Exploring Airfoil Theory through Complex Analysis

Austin Vera, Undergraduate CU-Boulder, Dept. Aerospace Engineering
Andrew Gusty, Undergraduate CU-Boulder, Dept. Applied Mathematics
Liane Carter, Undergraduate CU-Boulder, Dept. Applied Mathematics

April 29, 2024

Abstract

This project aims to explore the theory of fluid mechanics using complex variables. Specifically, we will explore how complex potential functions can be used to model fluid flow around objects. Joukowski transformations will be utilized to transform cylinder and rotating cylinder flow into airfoil flow with conformal mapping. Additionally, we will derive the Kutta-Joukowski theorem to derive lift of an airfoil. We will then use properties of conformal maps as well as the angle of attack to mathematically model the flow behavior around an airfoil using simulations. Finally, we will explore how the Kutta-condition can be used to find the lift of an airfoil alongside the Kutta-Joukowski theorem.

1 Introduction

Before the advent of powerful computers, mathematical techniques were entirely relied on to model and understand real-world phenomena such as heat transfer, fluid dynamics, and aerodynamics. Thus, innovative mathematical tools were created to address the complex geometry found in those affairs. Conformal mapping is a powerful tool that allows a mathematical problem and its solution to be mapped onto another simpler shape. For complex variables, this means one can define a function that takes all points in the complex plane, and map it onto a different plane.

Nikolai Zhukovsky (transliterated to Joukowski), born in 1847 and deceased in 1921 was a Russian mathematician, physicist, and engineer who made instrumental contributions to aerodynamic theory and aeronautical engineering. In 1910, he published a conformal map (Joukowski transform) that could transform a cylinder into a class of airfoils that have a cusp and a trailing edge. This can be used in aerodynamics to solve for two-dimensional potential flow.

Let the points z in the plane for a cylinder be represented by complex coordinates x for horizontal and y for vertical (Clancy, Section 4.5):

$$z = x + iy$$

The center of the cylinder can be adjusted to modify the shape of the resulting airfoil.

Let the points ζ in the plane for an airfoil be represented by the complex coordinates χ for horizontal and η for vertical:

$$\zeta = \chi + i\eta$$

Then every point for the airfoil can be related to the cylinder by the function:

$$\zeta = z + \frac{c_1^2}{z}$$

The transformation constant a, is used to control stretching in the flow field. A small a results in almost a cylindrical shape, while a large a close to the radius of the cylinder r results in a thin streamlined shape, and $a = r$ results in a flat plate. When $a > r$ the mapping is no longer conformal, meaning no valid flow is represented. For the mapping to remain conformal, the cylinder must enclose the point where the derivative is zero, $z = -1$ and intersect the point $z = 1$. Thus any Joukowski airfoil can be achieved by varying the center coordinates and adjusting the radius accordingly.

Instead of directly calculating the velocities and pressures around the airfoil, the Joukowski transformation can be used so only the velocity and pressures around a cylinder must be found. Applying the Kutta-Joukowski theorem allows the lift of an airfoil to be calculated with the flow field, velocity, and pressures around it.

Martin Wilhelm Kutta (1867-1944) was a German mathematician who worked in numerical analysis and aerodynamics. He's best known for his contributions to the development of numerical methods for solving differential equations and the flow of air around wings. Independently both Kutta and Joukowski found a relation between circulation in flow and lift in the early 20th century. This relation is called the Kutta-Joukowski Theorem, which is fundamental for calculating lift and other two-dimensional bodies in aerodynamics.

The Kutta-Joukowski Theorem relies on the Kutta condition which states that a body with a sharp trailing edge will create circulation about itself when moving through a fluid. This circulation is of sufficient strength to hold the rear stagnation point at the trailing edge of the airfoil. Using this condition, the Kutta-Joukowski theorem states the lift per unit span L:

$$L = \rho_\infty v_0 \Gamma$$

Where ρ_∞, V_∞ are the density and velocity far away from the airfoil, and Γ is the circulation which is defined as the line integral:

$$\Gamma = \oint_C v \cdot ds = \oint_C v \cos\theta ds$$

In other words, the force per unit length acting on a right cylinder of any cross section whatsoever is equal to $\rho_\infty v_\infty \Gamma$ and is perpendicular to v_∞ (Kuethe, Section 4.9). C is a closed contour enclosing the airfoil, going in the clockwise direction (negative). $v \cos\theta$ is the component of fluid velocity tangent to C, and ds is an infinitely small length on C. The path of flow used for this calculation must not be on the boundary of the cylinder.

