu \phi^2)(D^\mu \phi^2) + \phi^2 D^2 \phi^2 \\ \Rightarrow \qquad & (D_\mu \phi^2)(D^\mu \phi^2) = -\phi^2 D^2 \phi^2 \,. \end{aligned} \tag{2.6}$$

Both of these operators could be written in a Lagrangian but they are identical up to a minus sign.

2. **Equations of motion:** Another relation is given by the equations of motion for the fields. These will allow us to replace terms of the form of the kinetic terms. For a real scalar, the equation of motion give:

$$\partial^2\varphi = -m^2\varphi + (\text{interactions})\,, \tag{2.7}$$

so each instance of $\partial^2\varphi$ can be replaced with the right-hand side of this equation. Similar replacements can be derived for fermion operators involving $\not{\partial}\psi$.

3. **Fierz relations:** For fermion fields, relations of the following type hold:

$$(\bar{\psi}_L\gamma_\mu\psi_L)(\bar{\chi}_L\gamma^\mu\chi_L) = (\bar{\psi}_L\gamma_\mu\chi_L)(\bar{\chi}_L\gamma^\mu\psi_L)\,. \tag{2.8}$$

Therefore, many four-fermion operators can be related to each other. A list of Fierz identities can be found in Ref. [4].

Using these three techniques, one can reduce the number of operators drastically. While the idea of continuing the SM Lagrangian to higher mass-dimension was originally proposed in Ref. [5], the reduction to a minimal set of operators was performed in Ref. [6], lowering the number of operators from 80 to 59, much to the delight of the authors, as can be read off from their conclusions:

It is really amazing that no author of almost 600 papers that quoted Ref. [5] over 24 years has ever decided to rederive the operator basis from the outset to check its correctness. As the current work shows, the exercise has been straightforward enough for an M. Sc. thesis. It has required no extra experience with respect to what was standard already in the 1980's.

2.3.2 Classification

Let us now discuss the classes of operators the theory can have. We will follow the naming scheme of Ref. [6], where ψ^n denotes the power of fermions in the operator, X^n the powers of field strength tensors, φ^n the number of Higgs doublets and D^n the number of covariant derivatives. First, note that by Lorentz invariance, fermions always come in pairs. These pairs are frequently referred to as fermion currents.

Four-fermion, ψ^4:
Since a pair of fermion counts as three powers of mass, the most obvious class of operators at dimension six is the one of the four-fermion operators. We can further subdivide them by the chirality of the four fermions. The five categories we have are:

$$\bar{L}L\,\bar{L}L,\ \ \bar{R}R\,\bar{R}R,\ \ \bar{L}L\,\bar{R}R,\ \ \bar{L}R\bar{L}R,\ \ \mathrm{B-violating}\,. \tag{2.9}$$

The first three categories are products of same-chirality fermion currents, meaning vector currents $J^\mu \sim \bar{\psi}_L\gamma^\mu\psi'_L$. The mixed-chirality currents in the fourth category can either by scalar currents $J_S \sim \bar{\psi}_L\psi'_R$ or tensor currents $J_T^{\mu\nu} \sim \bar{\psi}_L\sigma^{\mu\nu}\psi'_R$. The

$(\bar{L}L)(\bar{L}L)$		$(\bar{R}R)(\bar{R}R)$		$(\bar{L}L)(\bar{R}R)$	
Q_{ll}	$(\bar{l}_p\gamma_\mu l_r)(\bar{l}_s\gamma^\mu l_t)$	Q_{ee}	$(\bar{e}_p\gamma_\mu e_r)(\bar{e}_s\gamma^\mu e_t)$	Q_{le}	$(\bar{l}_p\gamma_\mu l_r)(\bar{e}_s\gamma^\mu e_t)$
$Q_{qq}^{(1)}$	$(\bar{q}_p\gamma_\mu q_r)(\bar{q}_s\gamma^\mu q_t)$	Q_{uu}	$(\bar{u}_p\gamma_\mu u_r)(\bar{u}_s\gamma^\mu u_t)$	Q_{lu}	$(\bar{l}_p\gamma_\mu l_r)(\bar{u}_s\gamma^\mu u_t)$
$Q_{qq}^{(3)}$	$(\bar{q}_p\gamma_\mu \tau^I q_r)(\bar{q}_s\gamma^\mu \tau^I q_t)$	Q_{dd}	$(\bar{d}_p\gamma_\mu d_r)(\bar{d}_s\gamma^\mu d_t)$	Q_{ld}	$(\bar{l}_p\gamma_\mu l_r)(\bar{d}_s\gamma^\mu d_t)$
$Q_{lq}^{(1)}$	$(\bar{l}_p\gamma_\mu l_r)(\bar{q}_s\gamma^\mu q_t)$	Q_{eu}	$(\bar{e}_p\gamma_\mu e_r)(\bar{u}_s\gamma^\mu u_t)$	Q_{qe}	$(\bar{q}_p\gamma_\mu q_r)(\bar{e}_s\gamma^\mu e_t)$
$Q_{lq}^{(3)}$	$(\bar{l}_p\gamma_\mu \tau^I l_r)(\bar{q}_s\gamma^\mu \tau^I q_t)$	Q_{ed}	$(\bar{e}_p\gamma_\mu e_r)(\bar{d}_s\gamma^\mu d_t)$	$Q_{qu}^{(1)}$	$(\bar{q}_p\gamma_\mu q_r)(\bar{u}_s\gamma^\mu u_t)$
		$Q_{ud}^{(1)}$	$(\bar{u}_p\gamma_\mu u_r)(\bar{d}_s\gamma^\mu d_t)$	$Q_{qu}^{(8)}$	$(\bar{q}_p\gamma_\mu T^A q_r)(\bar{u}_s\gamma^\mu T^A u_t)$
		$Q_{ud}^{(8)}$	$(\bar{u}_p\gamma_\mu T^A u_r)(\bar{d}_s\gamma^\mu T^A d_t)$	$Q_{qd}^{(1)}$	$(\bar{q}_p\gamma_\mu q_r)(\bar{d}_s\gamma^\mu d_t)$
				$Q_{qd}^{(8)}$	$(\bar{q}_p\gamma_\mu T^A q_r)(\bar{d}_s\gamma^\mu T^A d_t)$

$(\bar{L}R)(\bar{R}L)$ and $(\bar{L}R)(\bar{L}R)$		B-violating	
Q_{ledq}	$(\bar{l}_p^j e_r)(\bar{d}_s q_t^j)$	Q_{duq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(q_s^{\gamma j})^T C l_t^k\right]$
$Q_{quqd}^{(1)}$	$(\bar{q}_p^j u_r)\varepsilon_{jk}(\bar{q}_s^k d_t)$	Q_{qqu}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(u_s^\gamma)^T C e_t\right]$
$Q_{quqd}^{(8)}$	$(\bar{q}_p^j T^A u_r)\varepsilon_{jk}(\bar{q}_s^k T^A d_t)$	Q_{qqq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jn}\varepsilon_{km}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(q_s^{\gamma m})^T C l_t^n\right]$
$Q_{lequ}^{(1)}$	$(\bar{l}_p^j e_r)\varepsilon_{jk}(\bar{q}_s^k u_t)$	Q_{duu}	$\varepsilon^{\alpha\beta\gamma}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(u_s^\gamma)^T C e_t\right]$
$Q_{lequ}^{(3)}$	$(\bar{l}_p^j \sigma_{\mu\nu} e_r)\varepsilon_{jk}(\bar{q}_s^k \sigma^{\mu\nu} u_t)$		

Table 2.1: Four-fermion operators in the SMEFT, taken directly from Ref. [6].

baryon-number violating operators are products scalar currents as well. The full list of four-fermion operators is laid out in Tab. 2.1. It is important to note that the operators in this list have four indices specifying the flavor of the fermions, defined in eq. (2.2). These indices are only shown in the definition as p, r, s, t. Each operator thus corresponds to twelve operators when flavor is distinguished.

The four-fermion operators offer a rich amount of phenomenology: They are relevant to flavor physics processes like hadron decays and meson-antimeson mixing, which are studied to understand the flavor structure of the SM and its possible extensions.

Fermion-boson, $\psi^2\varphi^3$, $\psi^2 X\varphi$, $\psi^2\varphi^2 D$:
There are a number of operators composed of one fermion current and non-fermion structures. For the phenomenological implications, it is important to note that we can replace each Higgs doublet by its vacuum expectation value (vev) once we go to the broken phase of the SM. The first type, referred to as $\psi^2\varphi^3$, is composed of a Yukawa-like current dressed with two additional Higgs doublets, so they are of the form

$$\sim (\bar{\psi}_L\, \varphi\, \psi_R)\,(\varphi^\dagger\varphi)\,. \tag{2.10}$$

After EWSB and replacing two Higgs doublets by the vev, these operators produce power-corrections to the Yukawa couplings of fermions to the Higgs, and to the fermion masses once we also replace the third Higgs doublet by the vev.

The operator type $\psi^2 X\varphi$ consists of a tensor current with the two Lorentz indices saturated by a field-strength tensor. An additional Higgs doublet is required by gauge invariance. The operators are of the form

$$(\bar{\psi}_L\, \varphi\, \sigma_{\mu\nu}\psi_R)F^{\mu\nu}\,. \tag{2.11}$$

X^3		φ^6 and $\varphi^4 D^2$		$\psi^2\varphi^3$	
Q_G	$f^{ABC} G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$	Q_φ	$(\varphi^\dagger\varphi)^3$	$Q_{e\varphi}$	$(\varphi^\dagger\varphi)(\bar{l}_p e_r \varphi)$
$Q_{\widetilde{G}}$	$f^{ABC} \widetilde{G}_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$	$Q_{\varphi\Box}$	$(\varphi^\dagger\varphi)\Box(\varphi^\dagger\varphi)$	$Q_{u\varphi}$	$(\varphi^\dagger\varphi)(\bar{q}_p u_r \widetilde{\varphi})$
Q_W	$\varepsilon^{IJK} W_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$	$Q_{\varphi D}$	$\left(\varphi^\dagger D^\mu\varphi\right)^\star \left(\varphi^\dagger D_\mu\varphi\right)$	$Q_{d\varphi}$	$(\varphi^\dagger\varphi)(\bar{q}_p d_r \varphi)$
$Q_{\widetilde{W}}$	$\varepsilon^{IJK} \widetilde{W}_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$				
$X^2\varphi^2$		$\psi^2 X\varphi$		$\psi^2\varphi^2 D$	
$Q_{\varphi G}$	$\varphi^\dagger \varphi\, G^A_{\mu\nu} G^{A\mu\nu}$	Q_{eW}	$(\bar{l}_p \sigma^{\mu\nu} e_r) \tau^I \varphi W^I_{\mu\nu}$	$Q_{\varphi l}^{(1)}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar{l}_p \gamma^\mu l_r)$
$Q_{\varphi\widetilde{G}}$	$\varphi^\dagger \varphi\, \widetilde{G}^A_{\mu\nu} G^{A\mu\nu}$	Q_{eB}	$(\bar{l}_p \sigma^{\mu\nu} e_r) \varphi B_{\mu\nu}$	$Q_{\varphi l}^{(3)}$	$(\varphi^\dagger i\overleftrightarrow{D}^I_\mu \varphi)(\bar{l}_p \tau^I \gamma^\mu l_r)$
$Q_{\varphi W}$	$\varphi^\dagger \varphi\, W^I_{\mu\nu} W^{I\mu\nu}$	Q_{uG}	$(\bar{q}_p \sigma^{\mu\nu} T^A u_r) \widetilde{\varphi}\, G^A_{\mu\nu}$	$Q_{\varphi e}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar{e}_p \gamma^\mu e_r)$
$Q_{\varphi\widetilde{W}}$	$\varphi^\dagger \varphi\, \widetilde{W}^I_{\mu\nu} W^{I\mu\nu}$	Q_{uW}	$(\bar{q}_p \sigma^{\mu\nu} u_r) \tau^I \widetilde{\varphi}\, W^I_{\mu\nu}$	$Q_{\varphi q}^{(1)}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar{q}_p \gamma^\mu q_r)$
$Q_{\varphi B}$	$\varphi^\dagger \varphi\, B_{\mu\nu} B^{\mu\nu}$	Q_{uB}	$(\bar{q}_p \sigma^{\mu\nu} u_r) \widetilde{\varphi}\, B_{\mu\nu}$	$Q_{\varphi q}^{(3)}$	$(\varphi^\dagger i\overleftrightarrow{D}^I_\mu \varphi)(\bar{q}_p \tau^I \gamma^\mu q_r)$
$Q_{\varphi\widetilde{B}}$	$\varphi^\dagger \varphi\, \widetilde{B}_{\mu\nu} B^{\mu\nu}$	Q_{dG}	$(\bar{q}_p \sigma^{\mu\nu} T^A d_r) \varphi\, G^A_{\mu\nu}$	$Q_{\varphi u}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar{u}_p \gamma^\mu u_r)$
$Q_{\varphi WB}$	$\varphi^\dagger \tau^I \varphi\, W^I_{\mu\nu} B^{\mu\nu}$	Q_{dW}	$(\bar{q}_p \sigma^{\mu\nu} d_r) \tau^I \varphi\, W^I_{\mu\nu}$	$Q_{\varphi d}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar{d}_p \gamma^\mu d_r)$
$Q_{\varphi\widetilde{W}B}$	$\varphi^\dagger \tau^I \varphi\, \widetilde{W}^I_{\mu\nu} B^{\mu\nu}$	Q_{dB}	$(\bar{q}_p \sigma^{\mu\nu} d_r) \varphi\, B_{\mu\nu}$	$Q_{\varphi ud}$	$i(\widetilde{\varphi}^\dagger D_\mu \varphi)(\bar{u}_p \gamma^\mu d_r)$

Table 2.2: Dimension-six operators in the SMEFT other than the four-fermion ones, taken directly from Ref. [6].

