EE5111 Mini Project 1

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1 Introduction

In this Mini-Project, we are going to analyse the performance of MLE as the sample size changes for different noise distributions. We will compare it with the theoretical convergence and discuss the expected value, varaince, CDF and PDF of the estimates.

$$x_i = A + n_i \quad \text{for } i = 1, \dots, N \tag{1}$$

For n_i(noise) we will experiment with different distributions like Gaussian $N(0,1)$, Laplace $(0, \frac{1}{\sqrt{2}})$ and Cauchy $(0, \sqrt{2 * 1.78})$

2 Gaussian - $N(0, 1)$

The data has been generated using noise sampled from $N(0,1)$, to get a smooth PDF and CDF distribution, we have analysed data with :

Number of Realisations = 100000

2.1 $E[a]$ and $Var[A]$

The MLE when noise has Gaussian distribution is the sample mean, as shown below

$$\hat{A} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

Using this expression and taking 100000 realisations for all sample sizes, we have calculated the mean and variance for $\hat{A}$ for each sample size.

The calculated values $E[\hat{A}]$ and $Var[\hat{A}]$ are in the table as follows :

Gaussian | Realizations = 100000				
sizes	**mean(A=1)**	**variance(A=1)**	**mean(A=10)**	**variance(A=10)**
1	1.001	0.997	10.002	0.996
10	0.999	0.0.316	10.000	0.314
100	1.000	0.100	10.000	0.100
1000	1.000	0.031	10.000	0.032
10000	1.000	0.01	10.000	0.010

Table 1: Variation in $E[\hat{A}]$ and $Var[\hat{A}]$ with increasing sample sizes for $A = 1$ and $A = 10$