This paper will thoroughly follow the mathematical derivations and theorems involved in using the Joukowski transform to map a typical airfoil, and derive an equation for lift.

2 Introduction To Flow Around a Cylinder

Recalling flow potential, let us assign a function:

$$\Omega(z) = v_0(z + \frac{a^2}{z})$$

Where v_0, a are real constants, and $|z| > a$.

$$\Omega(z) = v_0(re^{i\theta} + \frac{a^2}{r}e^{-i\theta})$$

And expanding with Euler's Formula:

$$= v_0(rcos(\theta) + irsin(\theta) + \frac{a^2}{r}cos(\theta) - \frac{a^2}{r}isin(\theta))$$

The real and imaginary parts are respectively:

$$\phi(r,\theta) = v_0 cos(\theta)(r + \frac{a^2}{r}) \tag{2.1}$$

$$\psi(r,\theta) = v_0 sin(\theta)(r - \frac{a^2}{r}) \tag{2.2}$$

Recall that ϕ is the velocity potential, and its derivative shows the tendencies of the flow's velocity in the complex plane. Ψ is the stream function, and gives the reader a sense of the flow direction in the complex plane.

Next, we derive the level curves of Ψ, and get of sense of direction of this flow. Set $\Psi =$ to some constant, let us start with zero.

$$\psi(r,\theta) = v_0 sin(\theta)(r - \frac{a^2}{r}) = 0 \tag{2.3}$$

This can occur when $\theta = 0, \pi n$, or when $r - \frac{a^2}{r} = 0$. Let us explore the second case. Solving for r, we get $r = a$, or the simple circle seen below.

Let us derive the equation to solve for the flow lines. We start by solving for r for the stream function. Set our constant to be c, which is a real positive number.

$$r = \frac{c \pm \sqrt{sin^2(\theta) + c}}{sin(\theta)} \tag{2.4}$$

To understand the tendencies of the flow at points very far away, let us investigate our original flow function. Taking the derivative of Omega, we get as follows:

$$\Omega' = v_0(1 - \frac{a^2}{z^2}) \tag{2.5}$$

Substituting Euler's equation for z:

$$\Omega' = v_0(1 - \frac{a^2}{r^2 e^{2i\theta}}) \tag{2.6}$$

Now, we are required to split this up into the vertical and horizontal components to understand the tendencies of its x and y components. We can do so by using the trig definition of log of z.

$$v_0[1 - \frac{a^2}{r^2}(cos(2\theta) - isin(2\theta)] \tag{2.7}$$

And our real component is the horizontal velocity, and respectively the imaginary is the vertical velocity.

$$v_{horizontal} = v_0(1 - \frac{a^2}{r^2} cos(2\theta)) \tag{2.8}$$

$$v_{vertical} = \frac{v_0}{r^2} a^2 sin(2\theta) \tag{2.9}$$

Notice the limit as we take r to infinity for the velocity's components:

$$\lim_{r\to\infty} v_{horizontal} = v_0 \tag{2.10}$$

$$\lim_{r\to\infty} v_{vertical} = 0 \tag{2.11}$$

$$\tag{2.12}$$

This should demonstrate to the reader that when taking a radius very far away from the cylinder, the flow becomes entirely horizontal. Deriving this using the stream function would have been more inconvenient, and this demonstrates the flow through it's velocity.

Lastly, let us discuss stagnation points. Specifically, we will now investigate the flow around the points of the cylinder. Realistically, the reader should consider a real cylinder in a river. When would the flow entirely stop? Around the cylinder? At the front? Or the back?

Starting at the front, when we set r to a, and θ to π or 0, both the horizontal and vertical velocity components vanish. However, on points besides these two around the cylinder, the reader might suspect that it moves around it. Or on a molecular level, the velocity is perfectly tangent to the shape of the cylinder. Let us prove this with complex analysis, starting with the gradient of Φ.

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{u}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{u}_\theta \tag{2.13}$$

$$= v_0\cos(\theta)\left(1 - \frac{a^2}{r^2}\right)\hat{u}_r - \frac{v_0\sin(\theta)}{r}\left(r + \frac{a^2}{r}\right)\hat{u}_\theta \tag{2.14}$$

The unit vectors represent velocity in the radial and circular respectively. As r becomes very large, the circular component vanishes, and the flow is completely radial, or tangential to the flow.