Once the Higgs is replaced by its vev, this operator generates the dipole-type interactions, entering observables like the anomalous magnetic moment of the the muon $(g-2)_\mu$, and flavor-changing neutral current transitions like $b \to s\gamma$ and $\mu \to e\gamma$, the latter being forbidden in the SM.

The last type of operators $\psi^2\varphi^2 D$, is simlar to the $\psi^2\varphi^3$ but with a vector current for the fermions. Only two Higgs doublets can be inserted here and the Lorentz index of the current is contracted with a covariant derivative:

$$\sim \left(\varphi^\dagger i \overleftrightarrow{D}_\mu \varphi\right) \left(\bar{\psi}_L \gamma^\mu \psi_L\right). \tag{2.12}$$

An important phenomenological implication of this type of operator is the alteration of the couplings of fermions to gauge bosons, as we will see below.

Bosonic, X^3, φ^6, $\varphi^4 D^2$, $X^2\varphi^2$:
Lastly, we have operators without fermions. The first type X^3 involves contractions of field strength tensors:

$$\sim f^{abc} G^a_{\mu\nu} G^{b\,\nu\rho} G^{c\,\mu}_{\rho}. \tag{2.13}$$

The modify three- and higher-point interactions of gauge bosons amongst themselves.

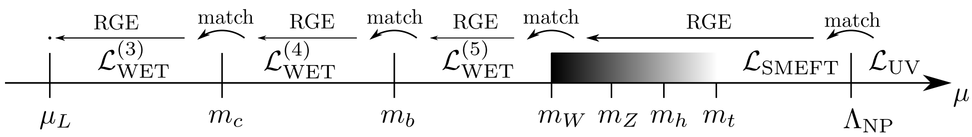
The φ^6 and $\varphi^4 D^2$ operators are self-interactions of the Higgs doublets, either as six-point interactions or quartic interactions with additional derivatives. They impact the self-interactions of the Higgs boson.

The last type of operators $X^2\varphi^2$ is then the product of a gauge-kinetic term and two Higgs doublets, impacting for example masses of the electroweak gauge bosons. All operators involving two or less fermions are given in Tab. 2.2.

2.4 Outlook

2.4.1 From the SMEFT to lower energies

As stated before, the SMEFT is written in the unbroken phase, meaning that the Higgs has not acquired a vev and thereby all fermions and gauge fields remain massless. This is clearly not appropriate for physics at very low energies, say at scales of $\mathcal{O}(\mathrm{GeV})$, where for example the W and Z bosons can be treated as infinitely heavy. Therefore one needs to match the SMEFT onto a generalization of the Fermi theory, which is very often called the *weak effective theory* (WET) or the *low-energy effective theory* (LEFT). In this theory, one decouples the electroweak bosons W, Z and h, and all fermions that are heavier than the corresponding scale. In the first step therefore, also the top-quark is integrated out. When the scale is then lowered, one undergoes a series of matching steps in which also the bottom-quark and eventually the charm-quark are decoupled. In between the matching steps, the Wilson coefficients are transformed usng their renormalization group running. The procedure is sketched here:

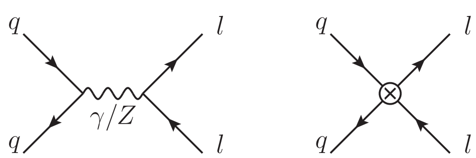


Here the index on the WET Lagrangian denotes the number of active quark flavors. It affects the possible quark flavors out of which the fermionic operators are composed and the renormalization group running of the couplings. A detailed list of operators along with their renormalization group equations are found in Ref. [7].

2.4.2 SMEFT and SM input parameters

When deriving predictions with the SMEFT Lagrangian, care needs to be taken about the consistency of the input parameters. Naively one would compute observables in terms of the SM pieces and then add contributions from the higher-dimensional operators. The SM contributions are then functions of the couplings and masses of the SM Lagrangian, which have been extracted from measurements. However, if we assume the presence of non-zero contributions from the dimension-six operators, the extraction of these parameters will be impacted by them. What we thought we measured as the value of a certain parameter is in reality the value of this parameter plus corrections from SMEFT operators.

Let us clarify this at an example and consider dilepton production at the LHC, generated by the parton-level process $q\bar{q} \to l^+l^-$. There are two types of diagrams: In the SM, this process occurs through Drell-Yan scattering with a virtual photon or Z-boson. Then there are corrections from four-fermion operators:



The SM contribution to the amplitude, given by the first diagram, will be a function of the electroweak gauge coupling $\alpha_{\rm ew}$, the Z-boson mass m_Z and the electroweak mixing-angle $s_W \equiv \sin\theta_W$. The contribution from four-fermion operators will be a function of their Wilson coefficients, namely $C_{lq}^{(1)}$, $C_{lq}^{(3)}$, C_{eu}, C_{ed}, C_{lu}, C_{qe} and C_{ld}. The input parameters can be extracted from measurements: The Z-boson mass and the electroweak coupling constant have been measured directly. The Weinberg mixing angle θ_W is typically extracted from the relation between the Z-boson mass, the gauge coupling and the Higgs vev:

$$m_Z = \frac{gv}{2\cos\theta_W}\,. \tag{2.14}$$

The Higgs vev can be directly related to the Fermi constant, as we have seen in section 1.4.1. We therefore find:

$$s_W^2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\pi\alpha_{\rm ew}}{\sqrt{2}G_F m_Z^2}}\,. \tag{2.15}$$

The Fermi constant can be very well determined from the decay of the muon, which we discussed in section 1.4.1. Using the results there we can determine the decay rate of the muon as a function of the Fermi constant, $\Gamma_\mu = \Gamma_\mu(G_F)$, and compare it to the experimental result to obtain a number for G_F. However, when computing muon decay in the SMEFT, there are additional contributions from the operators Q_{ll} and $Q_{\varphi l}^{(3)}$. The first operator generates the four-fermion interaction directly whereas the second one alters the coupling of the W boson to leptons. That means, the muom decay rate actually is a function of G_F and the Wilson coefficients of these operators, $\Gamma_\mu = \Gamma_\mu(G_F, C_{ll}, C_{\varphi l}^{(3)})$. The SM value extracted from measurements therefore changes to account for this fact. In the end, this means that the SMEFT cross section of $q\bar{q} \to l^+l^-$ depends for example on the Wilson coefficients of four-lepton operators C_{ll}, which cannot possibly enter the process directly as a Feynman diagram at tree-level. The same considerations apply also to the parameters $\alpha_{\rm ew}$ and m_Z. These effects have been studied for example in Ref. [8], where the shifts in the input parameters due to SMEFT Wilson coefficients are tabulated.

CHAPTER 3 Chiral Perturbation Theory

> All you really need to know for the moment is that the universe is a lot more complicated than you might think, even if you start from a position of thinking it's pretty damn complicated in the first place.
>
> — *Douglas Adams*, The Hitchhiker's Guide to the Galaxy

In this part of the course we will introduce the basic ideas and methods of Chiral Perturbation Theory (χPT). This lecture summarizes some of the contents in [9–13], as well as Iain stewart's EFT course in `https://courses.edx.org/courses/MITx/8.EFTx/3T2014/course/`. The interested student is encouraged to follow these materials for a deeper discussion on the topics of the lecture, as well as for some other topics that will not be covered due to time limitations.

3.1 Introduction

It is well-established that Quantum Chromodynamics (QCD) is the theory describing the strong interactions. This theory is asymptotically free [14, 15] meaning that the strong coupling decreases as the energy increase, and perturbation theory can be applied at short distances. The resulting short-distance predictions from QCD have been tested and validated to a remarkable accuracy. However, the growing running of the QCD coupling make the theory non-perturbative at low energies, see Figure 3.1 (left). In the non-perturbative regime, the strong interactions lead to the confinement quarks and gluons, making very difficult to perform a QCD analysis in terms of these degrees of freedom. At such energies, a theoretical description in terms of composite quark and gluon states (hadrons) becomes more adequate.

The construction of such a theory is no simple task, given the richness of the hadronic spectrum [see Figure 3.1 (right)]. At very low-energies, below the resonance region ($E < M_\rho$), a great simplification takes place, and the hadronic spectrum reduces to only an octet of very light pseudo-scalar particles (π, K, η). The interactions of such light particles can be neatly described using χPT.

χPT is an effective theory (EFT) based on the global symmetry properties of QCD. This theory provides a clear example of bottom-up effective field theory, and will help us further explore other important theoretical concepts such as Goldstone

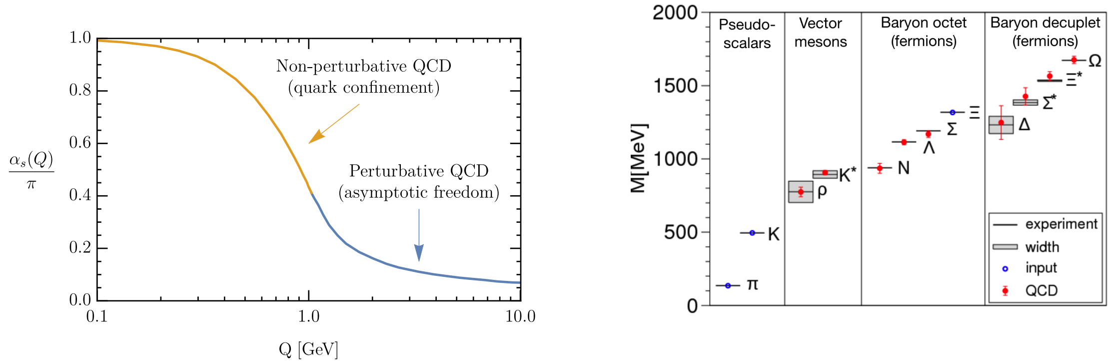


Figure 3.1: *Left:* Running of the QCD coupling. *Right:* The light hadron spectrum of QCD. Horizontal lines and bands correspond to the experimental values with their decay widths. The red dots denote lattice QCD predictions using the blue dots as input values. [Figure taken from [16]]

bosons and non-linear symmetry representations. Furthermore, χPT provides an example of EFT where loops are not suppressed by a coupling constant, but instead they are suppressed by powers in a power expansion parameter. This different to what we saw in the previous chapters, where we were integrating out heavy particles, yielding contributions that are suppressed in powers of the coupling, but are the same order in the power expansion of the heavy scale Λ. So this will give us an example of a theory with a non-trivial power counting and we will allow us further develop the concept of power counting in EFTs.

3.2 The chiral symmetry in massless QCD

In the absence of quark masses and EW interactions, the QCD Lagrangian for N_f light-flavors arranged in a flavor multiplet $q_{L,R} = (q^1_{L,R}, q^2_{L,R}, \dots, q^{N_f}_{L,R})^\intercal$ reads

$$\mathcal{L}^0_{\rm QCD} = -\frac{1}{4}\, G^a_{\mu\nu}\, G^{\mu\nu\, a} + i\,\bar{q}_L \not{D}\, q_L + i\,\bar{q}_R \not{D}\, q_R\,, \tag{3.1}$$

with $D_\mu = \partial_\mu - i g_s T^\alpha G^\alpha_\mu$, with $T^\alpha\,(\alpha = 1,\dots,8)$ being the Gell-Mann matrices. In addition to $SU(3)_c$ local invariance, the QCD Lagrangian is invariant under the *global* symmetry in flavor space[1]

$$\mathcal{G}_{N_f} \equiv SU(N_f)_L \times SU(N_f)_R \times U(1)_V\,, \tag{3.2}$$

under which left- and right-handed transform as

$$q_L \xrightarrow{G} g_L\, q_L\,, \qquad q_R \xrightarrow{G} g_R\, q_R\,, \qquad g_{L,R} \in SU(N_f)_{L,R}\,. \tag{3.3}$$

[1]The $U(1)_V$ symmetry, which survives also in the case of non-vanishing masses, correspond to baryon number and it is trivially realized in the meson sector. The global $U(1)_A$, which is a limit of the classical QCD action, is explicitly broken at the quantum level. This is the so-called $U(1)_A$ anomaly. The discussion of this is beyond the scope of this lecture, but the interested reader is encouraged to explore [11], where the effects of the $U(1)_A$ anomaly in the context of χPT are discussed.