2.2 CDF and PDF

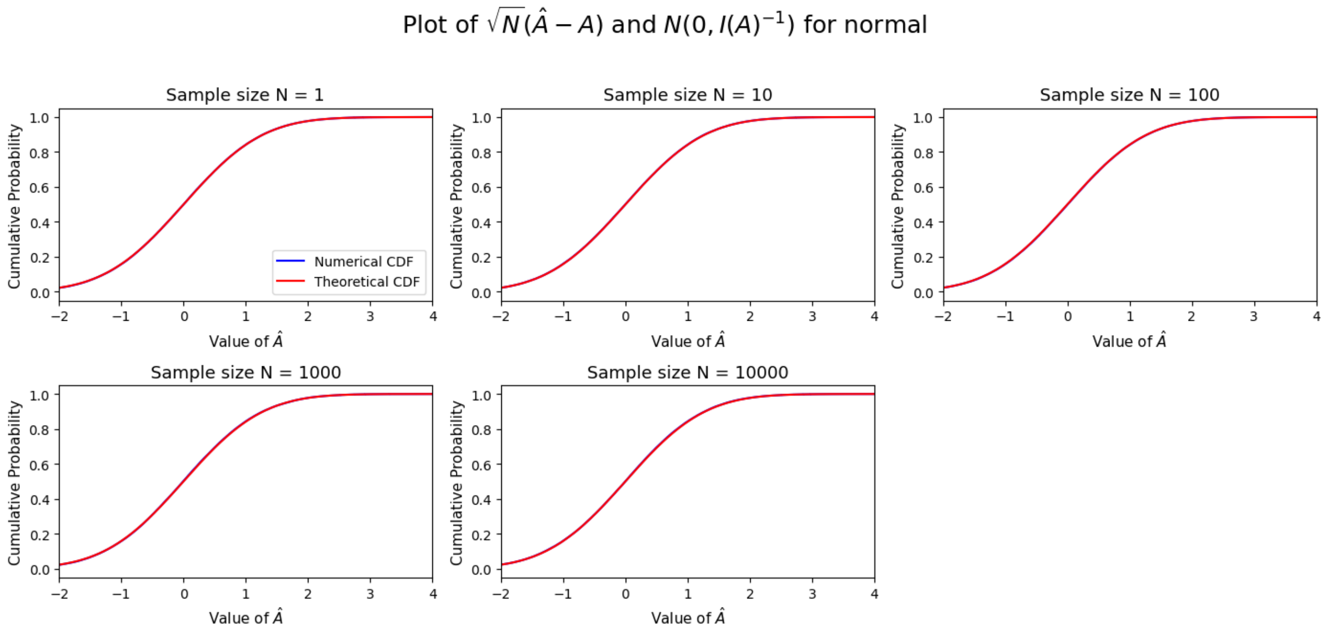

Figure 1: CDF and Convergence Plots For Gaussian noise | Realizations = 100000

3 Laplace - $Lap(0, \frac{1}{\sqrt{2}})$

The MLE for noise with Laplace distribution is given by

$$\hat{A} = median(x_i)$$

3.1 $E[a]$ and $Var[A]$

With 1000 realisations, the calculated values $E[\hat{A}]$ and $Var[\hat{A}]$ are in the table as follows:

Laplace | Realizations = 100000				
sizes	**mean(A=1)**	**variance(A=1)**	**mean(A=10)**	**variance(A=10)**
1	0.997	1.002	10.003	0.997
10	1.001	0.269	9.998	0.270
100	1.000	0.076	10.000	0.076
1000	1.000	0.023	10.000	0.023
10000	1.000	0.007	10.000	0.007

Table 2: Variation in $E[\hat{A}]$ and $Var[\hat{A}]$ with increasing sample sizes for $A = 1$ and $A = 10$

3.2 CDF and PDF

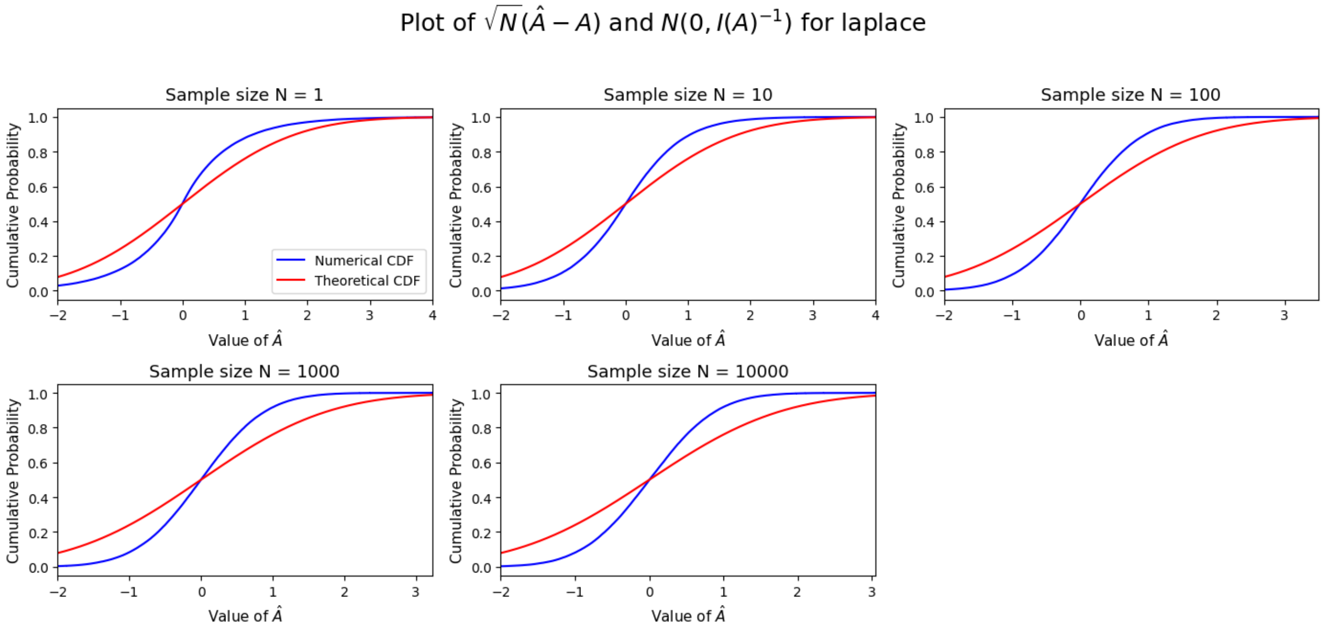

Figure 2: CDF and Convergence Plots For Laplacian noise | Realizations = 100000

4 Cauchy distribution: $Cauchy(0, \gamma)$

The data was generated from $Cauchy(0, \gamma)$ where $\gamma = \sqrt{2C_g}$ and $C_g = 1.78$.