3 Flow around a Rotating Cylinder

The following formulas are relevant to the the following's section's derivation.
The potential function for a circular cylinder with radius a and circulation Γ is,

$$\Omega(z) = -v_0\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\log\frac{z}{a} \tag{3.1}$$

Substituting $z = re^{i\theta}$ and taking the imaginary part, the stream function for a rotating cylinder is similar to a normal cylinder, except for the contribution of the vortex, created the the associated rotation:

$$\psi(r,\theta) = v_0\sin(\theta)(r - \frac{a^2}{r}) + \frac{\Gamma}{2\pi}\log(\frac{r}{a}) \tag{3.2}$$

Similarly, by taking the real part of equation 3.1, we can find Φ, and take the partial derivatives to find the corresponding velocity field as:

$$v_r = v_0\cos(\theta)(1 - \frac{a^2}{r^2}) \tag{3.3}$$

$$v_\theta = -v_0\sin(\theta)(1 + \frac{a^2}{r^2}) - \frac{\Gamma}{2\pi r} \tag{3.4}$$

A visualization for flow around a rotating cylinder can be acheived in Matlab, plotting the countour lines of the stream function alongside the cylinder. This figure can be found in Appendix D.

3.1 Kutta-Joukowski Theorem

Recall the flow around a rotating cylinder(enter figure here) from the previous section. Let us derive the lift force around this cylinder. Generally, a difference in pressure on an object will generate lift, and due to the asymmetrical distribution of flow lines vertically on this, this is quite possible.

$$v_r = v_0\cos(\theta)(1 - \frac{a^2}{r^2}) \tag{3.5}$$

$$v_\theta = -v_0\sin(\theta)(1 + \frac{a^2}{r^2}) - \frac{\Gamma}{2\pi r} \tag{3.6}$$

Intuition will drive us towards knowing that velocity will generate pressure, and from that will come a corresponding body force. Basic aerodynamics tell us that:

$$C_p = \frac{\Delta P}{1/2\rho v^2} \tag{3.7}$$

Where the pressure difference between the atmospheric pressure and the pressure on the aerodynamic body. Recalling our assumptions that our flow is in-compressible and inviscid, we know that Bernoulli's equation will hold.

$$P_\infty + \frac{1}{2}\rho v_0^2 = P_2 + \frac{1}{2}\rho v^2 \tag{3.8}$$

And we let our initial conditions for this be the free stream values. Solving for the pressure difference, we notice:

$$\Delta P = \frac{1}{2}\rho(v_0^2 - v^2) \tag{3.9}$$

Subbing this into Bernoulli's equation and using our definition of our pressure coefficient:

$$C_p = 1 - \frac{v_2^2}{v_0^2} \tag{3.10}$$

The reader should be aware that we know the free stream velocity based on how we set up our problem, and that along a cylinder the flow is entirely tangential, or equivalent to u_θ.

Recalling the equations for N' and A', which are the normal and tangential forces with respect to a point on the airfoil(

$$N' = -\int_{LE}^{TE} (P_n \cos\theta + \tau_u \sin\theta)\, ds_u + \int_{LE}^{TE} (P_a \cos\theta - \tau_u \sin\theta)\, ds \tag{3.11}$$

$$A' = \int_{LE}^{TE} (-P_u \sin\theta + \tau_n \cos\theta)\, ds + \int_{LE}^{TE} (P_u \sin\theta + \tau_l \cos\theta)\, ds \tag{3.12}$$

Recalling that we assumed inviscid flow, everything with a tau dissipates from our integral.

$$N' = -\int_{LE}^{TE} (P_u \cos\theta)\, ds_u + \int_{LE}^{TE} (P_l \cos\theta)\, ds \tag{3.13}$$

$$A' = \int_{LE}^{TE} (-P_u \sin\theta)\, ds + \int_{LE}^{TE} (P_u \sin\theta)\, ds \tag{3.14}$$