This global symmetry is commonly denoted as the QCD *chiral symmetry.* The chiral symmetry, which should be approximately good for the light quark sector, $q = (u, d, s)^\intercal$, is however not realized in the hadronic spectrum. Even though hadrons nicely fit into $SU(3)_V \equiv SU(3)_{L+R}$ multiplets, degenerate multiplets with opposite parity do not exist. Moreover, the octet of pseudo-scalar mesons is much lighter than the other hadronic states. This provides an strong indication that the ground state (vacuum) of the theory should not be symmetric under the chiral group. More specifically, it is expected that the symmetry $\mathcal{G}$ is spontaneously broken at low energies by the quark condensate[2]

$$\langle 0|\bar{q}_L^i q_R^j|0\rangle = \delta_{ij}\,\Lambda_\chi^3\,, \tag{3.4}$$

resulting in the Spontaneous Chiral Symmetry Breaking (SCSB) pattern

$$\mathcal{G} \equiv\, SU(N_f)_L \times SU(N_f)_R \times U(1)_V \overset{SCSB}{\longrightarrow} SU(N_f)_V \times U(1)_V\,. \tag{3.5}$$

According to the Goldstone theorem [17] (see below), the SCSB gives rise to a massless scalar field for each broken generator, referred as *Goldstone bosons* (GB). For $N_f = 3$, this corresponds precisely to eight GBs that we can identify with the eight lightest hadronic states (π^+, π^-, π^0, η, K^+, K^-, K^0 and $\bar{K}^0$). As we will see later, their small masses are generated by the small quark-masses, which explicitly break the chiral symmetry.

3.3 The Goldstone theorem

Goldstone theorem: If there is a continuous symmetry transformation under which the Lagrangian is invariant, then either the vacuum state is also invariant under the transformation, or there must exist spinless particles of zero mass or *Goldstone bosons.* The number of Goldstone Bosons corresponds to the number of continuous symmetries under which the vacuum is not invariant.

Proof: Consider a theory involving several scalar fields ϕ_i arranged into a multiplet ϕ,[3] and with a Lagrangian of the form

$$\mathcal{L} = (\text{terms with derivatives}) - V(\phi)\,. \tag{3.6}$$

The vacuum state of the theory is defined by the vacuum expectation values of ϕ, which we denote as $\langle\phi\rangle$, corresponding to the constant field configuration that minimizes the potential, i.e.

$$\left.\frac{\partial V(\phi)}{\partial \phi_i}\right|_{\phi=\langle\phi\rangle} = 0\,. \tag{3.7}$$

[2]Other operators with the quantum numbers of the vacuum, such as e.g. $\langle 0|G^a_{\mu\nu}\,G^{\mu\nu\,a}|0\rangle$, also form condensates. These do not play a relevant role in the spontaneous chiral symmetry breaking.

[3]The scalars ϕ could be either elementary fields or composite objects, e.g. arising from the confinement of fermion fields as in the QCD case.

The set of field configurations where $\phi = \langle\phi\rangle$ is known as the vacuum manifold. Expanding around this minimum we find

$$V(\phi) = V(\langle\phi\rangle) + \cancel{\left.\frac{\partial V(\phi)}{\partial\phi_i}\right|_{\phi=\langle\phi\rangle}}(\phi - \langle\phi\rangle)^i + \frac{1}{2}(\phi - \langle\phi\rangle)^i(\phi - \langle\phi\rangle)^j \left.\frac{\partial^2 V(\phi)}{\partial\phi_i\partial\phi_j}\right|_{\phi=\langle\phi\rangle} + \dots \tag{3.8}$$

Here the quadratic term

$$m_{ij} \equiv \left.\frac{\partial^2 V(\phi)}{\partial\phi_i\partial\phi_j}\right|_{\phi=\langle\phi\rangle}, \tag{3.9}$$

is a symmetric matrix whose eigenvalues give the mass of the scalars. These eigenvalues cannot be negative since $\langle\phi\rangle$ is a minimum. To prove the Goldstone theorem we need to show that every continuous symmetry transformation of the Lagragian that is not a symmetry of $\langle\phi\rangle$ yields a zero eigenvalue in m_{ij}.

Let us assume that the Lagrangian is invariant under a symmetry group $\mathcal{G}$ defined by the set of infinitesimal transformations

$$\delta\phi_i = \epsilon^\alpha\, T^\alpha_{ij}\, \phi_j\,. \tag{3.10}$$

where ϵ^α are infinitesimal parameters that define the transformation and T^α ($\alpha = 1, \dots, \dim(\mathcal{G})$) are the generators of $\mathcal{G}$. If the vacuum state were also invariant under these transformations, the vacuum expectation values of ϕ_i would satisfy[4]

$$T^\alpha_{ij}\,\langle\phi_j\rangle = 0\,. \tag{3.11}$$

We are going to examine the possibility that the vacuum is only invariant under a subgroup $\mathcal{H}$ of these transformations in which case

$$T^\alpha_{ij}\,\langle\phi_j\rangle \begin{cases} = 0\,, & \text{for } T^\alpha \in \mathcal{H} \\ \neq 0\,, & \text{for } T^\alpha \in \mathcal{G}/\mathcal{H} \end{cases}. \tag{3.12}$$

If we restrict to constant fields, the derivative terms in (3.19) vanish and the invariance of the Lagrangian under $\mathcal{G}$ implies that the potential alone must also be invariant. This invariance implies

$$\frac{\delta V(\phi)}{\delta\epsilon^\alpha} = 0 \Longrightarrow \frac{\partial V}{\partial\phi_i} T^\alpha_{ij}\,\phi_j = 0\,. \tag{3.13}$$

Differentiating with respect to ϕ_k and setting $\phi = \langle\phi\rangle$, we have

$$\left.\frac{\partial V}{\partial\phi_i}\right|_{\phi=\langle\phi\rangle} T^\alpha_{ik} + \left.\frac{\partial^2 V}{\partial\phi_k\partial\phi_i}\right|_{\phi=\langle\phi\rangle} T^\alpha_{ij}\,\langle\phi_j\rangle = 0\,. \tag{3.14}$$

[4] A vacuum is invariant under a general group transformation $g \in \mathcal{G}$ if $g\,\langle\phi\rangle = \langle\phi\rangle$. Under an infinitesimal group transformation this implies $(\delta_{ij} + i\epsilon^\alpha\, T^\alpha_{ij} + \dots)\,\langle\phi_j\rangle = \langle\phi_i\rangle \Leftrightarrow T^\alpha_{ij}\,\langle\phi_j\rangle = 0$, which is the condition we wrote. Similarly, if the vacuum is not invariant under $g \in \mathcal{G}$, we expect $T^\alpha_{ij}\,\langle\phi_j\rangle \neq 0$ for some α.

The first term vanishes since $\langle\phi\rangle$ is a minimum, yielding

$$m_{ki}\, T^{\alpha}_{ij}\, \langle\phi_j\rangle = 0\,. \tag{3.15}$$

If the transformation leaves the vacuum state unchanged, then from (3.11) we see that the relation above is trivial. However, if the vacuum is not invariant other the transformation, then (3.12) and the equation above imply that m_{ij} has a zero eigenvalue associated to the broken symmetry generators T^{α}. It is now clear that the number of massless particles is determined by the dimension of the coset $\mathcal{G}/\mathcal{H}$, or the number of spontaneously broken symmetries, which defines the vacuum manifold.

3.4 Linear and non-linear sigma models

Consider the so-called linear sigma model given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\cdot\partial^\mu\phi - \frac{\lambda}{4}\,(\phi\cdot\phi - v^2)^2\,, \tag{3.16}$$

where $\phi = (\phi_1,\ldots,\phi_N)^\intercal$ is a real N-component scalar field. This Lagrangian has a global $SO(N)$ symmetry where ϕ transforms as an $SO(N)$ vector. The potential has been chosen so that it is minimized at $|\,\langle\phi\rangle\,| = v$.[5] In this example, the vacuum manifold corresponds to the set of field configurations with $\phi_1^2+\phi_2^2+\cdots+\phi_N^2 = v^2$, which correspond to the $N-1$ dimensional sphere S^{N-1}. We can use the $SO(N)$ symmetry to rotate the vector $\langle\phi\rangle$ to a standard direction, which we fix to $(0,0,\ldots,v)$. The vacuum of the Lagrangian spontaneously break the $SO(N)$ symmetry to the $SO(N-1)$ subgroup acting on the first $N-1$ components. The group $SO(N)$ has $N(N-1)/2$ generators, so the number of GBs, which is equal to the number of broken generators, is: $N(N-1)/2-(N-1)(N-2)/2 = N-1$. The $N-1$ GBs correspond to rotations of the vector ϕ, which leave its length unchanged, i.e. to excitations along the vacuum manifold. The potential energy is unchanged under rotations of ϕ, so these modes are massless. The remaining mode is a radial excitation which changes the length of ϕ, and produces a massive excitation, with mass $\sqrt{2\lambda}\,v$.

The geometric interpretation of the Goldstone fields as excitations along the vacuum manifold S^{N-1} is not manifest in the Lagrangian of (3.16). A more clever (and simpler) way to capture this geometrical interpretation is obtained by using a exponential representation for the ϕ field[6]

$$\phi = \frac{1}{\sqrt{2}}\,(v+\rho)\,U(\pi)\,, \qquad\qquad U(\pi) = e^{i\sqrt{2}\,J^\alpha\,\pi^\alpha/v}\,, \tag{3.17}$$

where J^α ($\alpha = 1,\ldots,N-1$) are the $SO(N)$ broken generators, and ρ and π^a are a new basis for the N ϕ_i fields. The Lagrangian in terms of the new fields is

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\rho\,\partial^\mu\rho - \frac{\lambda}{4}\,(\rho^2+2\rho v)^2 + \frac{1}{2}\,(\rho+v)^2\,\langle\partial_\mu U^\intercal\,\partial^\mu U\rangle\,, \tag{3.18}$$

[5] Note that the constant field configuration $|\,\langle\phi\rangle\,| = 0$ correspond to a maximum of the potential.

[6] It is easy to show that this change of variables, which is only defined for small π^a, has unit Jacobean determinant. This warranties that this field redefinition do not modify the S-matrix elements, i.e. the physical predictions do not get affected by the field reparametrization.

where $\langle\cdot\rangle$ denotes the trace. In this new basis, it is clear that only ρ, which corresponds to the radial excitation, has a potential. As expected, the radial excitation gets a mass $m_\rho = \sqrt{2\lambda}\, v$ and the π fields remain massless. At small momenta compared to m_ρ, i.e. $p \ll m_\rho$, the ρ field decouples and can be integrated out. The Lagrangian reduces to

$$\mathcal{L} = \frac{1}{2}\, v^2\, \langle \partial_\mu U^\intercal\, \partial^\mu U \rangle + \mathcal{O}\left(\frac{p^2}{m_\rho^2}\right), \tag{3.19}$$

which describe universal (model-independent) self-interactions of the GB at very low energies. In order to be sensitive to the particular dynamical structure of the potential, and not just to its symmetry properties, one needs to test the model-dependent part involving the scalar field ρ (see exercise sheet 2).

From this example we can extract a few generic properties of GBs:

i) GBs are derivatively coupled. They describe the (local) orientation of the field ϕ in the vacuum manifold. A constant GB field would correspond to a configuration in which the field ϕ has been rotated by the same angle everywhere in spacetime, and corresponds to a vacuum that is equivalent to the standard vacuum, $\langle\phi\rangle = (0, 0, \ldots, v)$. Therefore, the Lagrangian must be independent of π^a when π^a is constant, so only gradients of π^a appear in the Lagrangian.