4.1 Log-Likelihood Function

The probability density function (PDF) of the Cauchy distribution is:

$$f(x_i; A, \gamma) = \frac{1}{\pi\gamma \left[1 + \left(\frac{x_i - A}{\gamma}\right)^2\right]}$$

The log-likelihood function (up to constants) is:

$$\log \mathcal{L}(A) = -\sum_{i=1}^{N} \log \left(1 + \left(\frac{x_i - A}{\gamma}\right)^2\right)$$

Finding the closed-form solution for the MLE is difficult, so we turn to Numerical methods like Newton-Raphson.

4.2 First Derivative (Score Function)

$$f(A) = \frac{d}{dA} \log \mathcal{L}(A) = \sum_{i=1}^{N} \frac{2(x_i - A)}{(x_i - A)^2 + \gamma^2}$$

4.3 Second Derivative

$$f'(A) = \frac{d^2}{dA^2} \log \mathcal{L}(A) = \sum_{i=1}^{N} \frac{2\left[(x_i - A)^2 - \gamma^2\right]}{\left[(x_i - A)^2 + \gamma^2\right]^2}$$

4.4 Newton-Raphson Iteration

Start with an initial guess $A^{(0)}$ (e.g., we used the median of the data) and iterate using:

$$A^{(k+1)} = A^{(k)} - \frac{f(A^{(k)})}{f'(A^{(k)})}$$

Stop when:

$$|A^{(k+1)} - A^{(k)}| < \epsilon$$

for a small tolerance ϵ.

4.5 Mean $E[\hat{A}]$ and Variance $Var(\hat{A})$

The calculated mean of $\hat{A}$ for different realisation numbers and numbers of samples are given below.

Sample Size	Mean (A=1)	Variance (A=1)	Mean (A=10)	Variance (A=10)
1	1.001	1641.064	10.002	236.054
10	0.999	1225.349	10.000	1228.972
100	1.000	0.027	10.000	0.271
1000	1.000	0.084	10.000	0.084
10000	1.000	0.027	10.000	0.027

Table 3: MLE mean and variance for **Cauchy** noise with different values of A and sample sizes. **Number of realisations** = 100000