Notice that our perpendicular force N is our lift, and our parallel force A is drag. We can additionally substitute our differentials dx and dy into polar coordinate form, leaving us with:

$$L' = \int_{LE}^{TE} (P_l - P_u) dx \tag{3.15}$$

$$D' = \int_{LE}^{TE} (P_u - P_l) dy \tag{3.16}$$

We convert back into coefficients, noticing that coefficient is related to dynamic pressure and area. Recall that D is twice the radius of our total cylinder's radius(r)

$$C_l' = \frac{1}{D}\int_{LE}^{TE} (C_{p_l} - C_{p_u})\, dx \tag{3.17}$$

$$C_d' = \frac{1}{D}\int_{LE}^{TE} (P_{p_u} - C_{p_l})\, dy \tag{3.18}$$

Pulling back the transformation back for the corresponding coordinate system:

$$\begin{bmatrix} y = R\sin(\theta) \\ dy = R\cos(\theta)d\theta \\ x = R\cos(\theta) \\ dx = -R\sin(\theta)d\theta \end{bmatrix} \tag{3.19}$$

And the following transformation gives us the result in cylindrical coordinates:

$$C_l = -\frac{1}{2}\int_{\pi}^{2\pi} C_{p_l}\sin(\theta)d\theta + \frac{1}{2}\int_{\pi}^{0} C_{p_u}\sin(\theta)d\theta \tag{3.20}$$

$$C_d = \frac{1}{2}\int_{\pi}^{0} C_{p_u}\cos(\theta)d\theta - \frac{1}{2}\int_{\pi}^{0} C_{p_l}\cos(\theta)d\theta \tag{3.21}$$

Noticing that the coefficient of pressure on the lower is equal to the coefficient of pressure on the upper half, so we unionize the integrals to get the following:

$$C_l = -\frac{1}{2}\int_{0}^{2\pi} C_p\sin(\theta)d\theta \tag{3.22}$$

$$C_d = -\frac{1}{2}\int_{0}^{2\pi} C_p\cos(\theta)d\theta \tag{3.23}$$

Recall the previous definition of coefficient of pressure:

$$C_p = 1 - \frac{v^2}{v_0^2} \tag{3.24}$$

Notice the previous definition of velocity along the body for a rotating a cylinder, and subbing it into the equation above:

$$C_p = -\,2v_0(\sin(\theta) - \frac{\Gamma}{2\pi R})^2 \tag{3.25}$$

$$C_p = 1 - (-2\sin(\theta) - \frac{\Gamma}{2\pi R v_0})^2 \tag{3.26}$$

$$C_p = 1 - 4\sin^2(\theta) - \frac{2\Gamma\sin(\theta)}{\pi r v_0} - (\frac{\Gamma}{2\pi r v_0})^2 \tag{3.27}$$

When we plug these into our integrals, we see notice:

$$C_l = -\frac{1}{2}\int_{0}^{2\pi}(1 - 4\sin^2(\theta) - \frac{2\Gamma\sin(\theta)}{\pi r v_0} - (\frac{\Gamma}{2\pi r v_0})^2)\sin(\theta)d\theta = \frac{\Gamma}{r v_0} \tag{3.28}$$

$$C_d = -\frac{1}{2}\int_{0}^{2\pi}(1 - 4\sin^2(\theta) - \frac{2\Gamma\sin(\theta)}{\pi r v_0} - (\frac{\Gamma}{2\pi r v_0})^2)\cos(\theta)d\theta = 0 \tag{3.29}$$

Drag becomes zero, and our coefficient of lift for a rotating cylinder, becomes: $\frac{\Gamma}{r v_0}$
Using our previously defined formula for dynamic pressure, we use the coefficient of lift to get the actual lift, or Kutta-Joukowski Formula:

$$L = \Gamma v_0 \rho \tag{3.30}$$

4 Conformal Mappings

Oftentimes two figures which appear entirely different are related to each other in some way. Consider shadows cast by an object, which are directly related to the object despite possibly having an unrecognizable shape. Let us analyze such a relationship mathematically. Consider a curve, C, in the z-plane. We can parameterize the real and imaginary parts of C with the variable t, over some interval.

$$z(t) = x(t) + iy(t), \quad t \in [a, b] \tag{4.1}$$

We can then perform a change of variables through $f(z) = w$ in an open neighborhood of $z_0 \equiv z(t_0)$ where $f(z)$ is analytic, so that,

$$w(t_0) = f(z(t_0)) \tag{4.2}$$

$$\frac{dw(t_0)}{dt} = f'(z_0)\frac{dz(t_0)}{dt} \tag{4.3}$$

The left and right-hand sides of the above equation are both complex numbers, so they can be written as,

$$R_w e^{i\arg[w'(t_0)]} = R_{f(z)} e^{i(\arg[f'(z_0)]+\arg[z'(t_0)])} \tag{4.4}$$