   Therefore, if we restrict ourselves to the low momentum regime, the theory is weakly coupled. This property is not clear from the Lagrangian in (3.16), and it implies that when computing processes with this Lagrangian one finds exact (and not very transparent) cancellations among different momentum-independent contributions (see Exercise sheet 2). The two functional forms of the Lagrangian should of course give the same physical predictions.

ii) The Goldstone Lagrangian is non-linear in the Goldstone fields, and it describes an infinite number of Goldstone self-interactions. GBs are constrained to live in the vacuum manifold, described by $\phi_1^2 + \phi_2^2 + \cdots + \phi_N^2 = v^2$ in our example, which is a non-linear constraint on the fields, hence the non-linear Lagrangian.

iii) The fields in the Lagrangian in (3.16) transform linearly under the $SO(N)$ symmetry, hence its name. On the other hand, the field ρ is a singlet of the global symmetry and the fields π^a transform non-linearly under $SO(N)$. That is why the representation of the field in (3.17) is said to be in a non-linear representation of the symmetry.

In general, non-linear symmetry realizations are the most effective way to describe GBs. The technology to do this was first worked out by Callan, Coleman, Wess and Zumino [18, 19], and therefore it is named after them as the CCWZ formalism.

3.5 The CCWZ formalism

The CCWZ formalism provides a general formalism to construct effective Lagrangians for spontaneously broken theories. Consider a theory with global symmetry group

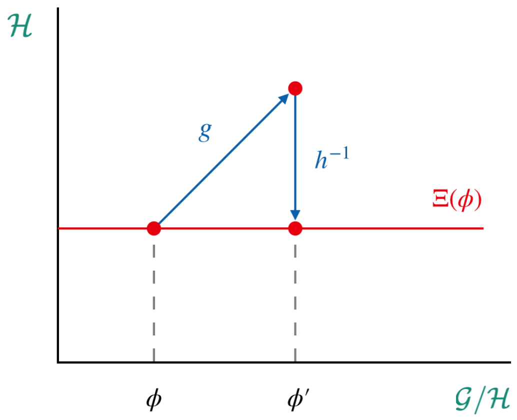


Figure 3.2: Under the action of $g \in \mathcal{G}$, $\Xi(\phi(x))$ transforms into some element of the ϕ' coset. A compensating $h(\phi, g) \in \mathcal{H}$ transformation is needed to go back to $\Xi(\phi')$.

$\mathcal{G}$ whose vacuum is only invariant under the subgroup $\mathcal{H}$, yielding the spontaneous symmetry breaking pattern $\mathcal{G} \to \mathcal{H}$. The vacuum manifold is given by the coset $\mathcal{G}/\mathcal{H}$. Vacuum manifolds are generically curved, as the S^{N-1} sphere in our example.

From the Goldstone theorem we know that the theory contains $N = \dim(\mathcal{G}) - \dim(\mathcal{H})$ GBs that correspond to excitations along the vacuum manifold. Our aim is to find a generic way to describe this set of GBs using only the symmetry information. We start by selecting a standard vacuum Φ_0 as the origin around which to do perturbation theory. We want to find the set of coordinates that describes small fluctuations in the vacuum manifold around the standard vacuum configuration. Let $\Xi(x) \in \mathcal{G}$ be an element that transform the standard vacuum configuration to the local field configuration. This element Ξ is not unique: $\Xi\, h$, with $h \in \mathcal{H}$, gives the same field configuration, since the standard vacuum is invariant under any transformation $h \in \mathcal{H}$, i.e. $h\,\Phi_0 = \Phi_0$. The CCWZ prescription consists in picking a set broken generators $\hat{T}^\alpha$, and choosing $\Xi(x)$ in what we denote as *canonical form*

$$\Xi(x) = \exp\left(i\,\hat{T}^\alpha\, \frac{\phi^\alpha(x)}{f}\right), \tag{3.20}$$

where ϕ^α are the GBs, and f is a normalization factor that makes the exponential dimensionless.

We want to determine how $\Xi(x)$ transforms under a global symmetry transformation $g \in \mathcal{G}$ (note that g is a global transformation, so it does not depend on x). The group element $\Xi(x)$ is transformed into a new element $g\,\Xi(x)$ that corresponds to a different point in the vacuum manifold. In general, the element $g\,\Xi(x)$ is not in its canonical form, but it can be written as[7]

$$g\,\Xi(x) = \Xi(x)'\, h(g, \Xi(x)), \tag{3.21}$$

with $h \in \mathcal{H}$ and where we made explicit the dependence of h on x trough its dependence on $\Xi(x)$. Choosing the canonical form for the transformed element, we

[7]Any element of the group $g \in \mathcal{G}$ can be written as $g = \Xi\, h$, where $\Xi \in \mathcal{G}/\mathcal{H}$ and $h \in \mathcal{H}$.

can therefore write the following transformation law for $\Xi(x)$ under $\mathcal{G}$

$$\Xi(x) \to \Xi(x)' = g\,\Xi(x)\,h(g,\Xi(x))^{-1}\,. \tag{3.22}$$

Figure 3.2 depicts the partition of the group elements (points in the plane) into cosets (vertical lines). Under a transformation $g \in \mathcal{G}$, the transformed element is not necessarily in its canonical form, and one needs a compensating transformation $h(g,\Xi) \in \mathcal{H}$ to bring it back to the canonical form. Note that the transformation in (3.32) is in general non-linear, except for when $g = h$, in which case it becomes linear. The CCWZ prescription provide the most general choice of non-linear representation which becomes linear when restricted to the unbroken subgroup. Any other choice gives the same results for all observables, but does not give the same off-shell Green functions.

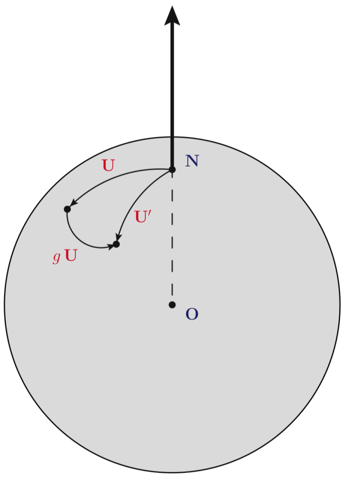


Figure 3.3: Geometrical representation of the S^2 vacuum manifold. The arrow indicates the chosen vacuum direction. [Figure taken from [9]]

Example 1: In the example of Section 3.4, $\mathcal{G} = SO(N)$, $\mathcal{H} = SO(N-1)$, and $\mathcal{G}/\mathcal{H} = S^{N-1}$. One can describe the orientation of the vector $\phi = (\phi_1, \ldots, \phi_N)^\intercal$ by giving the $SO(N)$ matrix $\Xi(x)$, where

$$\phi(x) = \Xi(x)\begin{pmatrix}0\\0\\\vdots\\v\end{pmatrix}. \tag{3.23}$$

The same configuration can also be described by $\Xi(x)\,h(x)$, where $h(x)$ is a matrix of the form

$$\begin{pmatrix}h'(x) & 0\\ 0 & 1\end{pmatrix}, \tag{3.24}$$

with $h'(x)$ an arbitrary $SO(N-1)$ matrix, since

$$\begin{pmatrix}h'(x) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\0\\\vdots\\v\end{pmatrix} = \begin{pmatrix}0\\0\\\vdots\\v\end{pmatrix}, \tag{3.25}$$

Let us take $N = 3$ for simplicity, so that the vacuum manifold is a sphere, which is easy to visualize. Figure 3.3 shows a geometrical representation of this vacuum manifold, with the north pole of the sphere indicating our choice for the standard vacuum. The unbroken symmetry group $\mathcal{H} = SO(2) = U(1)$ corresponds to rotations around the ON axis, where O is the center of the sphere. The group generators are J_1, J_2, J_3, where J_K generate the rotations along the kth axis, and the unbroken generator is J_3. The CCWZ prescription consists in choosing

$$\Xi(x) = \exp\left[\frac{i}{f}\left(J^1\pi^1(x) + J^2\pi^2(x)\right)\right]. \tag{3.26}$$

The matrix $\Xi(x)$ rotates a vector pointing along the z-axis to a different point in the sphere by rotating along longitude lines. Under a $g \in SO(3)$ transformation, the matrix $\Xi(x)$ is transformed to a new matrix $g\,\Xi(x)$ (see Figure 3.3)

$$g\,\Xi(x) = \Xi'(x)h(g,\Xi) \tag{3.27}$$

with $h \in SO(2)$. That h is non-trivial is a well-know property of rotations in three-dimensions. Take a vector and rotate it from A to B and then to C. This transformation is not the same as a direct rotation from A to C, but can be written as a rotation about OA, followed by a rotation from A to C. The transformation h is non-trivial the vacuum manifold $\mathcal{G}/\mathcal{H} = S^2$ is curved.

Example 2: For the second example, let us choose the chiral symmetry of QCD with N_f light quarks: $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$. The symmetry group is now $\mathcal{G} = SU(N_f)_L \times SU(N_f)_R$ and the unbroken subgroup $\mathcal{H} = SU(N_f)_V$. The vacuum manifold is $\mathcal{G}/\mathcal{H} = SU(N_f)_L \times SU(N_f)_R/SU(N_f)_V$, which is isomorphic to $SU(N_f)$. The generators of $\mathcal{G}$ are T_L^α and T_R^α ($\alpha = 1, 2, \ldots, N_f^2-1$), which act on left- and right- handed quarks respectively, and the generators of $\mathcal{H}$ are $T_V^\alpha = T_L^\alpha + T_R^\alpha$. The unbroken generators of $\mathcal{H}$ plus the broken generators $\hat{T}^\alpha$ span the space of all symmetry generators of $\mathcal{G}$, but the possible set of unbroken generators is not fixed, so we have to choose one. There are two commonly used bases for the QCD chiral Lagrangian, the ξ-basis and the Σ-basis, and we will consider them both. There are many simplifications that occur for QCD because the coset space $\mathcal{G}/\mathcal{H}$ is isomorphic to a Lie group. This is not true in general: as we saw, in the $SO(N)$ model, the coset space is S^{N-1} which is not isomorphic to a Lie group for $N \neq 4$.

1. The ξ basis.

One choice of broken generators is to pick $\hat{T}_A^\alpha = T_L^\alpha - T_R^\alpha$. Let a $SU(N_f)_L \times SU(N_f)_R$ transformation be represented in block diagonal form,

$$g = \begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix}, \tag{3.28}$$

where L and R are the $SU(N_f)_L$ and $SU(N_f)_R$ transformations, respectively. The unbroken transformation have the same form with $g_L = g_R = g_V$,

$$h = \begin{pmatrix} g_V & 0 \\ 0 & g_V \end{pmatrix}. \tag{3.29}$$

The Ξ field is then defined using the CCWZ prescription as

$$\Xi = e^{i\,\hat{T}^\alpha \pi^\alpha(x)/f_\xi} = \exp\left[\frac{i}{f_\xi}\begin{pmatrix} T^\alpha\pi^\alpha(x) & 0 \\ 0 & -T^\alpha\pi^\alpha(x) \end{pmatrix}\right] = \begin{pmatrix} \xi(x) & 0 \\ 0 & \xi(x)^\dagger \end{pmatrix}, \tag{3.30}$$

with $\xi(x) = \exp(i\,T^\alpha\pi^\alpha(x)/f_\xi)$. The transformation rule in (3.32) gives

$$\begin{pmatrix} \xi(x) & 0 \\ 0 & \xi(x)^\dagger \end{pmatrix} \to \begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix}\begin{pmatrix} \xi(x) & 0 \\ 0 & \xi(x)^\dagger \end{pmatrix}\begin{pmatrix} g_V^{-1} & 0 \\ 0 & g_V^{-1} \end{pmatrix}. \tag{3.31}$$

This implies the transformation law for $\xi(x)$

$$\xi(x) \to g_L\, \xi(x)\, g_V^{-1}(x) = g_V(x)\, \xi(x)\, g_R^\dagger\,, \tag{3.32}$$

which defines g_V in terms of g_L, g_R and ξ.

2. The Σ basis.

The σ basis is obtained using the choice $\hat{T}^\alpha = T_L^\alpha$ for the broken generators. In this case the CCWZ prescription gives

$$\Xi = e^{i\hat{T}^\alpha \pi^\alpha(x)/f_\Sigma} = \exp\left[\frac{i}{f_\Sigma}\begin{pmatrix} T^\alpha \pi^\alpha(x) & 0 \\ 0 & 0\end{pmatrix}\right] = \begin{pmatrix}\Sigma(x) & 0 \\ 0 & 1\end{pmatrix}\,, \tag{3.33}$$

where $\Sigma(x) = \exp(i\,T^\alpha \pi^\alpha(x)/f_\Sigma)$. The transformation law gives

$$\begin{pmatrix}\Sigma(x) & 0 \\ 0 & 1\end{pmatrix} \to \begin{pmatrix} g_L & 0 \\ 0 & g_R\end{pmatrix}\begin{pmatrix}\Sigma(x) & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} g_V^{-1} & 0 \\ 0 & g_V^{-1}\end{pmatrix}\,. \tag{3.34}$$

which implies $g_V = g_R$, and

$$\Sigma(x) \to g_L\, \Sigma(x)\, g_R^\dagger\,. \tag{3.35}$$

Comparing with (3.32), we see that Σ and ξ are related by[8]

$$\Sigma(x) = \xi^2(x)\,. \tag{3.36}$$

and hence $f_\Sigma = f_\xi/2$.

