As shall be seen, our approximation for the motion of a peristaltic crawler is a non-linear partial differential equation. Non-linear problems are too complex to allow a uniform theory for their analytical solutions. Some analytical solutions do exist for certain non-linear problems, but there does not exist a consistent theory. Thus, when arriving at a perfect analytic solution is not reasonably achievable, alternative methods should be applied.

The most obvious methods then are numerical solutions. Particularly in the age of computers, a numerical solution can be relatively easy to arrive at. In this paper we have included our numerical solutions for comparison. However, a numerical solution does not represent a full understanding of any given problem. In many cases it can be difficult to assess the accuracy of a numerical approximation without some analytical knowledge, and there can be many general features that arise from a problem which are more difficult to see with only a numerical approximation. Thereby some form of analytical solution is important in understanding a problem.

The method we have chosen to use in our analysis here is a perturbation method. This means that in order for us to solve our problem we will take an approximation based on a known linear solution. Using that known solution we can find nearby solutions to our non-linear problem by modifying that known solution with respect to a small parameter. In our case, the wet friction model provides a similar linear situation to base our ultimate perturbative solution off of and our model can also be constructed to contain a necessary small parameter epsilon. A perturbation in this case means a deviation from our value. A common application of perturbation theory for example, is the small perturbation upon a two body orbital system by a small third body, where we can easily find a solution for the two body motion but wish to account for the small gravitational effect of a third body. In our application, the small perturbation is non-linear dry friction. The feasibility of an approximation based on a deviated corrected is quite obvious. It is trivial that between any two functions there can exist a third function to correct the difference. If $U(x,t) = U_0(x,t) + U_n(x,t)$, $U_n(x,t)$ can merely be defined to be $U(x,t) - U_0(x,t)$ through basic algebra. In our case, we are trying to find $U(x,t)$ and we know the nearby linear solution $U_0(x,t)$ so we have to find $U_n(x,t)$. The methodology we can use for finding our $U_n(x,t)$ depends on what we know about $U(x,t)$. In many cases, including perturbation theory, $U_n(x,t)$ takes the form of a series of functions. If we were to know the derivatives of $U(x,t)$ then the right side $U_0(x,t) + U_n(x,t)$ can take a form identical to a Taylor series. Taylor series are also a series of corrections to an approximation, and so using less terms of the series makes* a less accurate approximation. However, truncating the series creates a closed form solution that is reasonably accurate as the order of Taylor terms decreases. This is part of why we need a small parameter in our perturbative series, because by increasing the order on our small parameter each term becomes successively less impactful. This is relevant for defining what we mean by “small”. “Small” is not an exact definition, and varies by how much accuracy we need. We need our epsilon to be sufficiently small that we feel comfortable neglecting terms with higher powers of epsilon. If for example we were to try to use an epsilon of 0.9, we would eventually need to neglect terms with a constant 0.81 when evaluating other terms with a 0.9, which is unreasonable in most cases. If we use an epsilon of 0.01, then we decrease by 2 orders of magnitude with each power, a difference reasonable enough to neglect in many cases. If we use a large parameter, then the order of which term can be neglected is flipped, which does not

work as the series can go to infinity. As such large parameters must be converted into small parameters.

Perturbative solutions take a different form than Taylor series as the information we use is contained in the partial differential equation for our U(x,t). They are formatted to allow a system of solvable differential equations. In the most basic form then, a perturbative solution is defined to take a form like

$U(x,t)=U_0(x,t)+eU_1(x,t)+e^2U_2(x,t)+...$

Our representation of this perturbation is defined by the small value e. $U_0(x,t)$ is our non-perturbed, and thus linear, solution. Our answer is derived through resubstitution of this general form into the original equation and rederivation of each U_n value up to the needed approximation. Since we should have a differential equation defining our U, we plug in our perturbative solution. Using an example equation

$dU/dy=eU^2$

$d(U_0(y)+eU_1(y)+e^2U_2(y))=e(U_0(y)+eU_1(y)+e^2U_2(y))^2$

We can now separate this equation by the powers of epsilon. For example, for our epsilon to the power of one equation.

$edU_1/dy=eU_0^2$

Which can be solved by first solving epsilon to the power of zero for U_0.

$dU_0/dy=0$, U_0 = constant

$edU_1/dy=eU_0^2$, $U_1=U_0^2y+c_1$

If we are deriving a perturbation value at scale $(epsilon)^2$, then the effect of $(epsilon)^3$ and higher powers should logically be small. This observation allows an approachable approximation to our problem, as the different scales of epsilon can create solvable problems. There are variations upon this general method, but the justification for why a perturbative solution works is ultimately the same. A small parameter in our equation allows us to situationally ignore parts of the equation and break it down into smaller pieces.