Where $R_w = |w'(t_0)|$, and $R_{f(z)} = |f'(z_0)z'(t_0)|$. Looking at equation (4.4), we see that,

$$\arg[w'(t_0)] = \arg[f'(z_0)] + \arg[z'(t_0)] \tag{4.5}$$

Thus, as long as $f'(z_0) \neq 0$ and $z'(t_0) \neq 0$, the tangent line through z_0 is rotated by the angle $\arg[f'(z_0)]$ in the w-plane. This means that if we have two intersecting curves in the z-plane, their tangent lines are both rotated by $\arg f'(z_0)$ when they are mapped to the w-plane, and thus the angle between them in preserved (Ablowitz 2003, 314). A mapping such as this one, where angles between directed lines are preserved, is called **conformal**.

5 Joukowski Transformation

In previous sections, flow about a circular cylinder, along with the lift generated on that cylinder was developed. We now seek a conformal transformation that transforms the flow around a cylinder to that around an airfoil. There are many possible transformations that could be employed, but one of the most well-developed is the Kutta-Joukowski transformation:

$$\zeta = z + \frac{c_1^2}{z} \tag{5.1}$$

To get a sense of how this transformation works, let $z = re^{i\Theta}$ and $\zeta = \chi + i\mu$, then,

$$\chi + i\mu = re^{i\theta} + \frac{c_1^2}{r}e^{-i\Theta} \tag{5.2}$$

Equating reals and imaginary parts yields,

$$\chi = (r + \frac{c_1^2}{r})\cos\Theta \tag{5.3}$$

$$\mu = (r - \frac{c_1^2}{r})\sin\Theta \tag{5.4}$$

Which shows that along the χ-axis, a circle in the z-plane is stretched, and along the μ-axis the circle is squashed.

Let us now shift the circle so that its center is to the right of the y-axis. Then, when we transform the circle, the quantity $(r - \frac{c_1^2}{r})$ is larger to the left of the y-axis than it is to the right. It is this feature that gives the transformed contour the shape of an airfoil. Using MATLAB, we demonstrate this transformation and its effect on the flow, which results from the same transformation applied to the stream function, Ψ.

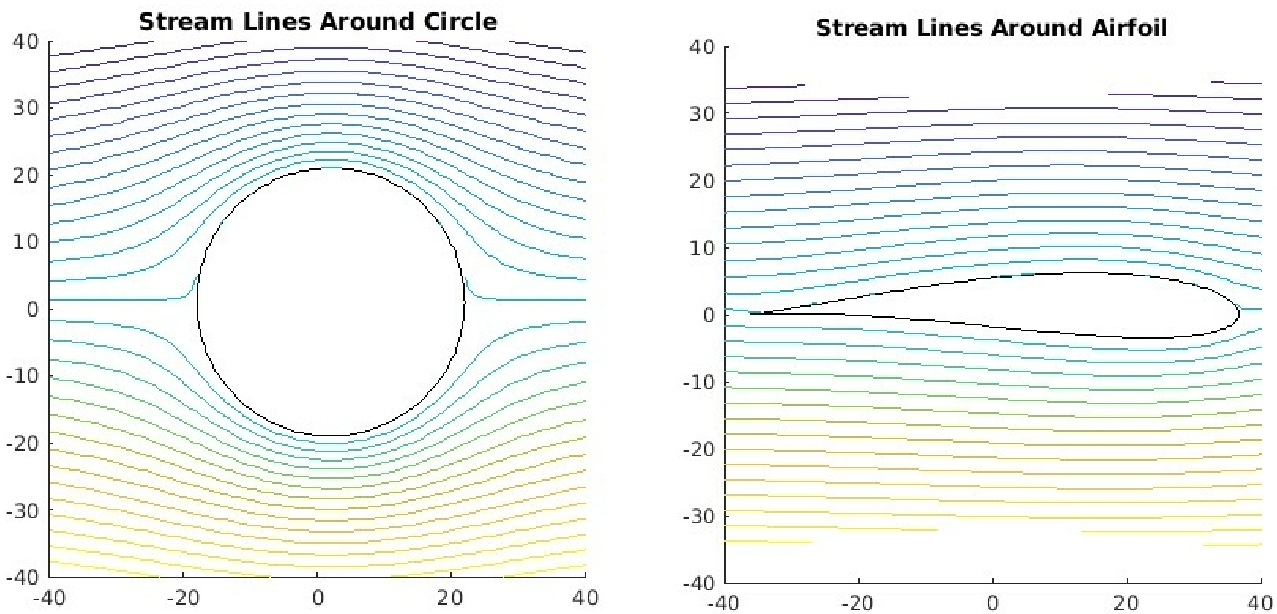


Figure 1: Streamlines around an off-center circle and corresponding airfoil after applying Joukowski-Kutta transformation.