It is interesting to check how the ξ and Σ fields transform under a parity transformation.[9] From the transformation properties derived in (3.32) and (3.36), we have that

$$\xi \xrightarrow{P} \xi^\dagger\,, \qquad\qquad \Sigma \xrightarrow{P} \Sigma^\dagger\,, \tag{3.38}$$

and therefore, for a Lagrangian in terms of these fields to be parity invariant, it needs to be invariant under the transformation $\Sigma \leftrightarrow \Sigma^\dagger$ ($\xi \leftrightarrow \xi^\dagger$). As we will show in the next section, this is the case for the χPT Lagrangian.

3.6 Effective chiral Lagrangian at lowest order

In Section 3.4, we studied an example of a renormalizable model, the linear sigma model in (3.16), that can be matched at low energies to a Goldstone Lagrangian

[8] Note that g_i $(i = V, L, R)$ denote transformations under the $SU(N_f)_i$ group, and hence $g_i^{-1} = g_i^\dagger$.

[9] The generators of the chiral symmetry group transform under a parity transformation as

$$P: \qquad T_{L,R} \to T_{R,L}\,, \qquad T_V \to T_V\,, \qquad T_A \to -T_A\,, \tag{3.37}$$

and so $\mathcal{G} \xrightarrow{P} \mathcal{G}$, as expected since the QCD Lagrangian is P invariant.

(see (3.19)). Contrary to this example, there is no known way of explicitly deriving the low energy QCD effective theory directly from the original QCD Lagrangian. That is, we do not know how to directly match strongly-coupled QCD to a theory of light pseudo-scalar mesons.[10] One must therefore write down the most general possible effective Lagrangian consistent with the chiral symmetries of the original QCD Lagrangian and with C, P and T invariance. The parameters of this effective Lagrangian could in principle be computed from QCD, but in practice, they have to be obtained by comparison with experiment. This is why χPT constitutes an example of a bottom-up EFT.

In the construction of the χPT Lagragian, we consider the QCD Lagrangian in the *chiral limit*, i.e. with massless quarks and no EW interactions, and take three quark flavors (that is $N_f = 3$) that we identify with the lightest SM quarks: u, d and s.[11] As we saw in section 3.2, the QCD Lagrangian in this limit, $\mathcal{L}^0_{\rm QCD}$, respects the following (global) chiral symmetry

$$\mathcal{G}_\chi = SU(3)_L \times SU(3)_R \times U(1)_V \,, \tag{3.39}$$

that at low energies, when QCD becomes strongly coupled, is spontaneously broken by the quark condensate,

$$\langle 0|\bar{u}_L u_R|0\rangle = \langle 0|\bar{d}_L d_R|0\rangle = \langle 0|\bar{s}_L s_R|0\rangle = \Lambda^3 \neq 0 \,, \tag{3.40}$$

down to the diagonal subgroup

$$\mathcal{H}_\chi = SU(3)_V \times U(1)_V \,. \tag{3.41}$$

We can use the CCWZ prescription to build the most general Lagrangian describing the Goldstone bosons arising from the spontaneous symmetry breaking $\mathcal{G}_\chi \to \mathcal{H}_\chi$. As we saw in the previous section, these can be described by

$$\Sigma(\phi) = \xi^2(\phi) = e^{2i\,\Phi(x)/f} \,, \tag{3.42}$$

where f is a normalization constant and[12]

$$\Phi(x) \equiv \phi^a\, T^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix} \,, \tag{3.43}$$

with T^a $(a = 1, \ldots, 8)$ being the $SU(3)$ generators with $\langle T^a\, T^b\rangle = \delta_{ab}/2$, where $\langle . \rangle$ denotes the trace. As we saw, the Σ field transforms under a global $\mathcal{G}_\chi$ transformation as

$$\Sigma \to \Sigma' = g_L^\dagger\, \Sigma\, g_R \,. \tag{3.44}$$

[10] It is interesting to note that there are simpler theories in $1+1$ dimensions where one can exactly relate a strongly coupled theory of fermions (Thirring model) to a weakly coupled theory of scalars (sine-Gordon model) [20].

[11] The reason for this choice will be clear in the next section, where we introduce quark masses explicitly in the chiral Langrangian.

[12] The η_8 field can be identified η meson, up to $\eta - \eta'$ mixing corrections.

It is easy to construct the most general Lagrangian in terms of the Σ field. The Lagrangian can be organized in terms of increasing powers of momentum or, equivalently, in terms of increasing number of derivatives (the need for an even number of derivatives follows from Lorentz invariance):

$$\mathcal{L}_\chi(\Sigma) = \sum_{n=0}^{\infty} \mathcal{L}_\chi^{(2n)}(\Sigma)\,. \tag{3.45}$$

This Lagrangian contains an infinite number of operators, so we need to apply a power counting to establish a hierarchy between them. Contrary to other types of theories, χPT does not have a small coupling around which to do an expansion. Instead, χPT is an expansion in powers of momenta. Higher-order operators in (3.45) contribute with higher powers of momenta to a given physical amplitude, so at low momenta, their contributions are suppressed. Or in other words, at sufficiently low momenta compared to a certain hadronic scale Λ_χ, an increasing number of derivatives necessarily implies a higher suppression for the associated operators. This let us establish the following power counting

$$\Sigma \sim \mathcal{O}(1)\,, \qquad\qquad \partial_\mu \sim \mathcal{O}\left(\frac{p}{\Lambda_\chi}\right)\,. \tag{3.46}$$

For this counting to make sense, we need to determine the value of the hadronic scale Λ_χ. Since we are ignoring the dynamics of the heavy QCD resonances in our χPT description (see discussion in 3.1), one can argue that the value of Λ_χ should correspond to that of the lightest QCD resonance we are not including: the ρ vector meson, and therefore[13]

$$\Lambda_\chi \sim m_\rho \approx 1 \text{ GeV}\,. \tag{3.47}$$

The most general invariant term with no derivatives must be the product of terms of the form $\langle \Sigma\Sigma^\dagger \ldots \Sigma\Sigma^\dagger \rangle$, where Σ and $\Sigma^\dagger$ alternate. However, $\Sigma\Sigma^\dagger = 1$, so all such terms are constant, and independent of the pion fields. This is just our old result that Goldstone bosons are derivatively coupled. The lowest-order chiral Lagrangian is therefore

$$\mathcal{L}_\chi^{(2)} = \frac{f^2}{4} \left\langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \right\rangle\,, \tag{3.48}$$

where the prefactor $f^2/4$ is completely fixed by the requirement that the fields in Φ are canonically normalized. Note that this Lagrangian contains as many Σ as $\Sigma^\dagger$ fields, showing the parity invariance of QCD. The Lagrangian in (3.50) contains an infinite number of interactions, which are all fixed in terms of a single parameter. Indeed, expanding Σ

$$\Sigma = \mathbb{1} + \frac{2i\Phi}{f} - \frac{2\Phi^2}{f^2} - \frac{4i\Phi^3}{3f^3} + \mathcal{O}\left(\frac{\Phi^4}{f^4}\right)\,, \tag{3.49}$$

[13]In Section 3.7, we will provide another estimate for this scale, based on the consistency of the theory.

in $\mathcal{L}_\chi^{(2)}$, and noting that $\Phi = \Phi^\dagger$, we get

$$\mathcal{L}_\chi^{(2)} = \langle \partial_\mu \Phi \, \partial^\mu \Phi \rangle + \frac{1}{3f^2} \langle [\Phi, \partial_\mu \Phi] \, [\Phi, \partial^\mu \Phi] \rangle + \mathcal{O}\left(\frac{\Phi^6}{f^4}\right) , \tag{3.50}$$

where $[\cdot,\cdot]$ is the commutator. As already argued, operators with more than two derivatives are suppressed by the hadronic scale Λ_χ. Using the power counting in (3.46), we can write $\mathcal{L}_\chi$ as

$$\mathcal{L}_\chi = \frac{f^2}{4} \left[\langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \rangle + \frac{1}{\Lambda_\chi^2} \mathcal{L}_\chi^{(4)} + \frac{1}{\Lambda_\chi^4} \mathcal{L}_\chi^{(6)} + \dots \right] , \tag{3.51}$$

from where the Λ_χ suppression is manifest. The contributions from these additional operators, which are small provided $p \ll \Lambda_\chi$, introduce new unspecified coupling constants that need to be determined experimentally. For the rest of the lecture, we will focus on the lowest-order chiral Lagrangian and postpone the discussion on the higher-order operators to Section 3.7.

Computing, for instance, the $\pi\pi$ scattering amplitude is now a trivial perturbative exercise. One then gets the well-known result [21]

$$\mathcal{A}(\pi^+ \pi^0 \to \pi^+ \pi^0)\big|_{m_q=0} = \frac{(p'_+ - p_+)^2}{f^2} \left[1 + \mathcal{O}\left(\frac{p^2}{\Lambda_\chi^2}\right) \right] , \tag{3.52}$$

where p_+ and p'_+ are the momenta of the incoming and outgoing π^+, respectively. The $\mathcal{O}(p^2/\Lambda_\chi^2)$ corrections stem from contributions arising from the next terms in the chiral Lagrangian in (3.51). Similar results to the one above can be obtained for $\pi\pi \to 4\pi, 6\pi, \dots$ from the $\mathcal{O}\left(\Phi^6/f^4\right)$ terms in (3.50), without introducing any new parameters. The χPT Lagrangian is therefore extremely predictive.

3.6.1 χPT with non-zero masses and EW interactions

The χPT Lagrangian can be extended to account for non-zero quark masses and electroweak interactions of the complete QCD Lagrangian. These terms break explicitly the global chiral symmetry of $\mathcal{L}_{\rm QCD}^0$ discussed in Section 3.2. While the electromagnetic and weak interactions only introduce a small breaking, and they can be treated as a perturbation, the same is only true for light-quark masses. By noting that χPT is an expansion in inverse powers of the hadronic scale Λ_χ, we can infer the expected size of the mass corrections by naive dimensional analysis

$$\frac{m_{u,d}}{\Lambda_\chi} = \mathcal{O}(10^{-2}), \qquad \frac{m_s}{\Lambda_\chi} = \mathcal{O}(10^{-1}) , \qquad \frac{m_c}{\Lambda_\chi} = \mathcal{O}(1) , \tag{3.53}$$

where we see that, while these are indeed small perturbations for the u, d and s quarks, this is not the case for the c quark (or any other heavier quark), whose mass completely breaks our power counting.

Before proceeding to systematically include these chiral symmetry breaking effects, it is illustrative to discuss how $m_q \neq 0$ $(q = u, d, s)$ *explicitly* break the chiral symmetry

i) If $m_u = m_d = m_s \Longrightarrow G_\chi \to \mathcal{H}_V = SU(3)_V \times U(1)_V$.

ii) If $m_u = m_d \neq m_s \Longrightarrow G_\chi \to SU(2)_V \times U(1)_V \times U(1)_S$, where $SU(2)_V$ is commonly referred to as *isospin* and $U(1)_S$ (corresponding to strange quark number) as *strangeness*.

iii) If $m_u \neq m_d \neq m_s \Longrightarrow G_\chi \to U(1)_V \times U(1)_Q \times U(1)_S$, where $U(1)_Q$ is the group associated to the electromagnetic interaction.

These explicit breaking patterns provide useful information when defining the symmetries of a realistic chiral Lagrangian. As we can see, while having non-zero masses will partially break some of the (global) symmetries of QCD, there are still certain residual symmetries that remain unbroken, and which should therefore be preserved also in the chiral Lagrangian.