4.6 CDF and PDF

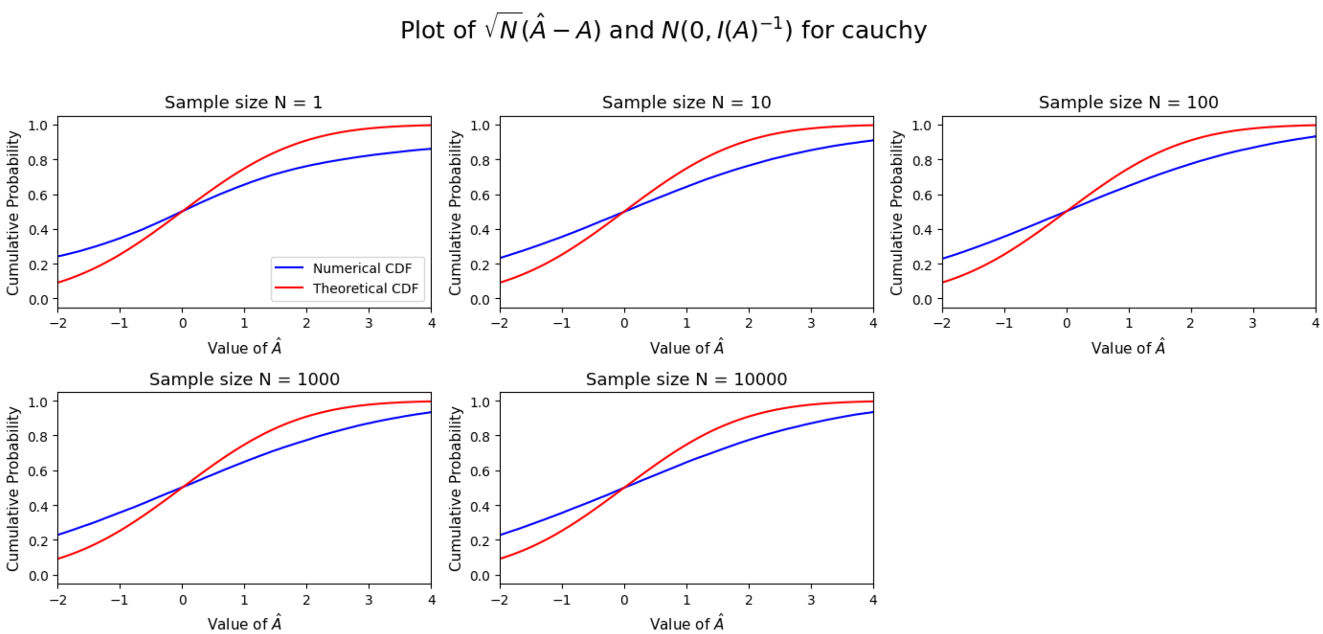

Figure 3: CDF and Convergence Plots For Cauchy noise | Realizations = 100000

5 Convergence of PDFs and CDFs for Different Values of N

PDF and CDF of MLE estimate for various number of samples, each using 1000 realizations.