Now, to verify that this transformation is conformal, recall that in two dimension, the transformation is conformal in a neighborhood of z_0 as long as $f(z_0)$ is analytic, single-valued, and $f'(z_0) \neq 0$. The only singularity in the Joukowski transformation occurs when $z_0 = 0$, which can be ignored since it lies inside the contour. We can then find the critical points of $f'(z)$,

$$\frac{d}{dz}f(z) = 1 - \frac{c_1^2}{z^2} = 0 \quad \text{when,} \quad z = \pm c_1 \tag{5.5}$$

To handle the point at $z = -c_1$, we simply make the radius of the circle large enough that it surrounds this point, so that we can ignore it. However, special attention must be paid to $z = +c_1$. Before this can be done, we must understand how the velocities of the stream are affected by our transformation, which will now be explored.

5.1 Transforming the Velocity

The amount of fluid that passes through two points in the z-plane must also pass through those points after they have been transformed to the ζ-plane. Thus, if two points are further apart in the ζ-plane than they are in the z-plane, then the velocity of the fluid between those two points is smaller in the ζ-plane. That is, the velocities in the two planes are inversely proportional to the distance between two points. If we have two points that are separated by dz in the z-plane, and by $d\zeta$ in the transformed plane, then we have,

$$|v_\zeta| = \frac{|v_z|}{\left|\frac{d\zeta}{dz}\right|} = \frac{|v_z|}{\left|1 - \frac{c_1^2}{z^2}\right|} \tag{5.6}$$

Where v_ζ and v_z are the velocities of the fluid in the ζ-plane and z-plane respectively. An important result of this equation is that when $z = \pm c_1$, $|v_\zeta|$ is infinite unless $v_z = 0$ at that point (Pope 1951, 79).

5.2 Finding Lift of an Airfoil with an Angle of Attack

In previous sections, we showed how to find the lift on a body in a fluid using the potential function for the flow about that body. Additionally, we have developed the methods to find the potential function about a cylinder and transform it to an airfoil, but we now must find an angle of attack α, of the airfoil before we can derive lift of the airfoil. The condition for the angle of attack suggested by Kutta is that "enough circulation arise to permit the flow to leave the trailing edge smoothly (i.e. without infinite velocities)" (Pope 1951, 79). Thus, we require finite velocity of the stream at the trailing edge.

Equation 5.6 tells that that if $d\zeta/dz = 0$ at a point on the airfoil, then the velocity at that point will be infinite unless the velocity of the corresponding point in the z-plane is 0. Therefore, we should seek to make the point which becomes the trailing edge of the airfoil a stagnation point. Thus, we make the rear stagnation point of the circle to be at $z = +c_1$.

Now that this has been established, we can rotate the circle on its axes by the angle α. Now, we can write down a potential function for the cylinder in terms of the new axes x' and y', which have their origin at the center of the circle and are rotated in the positive direction by α. Using equation 3.1, we have

$$\Omega(z) = -v_0\left(z' + \frac{a^2}{z'}\right) - i\frac{\Gamma}{2\pi}\log\frac{z'}{a} \tag{5.7}$$

Such that,

$$\frac{d\Omega(z)}{dz'} = -v_0\left(1 - \frac{a^2}{(z')^2}\right) - i\frac{\Gamma}{2\pi z'} \tag{5.8}$$

Now we can utilize the fact that $z = c_1$ is a stagnation point. In terms of the new axes, before being rotated by α, this point is $z'_{c_1} = ae^{i\beta}$, where β is the angle between the center of the off-center circle and the point $z = c_1$. We then rotate this by α to achieve $z'_{c_1} = -ae^{i(\alpha+\beta)}$. We can substitute this into equation 5.8 to get,

$$\frac{d\Omega(z)}{dz'} = -v_0\left(1 - \frac{a^2}{(a^2e^{2i(\alpha+\beta)})}\right) - i\frac{\Gamma}{2\pi ae^{i(\alpha+\beta)}} = 0 \tag{5.9}$$