Spurion analysis

The best way to systematically introduce the (small) breakings of the chiral symmetry is by using the so-called spurion analysis. This approach consists in considering the explicit symmetry-breaking terms as *spurions* to which we assign transformation properties such that the symmetry is restored. Since these terms are not necessarily fields, they are not allowed to transform under the symmetry, and the transformations are spurious, hence their name. However this approach serves as a very good organizing principle when translating the breaking of the chiral symmetry in the QCD Lagrangian to our chiral Lagrangian. The most general extension of the QCD Lagrangian in the chiral limit is given by (in terms of the Dirac spinor $q = (u\ d\ s)^\intercal$)

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}^0_{\text{QCD}} + \bar{q}\,\gamma^\mu\,(v_\mu + \gamma_5\,a_\mu)\,q - \bar{q}\,(s - i\gamma_5\,p)\,q\,, \tag{3.54}$$

with $\mathcal{L}^0_{\text{QCD}}$ as in (3.1) and where the v_μ, a_μ, s and p spurions are 3×3 hermitian matrices. We can identify the spurions in terms of the electroweak interactions and quark masses in the full QCD Lagrangian, with $N_f = 3$, by taking:

$$\begin{aligned} r_\mu &\equiv v_\mu + a_\mu = e\,Q\,A_\mu + \dots\,, \\ l_\mu &\equiv v_\mu - a_\mu = e\,Q\,A_\mu + \frac{g}{\sqrt{2}}(W^\dagger_\mu\,T_+ + h.c.) + \dots\,, \\ s &= \mathcal{M} + \dots\,, \\ p &= 0\,. \end{aligned} \tag{3.55}$$

where the dots denote interactions, such as the ones of the Z boson and the Higgs, that are not phenomenological relevant. Here Q and $\mathcal{M}$ denote the quark charge and mass matrices, respectively,

$$Q = \frac{1}{3}\,\text{diag}(2,-1,-1)\,, \quad \mathcal{M} = \text{diag}(m_u, m_d, m_s)\,, \quad T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{3.56}$$

As we argued before, we assign transformation properties to the spurions to keep the chiral symmetry invariant. Actually, we will go one step further and assign appropriate transformation properties such that the chiral symmetry is promoted to a *local* symmetry of the QCD Lagrangian. This way, we can introduce the SM gauge fields as external spurionic fields. The Lagrangian in (3.54) is invariant under the local $SU(3)_L \times SU(3)_R$ chiral symmetry if

$$\begin{aligned} q_L &\to g_L\, q_L\,, & q_R &\to g_R\, q_R\,,\\ l_\mu &\to g_L\, l_\mu\, g_L^\dagger + i g_L\, \partial_\mu\, g_L^\dagger\,, & r_\mu &\to g_R\, r_\mu\, g_r^\dagger + i g_R\, \partial_\mu\, g_R^\dagger\,,\\ s + ip &\to g_R\,(s+ip)\, g_L^\dagger\,, & s - ip &\to g_L\,(s-ip)\, g_R^\dagger\,. \end{aligned} \tag{3.57}$$

As anticipated, we can use the transformation properties of these spurions to build a generalized chiral Lagrangian. The first thing to note is that, in order to preserve *local* chiral symmetry in the χPT Lagrangian, we need to promote the derivative to a covariant derivative defined as

$$D_\mu\Sigma = \partial_\mu\Sigma - i r_\mu\Sigma + i\Sigma\, l_\mu\,, \qquad D_\mu\Sigma^\dagger = \partial_\mu\Sigma^\dagger + i\Sigma^\dagger\, r_\mu - i l_\mu\Sigma^\dagger\,. \tag{3.58}$$

Moreover, local chiral symmetry invariance also allows us to introduce new operators in terms of the field strength tensors

$$F_L^{\mu\nu} = \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu]\,, \qquad F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu]\,. \tag{3.59}$$

From (3.58), it is clear how we can extend our chiral counting also to the l_μ and r_μ spurions

$$l_\mu, r_\mu \sim \mathcal{O}\left(\frac{p}{\Lambda_\chi}\right)\,, \qquad F_{L,R}^{\mu\nu} \sim \mathcal{O}\left(\frac{p^2}{\Lambda_\chi^2}\right)\,. \tag{3.60}$$

Concerning the spin-0 spurions, in principle one can formulate χPT as an independent expansion in both derivatives and quark masses. However, it is convenient to combine the two expansions into a single one by making use of the relations between meson and quark masses. As we will see below, the choice of counting rule

$$s, p \sim \mathcal{O}\left(\frac{p^2}{\Lambda_\chi^2}\right)\,, \tag{3.61}$$

implies the relation $M_\pi^2 \propto m_q$, and leads to the Gell-Mann–Okubo mass formula (3.70), which is well reproduced experimentally.

We can now write a general χPT Lagrangian including the spurion sources. At lowest order in the chiral counting, this Lagrangian reads

$$\mathcal{L}_\chi^{(2)} = \frac{f^2}{4}\, \langle D_\mu\Sigma^\dagger D^\mu\Sigma + \chi\Sigma^\dagger + \Sigma\chi^\dagger\rangle\,, \tag{3.62}$$

with

$$\chi = 2B\,(s+ip)\,, \tag{3.63}$$

where B is an undetermined constant with dimensions of energy. Thanks to the spurion analysis, it is now trivial to read off the light-meson interactions with the SM gauge fields. For instance, for the electromagnertic interactions, we have (taking the A_μ piece in (3.55) from l_μ and r_μ)

$$\begin{aligned} \mathcal{L}_\chi^{(2)} &\supset -2ie\, A_\mu\, \langle \partial^\mu\, \Phi[Q,\Phi]\rangle + e^2\, A_\mu A^\mu\, \langle [Q,\Phi]\,[Q,\Phi]\rangle\\ &\supset [ie\, A_\mu\,(\pi^+\partial^\mu\pi^- + K^+\partial^\mu K^-) + h.c.] + e^2\, A_\mu A^\mu\,(\pi^+\pi^- + K^+K^-)\,, \end{aligned} \tag{3.64}$$

which defines the QED interactions with the light-mesons.

Connecting the chiral parameters to measurable quantities

The two undetermined quantities of the generalized chiral Lagrangian, f and B, are connected to two fundamental order parameters of the spontaneous chiral symmetry breaking: the pion decay constant f_π, defined by $\langle 0|\bar{u}\,\gamma_\mu\gamma_5\,d|\pi^+\rangle = i\sqrt{2}\,f_\pi\,p_\mu$, and the quark condensate $\langle 0|\bar{q}_L q_R|0\rangle$. Indeed, by differentiating with respect to the external sources we have

$$\begin{aligned}
\langle 0|\bar{u}\,\gamma^\mu\gamma_5\,d|\pi^+\rangle &= \langle 0|\frac{\delta\mathcal{L}_{\rm QCD}}{\delta a_\mu^{12}}|\pi^+\rangle = \langle 0|\frac{\delta\mathcal{L}_\chi}{\delta a_\mu^{12}}|\pi^+\rangle \\
&= \frac{if^2}{2}\,\langle 0|\left[\Sigma^\dagger\partial^\mu\Sigma - \Sigma\,\partial^\mu\Sigma^\dagger + \ldots\right]_{12}|\pi^+\rangle \\
&= -\frac{2f}{\sqrt{2}}\,\langle 0|\left[\partial^\mu\phi + \ldots\right]|\pi^+\rangle \\
&\approx \sqrt{2}f\,p_\mu\,,
\end{aligned} \tag{3.65}$$

from where we get $f = f_\pi$, up to higher order corrections in the chiral expansion. The value of f_π can be determined experimentally from the W-mediated semileptonic decay of pion $\pi^+ \to \ell\nu$, hence its name, yielding $f = f_\pi = 92.4$ MeV. One could do the same derivation to measure the corresponding Kaon decay constant, defined analogously to f_π and determined experimentally to be $f_K = 114$ MeV. The difference between f_π and f_K is an $\mathcal{O}(p^4)$ effect, which goes beyond the lowest order. However, since higher-order effects are expected to be larger in the case of the Kaon (due to the larger mass of the strange quark), the most natural determination of f at lowest order is provided by f_π.

Proceeding in a similar way with the quark condensate, we find that

$$\begin{aligned}
\langle 0|\bar{q}_L^i q_R^j|0\rangle &= -\,\langle 0|\frac{\delta\mathcal{L}_{\rm QCD}}{\delta(s-ip)^{ij}}|0\rangle = -\,\langle 0|\frac{\delta\mathcal{L}_\chi}{\delta(s-ip)^{ij}}|0\rangle \\
&= -\frac{f^2}{2}B\,\langle 0|\Sigma_{ij} + \ldots|0\rangle \\
&= -\frac{f^2}{2}B\,\langle 0|\delta_{ij} + \ldots|0\rangle \\
&\approx -\frac{f^2}{2}B\,\delta_{ij}\,.
\end{aligned} \tag{3.66}$$

Note that contrary to f, B is not an observable quantity, so we cannot determine its value experimentally. However, we can relate the product $B \times m_q$ to the meson masses. Expanding the $\langle\chi\Sigma^\dagger + \Sigma\chi^\dagger\rangle$ piece in $\mathcal{L}_\chi^{(2)}$, and taking $s = \mathcal{M}$ and $p = 0$, we have

$$\begin{aligned}
\mathcal{L}_\chi^{(2)} \supset \frac{f^2}{4}\,\langle\chi\Sigma^\dagger + \Sigma\chi^\dagger\rangle &= \frac{f^2}{4}2B\,\langle\mathcal{M}(\Sigma^\dagger + \Sigma)\rangle \\
&= 2B\left[\frac{f^2}{2}\,\langle\mathcal{M}\rangle - \langle\mathcal{M}\Phi^2\rangle + \frac{1}{3f^2}\,\langle\mathcal{M}\Phi^4\rangle + \mathcal{O}\left(\frac{\Phi^6}{f^4}\right)\right].
\end{aligned} \tag{3.67}$$

The constant term $Bf^2 \langle \mathcal{M} \rangle$ is unphysical and can be dropped. From the term quadratic in the light-meson fields, we find[14]

$$\begin{aligned} M_{\pi^0}^2 &= M_{\pi^+}^2 = 2B\,\hat{m} + \mathcal{O}(\epsilon)\,, & M_{K^+}^2 &= B\,(m_u + m_s)\,, \\ M_{K^0}^2 &= B\,(m_d + m_s)\,, & M_{\eta_8}^2 &= \frac{2}{3}B\,(\hat{m} + 2m_s) + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{3.69}$$

with $\hat{m} = (m_u + m_d)/2$. Since we have four meson masses written in terms of three quark masses, we can obtain an absolute prediction for one of the meson masses in terms of the others. In the limit $m_u = m_d$, this is the famous Gell-Mann–Okubo mass formula [22, 23]

$$4M_{K^0}^2 = 3M_{\eta_8}^2 + M_{\pi^0}^2\,, \tag{3.70}$$

which gives 0.99 GeV^2 = 0.92 GeV^2 if we take $M_{\eta_8} = M_\eta$. The validity of the Gell-Mann–Okubo relation provides an important consistency check for our power counting assignment in (3.61). Moreover, it shows that $\mathcal{O}(m_q^2)$ corrections to the meson masses (which would arise from higher-order operators in $\mathcal{L}_\chi$) are small. In addition to $\mathcal{O}(m_q^2)$ corrections, the mass relations in (3.69) are affected by QED effects. At leading order in the chiral expansion, they only depend on meson charges and we can write[15]

$$\begin{aligned} M_{\pi^0}^2 &= 2B\,\hat{m} + \mathcal{O}(\epsilon, m_q^2) & M_{\pi^+}^2 &= 2B\,\hat{m} + \alpha\Delta_{\text{em}} + \mathcal{O}(m_q^2, \alpha m_q)\,, \\ M_{K^0}^2 &= B\,(m_d + m_s) + \mathcal{O}(m_q^2)\,, & M_{K^+}^2 &= B\,(m_u + m_s) + \alpha\Delta_{\text{em}} + \mathcal{O}(m_q^2, \alpha m_q)\,, \\ M_{\eta_8}^2 &= \frac{2}{3}B\,(\hat{m} + 2m_s) + \mathcal{O}(\epsilon, m_q^2)\,, \end{aligned} \tag{3.71}$$

Although the absolute value of the quark masses cannot be determined within χPT, this theory does provide information about the quark-mass ratios. Neglecting the small $\mathcal{O}(\epsilon, m_q^2, \alpha m_q)$ corrections, we find

$$\begin{aligned} \frac{m_d - m_u}{m_u + m_d} &= \frac{(M_{K^0}^2 - M_{K^+}^2) - (M_{\pi^0}^2 - M_{\pi^+}^2)}{M_{\pi^0}^2} = 0.29\,, \\ \frac{m_s - m_u}{m_u + m_d} &= \frac{M_{K^0}^2 - M_{\pi^0}^2}{M_{\pi^0}^2} = 12.6\,. \end{aligned} \tag{3.72}$$

Interestingly, the three light-quark masses turn out to be very different. From the expressions above, we have

$$\frac{m_u}{m_d} = 0.55\,, \qquad \frac{m_s}{m_d} = 20.3\,. \tag{3.73}$$

[14] The $\mathcal{O}(\epsilon)$ isospin-breaking corrections, with

$$\epsilon = \frac{B}{4}\frac{(m_u - m_d)^2}{m_s - \hat{m}}\,, \tag{3.68}$$

originate from a small mixing term between the π^0 and the η_8 fields.