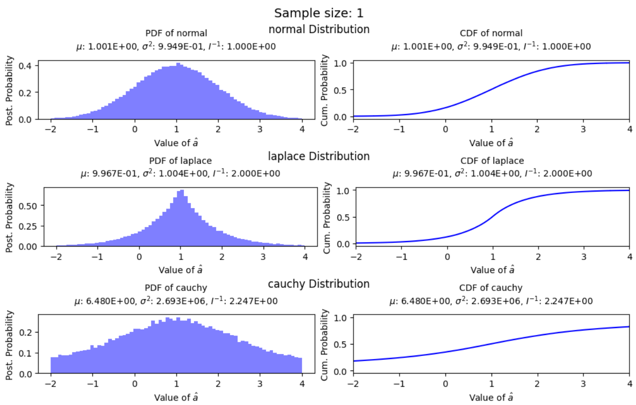

Figure 4: Converged PDF and CDF for $N = 1$

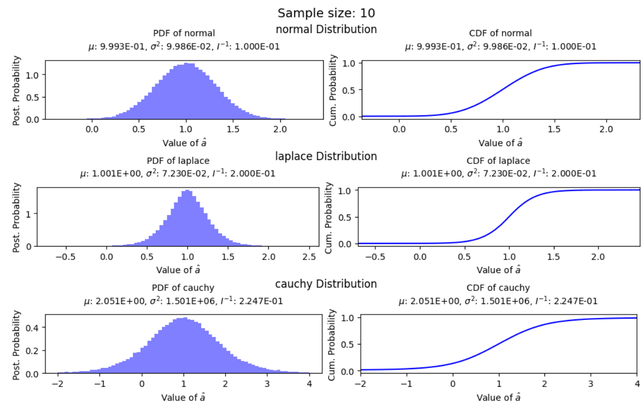

Figure 5: Converged PDF and CDF for $N = 10$

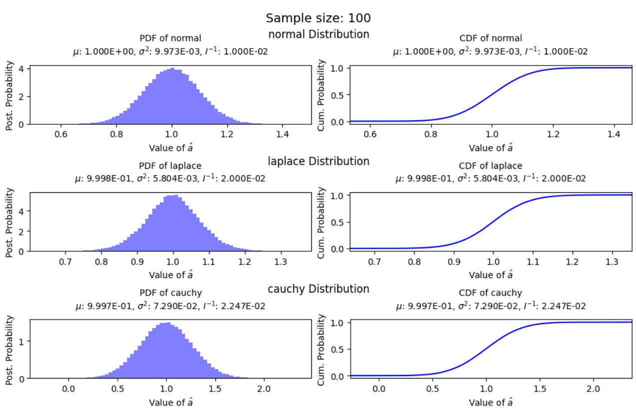

Figure 6: Converged PDF and CDF for $N = 100$

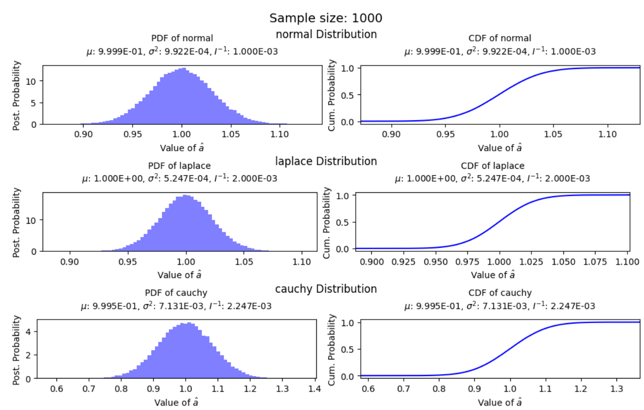

Figure 7: Converged PDF and CDF for $N = 1000$

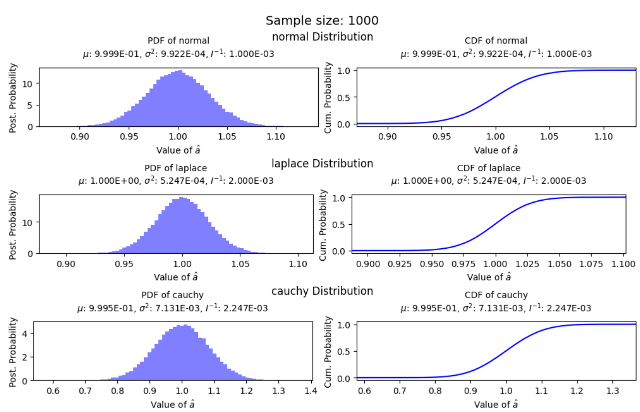

Figure 8: Converged PDF and CDF for $N = 10000$

6 Inferences about the estimates:

- For normal distribution, the MLE estimator is unbiased and is evident from our results. The variance also converges to $I^{-1} = \sigma^2/N$ and hence as N increases, the variance tends to 0
- The MLE estimate of the location parameter of the Laplace distibution is the median, which we discovered upon reading a bit, is biased. But the bias is too small to be noticed, and hence the estimator seems more or less unbiased for higher N. It becomes unbiased as N increases, consistent with MLE's asymptomatic distribution and variance converges to $I^{-1} = \frac{1}{2N}$ (as N increases, variance tends to 0).
- The location parameter of the Cauchy distribution cannot be statistically analysed as there is no closed form MLE. Hence we used Newton Raphson and upon reading up a bit, we settled for the sample median as the intial guess instead of the sample mean to give a bounded guess.
- However, for N=1 sample, the MLE estimator is the sample itself. Hence the mean $E[\hat{A}]$ and variance $Var(\hat{A})$ are undefined.
- The mean however remains small owing to the bell-like distribution for Cauchy.
- For N=10, we have the mean shooting up due to its heavy tailed nature. As N further increases, we observe the estimator being unbiased with variance converging to $I^{-1} = 2\gamma^2/N$ tending to 0 as N becomes larger.

7 Regarding the CDF and PDF:

- The CDF of Normal distribution converges to the asymptotic MLE distribution because the estimator is unbiased and satisfies the Cramer Rao Lower Bound.
- Laplace matches closer than Cauchy as it's bias is very small and the fact that a closed form MLE for Cauchy doesn't exist with bias shooting up incredibly for smaller samples.

$$\sqrt{N}(\hat{A} - A) \sim N(0, I(A)^{-1})$$

- The PDFs converge to the above mentioned normal distribution as sample size increases