And solving for Γ yields,

$$i\Gamma = 2\pi av_0(e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}) \tag{5.10}$$

$$\Gamma = 4\pi av_0\sin(\alpha+\beta) \tag{5.11}$$

And thus we have an expression for the circulation, Γ. Finally, we can plug this into the Kutta-Joukowski Theorem from equation 3.30 to get the lift of the airfoil as,

$$L = 4\rho\pi av_0^2\sin(\alpha+\beta) \tag{5.12}$$

6 Conclusion and Future Studies

To summarize, we started by analyzing ideal fluid flow around a cylinder using complex variables, and then extended this model to include circulation. We then derived The Kutta-Joukowski theorem, which gives us a simple expression for the lift of a cylinder when circulation is present, $L = \Gamma v_0\rho$. Next, we explored conformal maps, showing how they preserve angles between directed lines, and then explored a convenient conformal map for the modeling of an airfoil. Finally, we utilized the Kutta-condition, which states the velocity at the trailing edge should not be infinite, to derive an expression for the circulation created by the airfoil, which was then used to calculate the lift using the Kutta-Joukowski theorem.

These methods demonstrate the convenience of complex variables at modeling real situations encountered in engineering. However, a number of caveats must be mentioned. Firstly, in practice, air is neither inviscid nor incompressible, which have both been assumed for the purpose of our studies. In reality, an airfoil moving through air experiences drag as a result of viscosity, and at high enough speeds has a compressible flow. These results can create both turbulence and cavitation, both of which are advanced topics that are the subjects of many modern studies. With those caveats mentioned, the methods employed here make good approximations for an airfoil moving at relatively low speeds. Furthermore, the Joukowski transformation can still be used in more complicated scenarios to greatly aid in the study of airfoils.

Even with the assumptions used in this paper, there are many further studies to pursue. For example, we might study how the pressure distribution around the airfoil changes as we change the camber, and could even explore the "flat-plate" airfoil, which has no camber at all. Finally, other methods for creating airfoils can be explored other than the Joukowski transformation. Many engineers have utilized the fact that airfoil thickness has a small effect on lift and moment to create "thin-airfoil theory", where airfoils are modeled as median lines. These studies have many rich mathematical results that could be very fruitful in future undertakings.

References

[1] Ablowitz, Mark J., and A. S. Fokas. 2003. Complex Variables: Introduction and Applications. Cambridge University Press.

[2] Clancy, L. J. (1975). Aerodynamics. London: Pitman. Section 4.5. ISBN 0-273-01120-0.

[3] Hall, Nancy. "Conformal Mapping." NASA, May 13, 2021. https://www.grc.nasa.gov/www/k-12/airplane/map.html#: :text=Conformal%20mapping%20is%20a%20mathematical,is%20taught%20in%20secondary%20schools.

[4] Kuethe, A. M.; Schetzer, J. D. (1959). Foundations of Aerodynamics. New York: John Wiley & Sons. Section 4.9. ISBN 0-471-50952-3.

[5] "Basic Wing and Airfoil Theory. Alan Pope. McGraw-Hill. New York. 1951. 312 Pp. 42s. 6d." 1951. Journal of the Royal Aeronautical Society 55 (491): 746–47. https://doi.org/10.1017/s0368393100122448.

Appendices

Relevant Variables for Kutta-Joukowski Theorem Proof

v_r	Radial velocity component.
v_θ	Tangential velocity component.
v_0	Free-stream velocity.
θ	Angular for polar coordinates.
a	Radius of cylinder
r	Cylindrical coordinate radius
Γ	Circulation in relevant flow field
C_p	Pressure coefficient.
ΔP	Pressure difference between the atmospheric pressure and the pressure on the aerodynamic body.
P_∞	Free-stream pressure.
P_2	Pressure at a point on the aerodynamic body.
ρ	Density of the fluid.
N'	Normal force per unit span (related to lift).
A'	Axial (tangential) force per unit span (related to drag).
LE	Leading edge of the airfoil.
TE	Trailing edge of the airfoil.
L'	Integral expression for lift.
D'	Integral expression for drag.
C_l'	Lift coefficient per unit span.
C_d'	Drag coefficient per unit span.
dx, dy	Cartesian Differentials
ds_u	Differential element along the upper surface.
ds_l	Differential element along the lower surface.
ds	Differential element along the surface.
x, y	Cartesian coordinates.
$\sin(\theta)$, $\cos(\theta)$	Trigonometric functions of the angle θ.
C_l, C_d	Lift and drag coefficients over the entire body.
L	Lift force based on Kutta-Joukowski theorem.
D	Axial Drag force along airfoil/cylinder