[15] QED corrections to the meson masses are computed in Section 3.9.

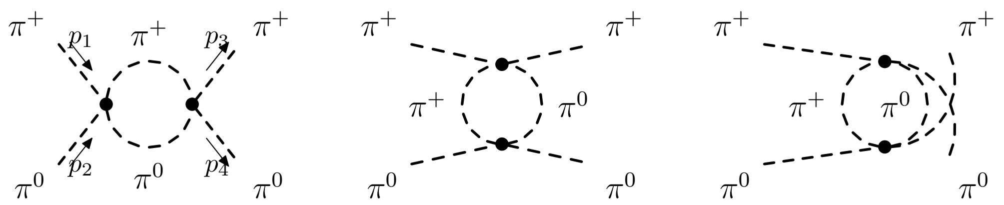


Figure 3.4: Diagrams contributing to $\pi-\pi$ scattering at one loop. The circle denotes a vertex from $\mathcal{L}_\chi^{(2)}$.

As we anticipated at the beginning of this section (see (3.53)), quark-mass corrections are dominated by m_s, which is large compared with m_u and m_d. Also, note that even though the difference $m_d - m_u$ is not small compared with the individual up- and down-quark masses, isospin breaking effects are controlled by the small ratio $(m_u - m_d)/\Lambda_\chi$, and therefore the isospin symmetry $SU(2)_V$ turns out to be a very good (approximate) symmetry of the strong interactions.

The new interactions in the Lagrangian of (3.62) introduce mass corrections to the $\pi\pi$ scattering amplitude derived in (3.52). Including these mass corrections (see (3.67)), we find now the full result [21]

$$\mathcal{A}(\pi^+\pi^0 \to \pi^+\pi^0) = \frac{(p'_+ - p_+)^2 - M_\pi^2}{f^2}\left[1 + \mathcal{O}\left(\frac{p^2}{\Lambda_\chi^2}\right)\right] . \tag{3.74}$$

Since $f = f_\pi$ is fixed from the pion decay, this result is an absolute prediction of χPT.

3.7 Loop corrections in χPT

χPT is a well-defined quantum field theory. As such, we must also consider quantum loops in this theory. Before discussing the most general case, it is illustrative to focus on the $\pi - \pi$ scattering amplitude at one-loop. There is a lot of information that can be obtained by a simple power counting analysis, without explicitly doing the computation. We can easily estimate the order in momenta of the corresponding loop graphs in Figure 3.4:

$$\mathcal{A}_{\text{loop}} \sim \int \frac{d^4p}{(2\pi)^4}\,\frac{p^2}{f^2}\,\frac{1}{p^4}\,\frac{p^2}{f^2} \sim \frac{p^2}{f^2} \times \frac{p^2}{(4\pi f)^2} \,. \tag{3.75}$$

Here p generically denotes any term that is of $\mathcal{O}(p_i, M_\pi)$, with p_i being the external momenta. The factor of $1/p^4$ is from the two internal propagators, and p^2/f^2 from the vertices. We see that the loop correction comes with a power suppression with respect to the tree level effect in (3.52), given by $p^2/(4\pi f)^2$. The additional factor of 4π is important because it gives some range to apply χPT to low energy processes involving pions and kaons. If the suppression factor were f instead of $4\pi f$, χPT would not be useful even for pions, since $M_\pi > f$. Note that there is an important difference here with respect to other quantum field theories: in χPT, loops are not suppressed by a small coupling but by powers of momentum over a hadronic scale. This is due to the fact that the chiral symmetry requires the pion couplings to vanish at zero momenta (and zero pion mass).

By simple power counting, it is also easy to see that the loop diagram is UV divergent. If we use a regularization that preserves the symmetries of the Lagrangian, such as dimensional regularization, the result of the loop calculation will necessarily be symmetric. Using dimensional regularization with $d = 4 - 2\epsilon$, we estimate the loop integral to give

$$\mathcal{A}_{\text{loop}} \sim \frac{p^4}{16\pi^2 f^4}\left[a\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{p^2}\right) + b\right], \tag{3.76}$$

with a, b being numerical constants that can be obtained from the explicit loop computation. Since, by construction, $\mathcal{L}_\chi$ contains all terms permitted by the chiral symmetry, all loop divergences can be absorbed by renormalizing the $\mathcal{L}_\chi$ couplings.[16] To absorb the divergence in (3.76), we need to add counterterms of $\mathcal{O}(p^4)$. Recalling our momentum expansion in (3.51), this means that we need to renormalize the couplings in $\mathcal{L}_\chi^{(4)}$. For our computation, in the $M_\pi = 0$ limit, we only need a subset of the $\mathcal{L}_\chi^{(4)}$ operators[17]

$$\begin{aligned}\frac{f^2}{4}\frac{1}{\Lambda_\chi^2}\mathcal{L}_\chi^{(4)} \supset\; & L_1 \left\langle D_\mu\Sigma^\dagger D^\mu\Sigma\right\rangle^2 + L_2 \left\langle D_\mu\Sigma^\dagger D_\nu\Sigma\right\rangle \left\langle D^\mu\Sigma^\dagger D^\nu\Sigma\right\rangle \\ & + L_3 \left\langle D_\mu\Sigma^\dagger D^\mu\Sigma\, D_\nu\Sigma^\dagger D^\nu\Sigma\right\rangle ,\end{aligned} \tag{3.77}$$

which produce a four-pion interaction of $\mathcal{O}(p^4)$, when expanding Σ. The total $\mathcal{O}(p^4)$ $\pi - \pi$ scattering amplitude is a combination of tree graphs involving these interactions, and the loop contribution in (3.76). Schematically, we have

$$\mathcal{A}^{(4)} \sim \frac{p^4}{16\pi^2 f^4}\left[a\,\log\frac{\mu^2}{p^2} + b'\right] + \frac{4\,p^4}{f^4}L_i^r(\mu)\,, \tag{3.78}$$

written in terms of the renormalized couplings $L_i^r(\mu)$, which absorb the loop divergence, i.e.

$$L_i^r(\mu) = L_i(\mu) + \frac{f^2}{4}\frac{1}{16\pi^2 f^2}\left(a\,\frac{1}{\epsilon} + b - b'\right). \tag{3.79}$$

The value of the constant $b - b'$ define a choice for the renormalization scheme. The total amplitude is μ independent. Therefore, in order to match the μ-dependence from the loop, $L_i^r(\mu)$ has to be of the form

$$L_i^r(\mu) = \frac{f^2}{4}\frac{1}{16\pi^2 f^2}\left[-a\,\log\frac{\mu^2}{\Lambda_\chi^2} + c_i\right], \tag{3.80}$$

[16] A mass-dependent regularization, such as a momentum cutoff, would explicitly break the chiral symmetry. As a result, we would obtain divergences that cannot be absorbed into a renormalization of the χPT couplings, making our loop computations more involved.

[17] The full $\mathcal{L}_\chi^{(4)}$ Lagrangian is presented in Section 3.8.

with c_i an undetermined numerical coefficient. This is the same as noting that the $L_i^r(\mu)$ couplings need to satisfy the (trivial) RGE equation

$$\frac{dL_i^r(\mu)}{d\log\mu} = -\frac{a}{32\pi^2}\,. \tag{3.81}$$

From (3.80), we see that a shift in the renormalization scale μ is compensated by a corresponding shift in $L_i^r(\mu)$, i.e.

$$L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{f^2}{4}\frac{a}{(4\pi f)^2}\log\frac{\mu_1^2}{\mu_2^2}\,, \tag{3.82}$$

and so we expect $L_i^r(\mu)$ to be of the same order as these shifts. This naive dimensional analysis estimate let us infer the value of the hadronic scale Λ_χ in (3.77) in terms of the loop corrections

$$\Lambda_\chi \sim 4\pi f \approx 1\ \text{GeV}\,, \tag{3.83}$$

which is compatible with our initial estimate of $\Lambda_\chi \sim m_\rho$, based on general EFT arguments.

3.7.1 Weinberg's power counting theorem

We can generalize the features we observed in $\pi-\pi$ scattering to an arbitrary χPT amplitude. A generic χPT connected diagram gives the amplitude

$$\mathcal{A} \sim \int \left[\frac{d^4p}{(2\pi)^4}\right]^{N_L} \left[\frac{1}{p^2}\right]^{N_I} \prod_k \left(p^k\right)^{N_k}\,, \tag{3.84}$$

where N_L is the number of loops, N_I the number of internal lines, N_k the number of vertices of $\mathcal{O}(p^k)$ ($k = 2, 4, \dots$), and p represents any term that is of $\mathcal{O}(p_i, M_j)$, with p_i and M_j being the external momenta and the light-meson masses, respectively. In a mass-independent regularization scheme, such as dimensional regularization, the only dimensional parameter is p. The amplitude must therefore have the following form $\mathcal{A} \sim p^D$, which defines its order in the chiral expansion. We can obtain the value of D by simple dimensional counting

$$D = 4N_L - 2N_I + \sum_k k\, N_k\,. \tag{3.85}$$

For any Feynman graph, one can show that

$$N_V - N_I + N_L = 1\,, \tag{3.86}$$

where $N_V = \sum_k N_k$ is the total number vertices. This is a mathematical theorem in graph theory known as Euler's formula. Using Euler's formula, we can remove the number of internal lines from (3.85). By doing so, we obtain Weinberg's power counting formula [24]

$$D = 2 + 2N_L + \sum_k (k-2)\, N_k\,. \tag{3.87}$$

This expression connects the chiral counting in $\mathcal{L}_\chi$ and the loop counting into a single power counting in terms of powers of p. The chiral Lagrangian starts at $\mathcal{O}(p^2)$, so $k \geq 2$, and each term is non-negative. As a result, only a finite number of terms in $\mathcal{L}_\chi$ are needed to work at a given order in p, and the Lagragian is renormalizable order by order in the p expansion. The lowest order in the chiral expansion is $D = 2$. This is obtained for $N_L = 0$ and $k = 2$, which correspond to only tree-level graphs with $\mathcal{L}_\chi^{(2)}$ insertions. Since N_2 drops from (3.87), we can have an arbitrary number of $\mathcal{L}_\chi^{(2)}$ interactions, without increasing the order in p. The next order correction is at $\mathcal{O}(p^4)$. As we saw in the $\pi - \pi$ scattering example, at this order we have two possible contributions: tree-level graphs with one insertion of $\mathcal{L}_\chi^{(4)}$ ($N_L = 0$ and $N_4 = 1$), and loop graphs with an arbitrary number of $\mathcal{L}_\chi^{(2)}$ insertions ($N_L = 1$ and $N_{k>2} = 0$). One can proceed similarly for any order in the momentum expansion to find the different contributions.

Weinberg's power counting formula has an additional implication. Since each loop increases the chiral order, a generic χPT amplitude at $\mathcal{O}(p^4)$ has the same form as in (3.78), with a single logarithm. This is different to the SM case where, as we saw, there is an infinite series of logarithms that need to be resummed (see Section 1.5.2). Since in χPT there is just a single logarithm, which is commonly referred to as *chiral logarithm*, there is no need to do resummation, and one normally defines all the χPT $\mathcal{L}_\chi^{(4)}$ couplings at the same scale, typically $\mu = m_\rho$, $4\pi f_\pi$ or 1 GeV. The chiral logarithm is completely determined in terms of the lowest order terms in the chiral Lagrangian. On the other hand, the $\mathcal{L}_\chi^{(4)}$ couplings introduce additional unknown parameters. One can hope that the chiral logarithm is numerically more important than the $\mathcal{L}_\chi^{(4)}$ contributions, when one picks $\mu \sim 1$ GeV. This is formally correct, since $p^4 \log(\Lambda_\chi^2/p^2) > p^4$. However, in practice $p \sim M_\pi$ or M_K, for which the logarithms give respectively 3.9 and 1.4. This is not a very large enhancement (especially for kaons). Still, the chiral logarithm provides useful estimates of the size of the $\mathcal{O}(p^4)$ corrections.

3.8 The $\mathcal{L}_\chi^{(4)}$ Lagrangian

In order to derive the $\mathcal{L}_\chi^{(4)}$ Lagrangian, one needs to list all possible operators of order $\mathcal{O}(p^4)$ invariant under parity, charge conjugation and local chiral symmetry. Although it is not strictly necessary, it is very convenient to reduce this list to a minimal basis of operators by using equations of motion, trace identities and integration by parts. The operator reduction procedure is similar to the one described in 2.3.1. The most general $\mathcal{L}_\chi^{(4)}$ Lagrangian reads [25]

$$\begin{aligned}
\mathcal{L}_4 = {} & L_1 \langle D_\mu \Sigma^\dagger D^\mu \Sigma \rangle^2 + L_2 \langle D_\mu \Sigma^\dagger D_\nu \Sigma \rangle \langle D^\mu \Sigma^\dagger D^\nu \Sigma \rangle \\
& + L_3 \langle D_\mu \Sigma^\dagger D^\mu \Sigma D_\nu \Sigma^\dagger D^\nu \Sigma \rangle + L_4 \langle D_\mu \Sigma^\dagger D^\mu \Sigma \rangle \langle \Sigma^\dagger \chi + \chi^\dagger \Sigma \rangle \\
& + L_5 \langle D_\mu \Sigma^\dagger D^\mu \Sigma \left(\Sigma^\dagger \chi + \chi^\dagger \Sigma \right) \rangle + L_6 \langle \Sigma^\dagger \chi + \chi^\dagger \Sigma \rangle^2 \\
& + L_7 \langle \Sigma^\dagger \chi - \chi^\dagger \Sigma \rangle^2 + L_8 \langle \chi^\dagger \Sigma \chi^\dagger \Sigma + \Sigma^\dagger \chi \Sigma^\dagger \chi \rangle \\
& - i L_9 \langle F_R^{\mu\nu} D_\mu \Sigma D_\nu \Sigma^\dagger + F_L^{\mu\nu} D_\mu \Sigma^\dagger D_\nu \Sigma \rangle + L_{10} \langle \Sigma^\dagger F_R^{\mu\nu} \Sigma F_{L\mu\nu} \rangle
\end{aligned}$$

i	$L_i^r(M_\rho) \times 10^3$	Source
1	0.4 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$
2	1.4 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$
3	-3.5 ± 1.1	K_{e4}, $\pi\pi \to \pi\pi$
4	-0.3 ± 0.5	Zweig rule
5	1.4 ± 0.5	$F_K : F_\pi$
6	-0.2 ± 0.3	Zweig rule
7	-0.4 ± 0.2	Gell-Mann–Okubo, L_5, L_8
8	0.9 ± 0.3	$M_{K^0} - M_{K^+}$, L_5, $(m_s - \hat{m}) : (m_d - m_u)$
9	6.9 ± 0.7	$\langle r^2 \rangle_V^\pi$
10	-5.5 ± 0.7	$\pi \to e\nu\gamma$

Table 3.1: Phenomenological values of the renormalized couplings $L_i^r(M_\rho)$. The last column shows the source used to extract this information. [Taken from [11]]

$$+ \; H_1 \, \langle F_{R\mu\nu} F_R^{\mu\nu} + F_{L\mu\nu} F_L^{\mu\nu} \rangle \; + \; H_2 \, \langle \chi^\dagger \chi \rangle \, . \tag{3.88}$$

As we see, the $\mathcal{O}(p^4)$ chiral Lagrangian introduce new couplings that are not determined by the chiral symmetry. These couplings parametrize our ignorance on the details of the underlying QCD dynamics. The operators with H_1 and H_2 couplings do not contain light-meson fields and cannot be measured directly. Therefore, we need to determine ten low-energy coefficients to specify the $\mathcal{O}(p^4)$ light-meson dynamics. In principle, these could be determined from QCD, for instance using lattice QCD. However, at present, the best determination of these couplings is obtained by fixing them from experimental measurements. As shown in Table 3.1, the experimental values for these coefficients are in reasonable agreement with the power counting estimate

$$L_i \sim \frac{f_\pi^2/4}{\Lambda_\chi^2} \sim \frac{1}{4(4\pi)^2} \sim 10^{-3} \, , \tag{3.89}$$

indicating a good convergence of the χPT momentum expansion below the resonance region, i.e. for $p < M_\rho$.

It seems reasonable to expect that the lowest-mass resonances, such as the ρ meson, should have an impact on the dynamics of the light-mesons. Interestingly, one can show that the values of the $\mathcal{L}_\chi^{(4)}$ couplings are mostly saturated by lowest-mass resonance exchange. For more details, the interested reader is encouraged to read the original article where this is shown [26].

3.9 QED corrections to the light-meson masses

As we saw in Section 3.6.1, we can use spurion analysis to compute the electromagnetic coupling to light mesons. We found that the $\mathcal{O}(p^2)$ QED couplings are given by

$$\begin{aligned} \mathcal{L}_\chi^{(2)} &\supset \frac{f^2}{4} [\, ie \, A_\mu \, \langle \partial^\mu \Sigma \, [Q, \Sigma^\dagger] \rangle + \text{h.c.}] + \frac{f^2}{4} \, e^2 \, A_\mu A^\mu \, \langle [\Sigma^\dagger, Q] \, [Q, \Sigma] \rangle \\ &\supset [ie \, A_\mu \, (\pi^+ \partial^\mu \pi^- + K^+ \partial^\mu K^-) + \text{h.c.}] + e^2 \, A_\mu A^\mu \, (\pi^+ \pi^- + K^+ K^-) \, , \end{aligned} \tag{3.90}$$

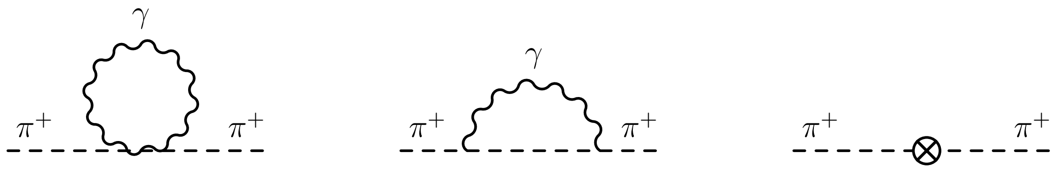


Figure 3.5: Diagrams contributing to the electromagnetic pion mass difference. Identical diagrams are obtained for the K^+.

with $Q = \mathrm{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$. We can evaluate the electromagnetic correction to the light-meson masses by computing one-photon loops involving these interactions (see Figure 3.5). In addition to the loop contributions, we also need to include the local counterterms, in analogy to what we did in Section 3.7 in the $\pi - \pi$ scattering example. To match the loop contribution, the effective Lagrangian that contains the counterterm must read as follows

$$\begin{aligned} \mathcal{L}_{\mathrm{em}}^{(2)} &= \frac{f^2}{4}\, e^2 \Delta^2 \,\langle [\Sigma^\dagger, Q]\,[Q, \Sigma]\rangle \\ &\supset e^2\,\Delta^2\,(\pi^+\pi^- + K^+K^-)\,, \end{aligned} \tag{3.91}$$

with Δ being an undetermined coupling. Note that these electromagnetic corrections only affect the charged light-mesons, and that they are the same for π and for K. The loop diagrams in Figure 3.5 are zero in the massless meson limit, so the coupling Δ does not get renormalized and it is independent of the renormalization scale, i.e. $\Delta \neq \Delta(\mu)$. Neglecting terms of $\mathcal{O}(\alpha\, M^2_{\pi,K})$, the electromagnetic correction to the light-meson masses is entirely given by (3.91), and thus we find

$$\hat{M}^2_{\pi^+} = \hat{M}^2_{K^+} = e^2\Delta^2\,, \qquad \hat{M}^2_{\pi^0} = \hat{M}^2_{K^0} = \hat{M}^2_{\eta} = 0\,, \tag{3.92}$$

We can extract the value of Δ experimentally from the pion mass difference

$$M^2_{\pi^+} - M^2_{\pi^0} = e^2\Delta^2\,. \tag{3.93}$$

Using naive dimensional analysis, we can estimate the value of Δ to be

$$\Delta = \frac{\Lambda_\chi}{4\pi}\, C = \frac{4\pi f_\pi}{4\pi}\, C = f_\pi\, C = 92.4\,\mathrm{MeV} \times C\,, \tag{3.94}$$

with C of $\mathcal{O}(1)$. This is in good agreement with experimental value of the pion mass difference $M^2_{\pi^+} - M^2_{\pi^0} = 1.26 \times 10^3\,\mathrm{MeV}^2$, from where one extracts the value $\Delta = 117$ MeV.

Bibliography

[1] S. Weinberg, *A Model of Leptons*, *Phys. Rev. Lett.* **19** (1967) 1264–1266.

[2] M. Neubert, *Heavy quark effective theory*, *Subnucl. Ser.* **34** (1997) 98–165, [hep-ph/9610266].

[3] T. Becher, A. Broggio, and A. Ferroglia, *Introduction to Soft-Collinear Effective Theory*, *Lect. Notes Phys.* **896** (2015) pp.1–206, [arXiv:1410.1892].

[4] J. F. Nieves and P. B. Pal, *Generalized Fierz identities*, *Am. J. Phys.* **72** (2004) 1100–1108, [hep-ph/0306087].

[5] W. Buchmuller and D. Wyler, *Effective Lagrangian Analysis of New Interactions and Flavor Conservation*, *Nucl. Phys.* **B268** (1986) 621–653.

[6] B. Grzadkowski, M. Iskrzynski, M. Misiak, and J. Rosiek, *Dimension-Six Terms in the Standard Model Lagrangian*, *JHEP* **10** (2010) 085, [arXiv:1008.4884].

[7] G. Buchalla, A. J. Buras, and M. E. Lautenbacher, *Weak decays beyond leading logarithms*, *Rev. Mod. Phys.* **68** (1996) 1125–1144, [hep-ph/9512380].

[8] L. Berthier and M. Trott, *Towards consistent Electroweak Precision Data constraints in the SMEFT*, *JHEP* **05** (2015) 024, [arXiv:1502.02570].

[9] A. Pich, *Effective Field Theory with Nambu-Goldstone Modes*, in *Les Houches summer school: EFT in Particle Physics and Cosmology Les Houches, Chamonix Valley, France, July 3-28, 2017*, 2018. arXiv:1804.05664.

[10] A. V. Manohar, *Effective field theories*, in *Quarks and colliders. Proceedings, 10th Winter Institute, Lake Louise, Canada, February 19-25, 1995*, pp. 274–315, 1995. hep-ph/9508245.

[11] A. Pich, *Chiral perturbation theory*, *Rept. Prog. Phys.* **58** (1995) 563–610, [hep-ph/9502366].

[12] G. Colangelo and G. Isidori, *An Introduction to ChPT*, *Frascati Phys. Ser.* **18** (2000) 333–376, [hep-ph/0101264].

[13] G. Ecker, *Chiral perturbation theory*, *Prog. Part. Nucl. Phys.* **35** (1995) 1–80, [hep-ph/9501357].

[14] D. J. Gross and F. Wilczek, *Ultraviolet Behavior of Nonabelian Gauge Theories*, *Phys. Rev. Lett.* **30** (1973) 1343–1346.

[15] H. D. Politzer, *Reliable Perturbative Results for Strong Interactions?*, *Phys. Rev. Lett.* **30** (1973) 1346–1349.

[16] S. Durr et al., *Ab-Initio Determination of Light Hadron Masses*, *Science* **322** (2008) 1224–1227, [`arXiv:0906.3599`].

[17] J. Goldstone, *Field Theories with Superconductor Solutions*, *Nuovo Cim.* **19** (1961) 154–164.

[18] S. R. Coleman, J. Wess, and B. Zumino, *Structure of phenomenological Lagrangians. 1.*, *Phys. Rev.* **177** (1969) 2239–2247.

[19] C. G. Callan, Jr., S. R. Coleman, J. Wess, and B. Zumino, *Structure of phenomenological Lagrangians. 2.*, *Phys. Rev.* **177** (1969) 2247–2250.

[20] S. R. Coleman, *The Quantum Sine-Gordon Equation as the Massive Thirring Model*, *Phys. Rev.* **D11** (1975) 2088. [,128(1974)].

[21] S. Weinberg, *Pion scattering lengths*, *Phys. Rev. Lett.* **17** (1966) 616–621.

[22] M. Gell-Mann, *Symmetries of baryons and mesons*, *Phys. Rev.* **125** (1962) 1067–1084.

[23] S. Okubo, *Note on unitary symmetry in strong interactions*, *Prog. Theor. Phys.* **27** (1962) 949–966.

[24] S. Weinberg, *Phenomenological Lagrangians*, *Physica* **A96** (1979), no. 1-2 327–340.

[25] J. Gasser and H. Leutwyler, *Chiral Perturbation Theory: Expansions in the Mass of the Strange Quark*, *Nucl. Phys.* **B250** (1985) 465–516.

[26] G. Ecker, J. Gasser, A. Pich, and E. de Rafael, *The Role of Resonances in Chiral Perturbation Theory*, *Nucl. Phys.* **B321** (1989) 311–342.