A Relevant Assumptions for Ideal Flow

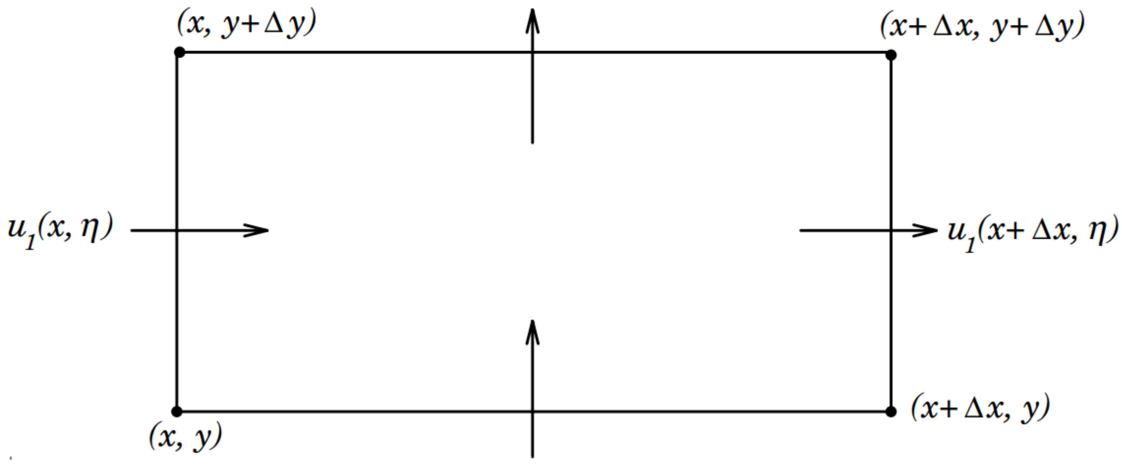
(a) Steady: The velocity of a fluid at any point is only dependent on position, not time.

(b) Incompressible: Velocity is non-divergent and density, ρ, is constant.

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0$$

Where the 2D vector $v = (v_1, v_2)$, and v_1, v_2 are the horizontal and vertical components, respectively.

If a flow is 2D, the fluid motion in any plane is identical to that of any other parallel plane. Consider flow through a rectangle with sides Δx, Δy. Let ρ and u represent the density and the velocity of the fluid



The rate the fluid accumulates in the rectangle is given by $\frac{d}{dt}\int_x^{x+\Delta x}\int_y^{y+\Delta y}\rho dxdy$. The rate of fluid entering along the left side (between points (x, y) and $(x, y+\Delta y)$ is given by $\int_y^{y+\Delta y}(\rho u_1)(x,\eta)d\eta$. The rate of fluid entering through the bottom side can be represented similarly.

To be safe, let ρ be a function of x, y, and t. Then by conservation of mass,

$$\frac{d}{dt}\int_x^{x+\Delta x}\int_y^{y+\Delta y}\rho dxdy = \int_y^{y+\Delta y}[(\rho u_1)(x,\eta)-(\rho u_1)(x+\Delta x,\eta)]d\eta+\int_x^{x+\Delta x}[(\rho u_2)(\xi,y)-(\rho u_2)(\xi,y+\Delta y)]d\xi$$

The equation can be divided by $\Delta x\Delta y$. Looking at the limit as Δx, Δy go to zero (assuming that ρ, u_1, and u_2 are smooth functions of x, y, and t, then

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u_1)}{\partial x}+\frac{\partial(\rho u_2)}{\partial y}=0$$

Since the flow is steady $\partial\rho/\partial t = 0$, and because the flow is incompressible, ρ is constant. Thus we have the equation from earlier

$$\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}=0$$

Where $v_1 = u_1$ and $v_2 = u_2$.

(c) Irrotational: Circulation of the fluid along any closed contour C is zero (velocity has no curl)

$$\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}=0$$

The circulation around C is given by $\oint_C u \cdot ds$ where ds is the vector element of arc length along C, and u is the velocity of the fluid. Using Green's Theorem where $u = (u_1, u_2)$ and $ds = (dx, dy)$.

$$\oint_C u_1 dx + u_2 dy = \iint_R \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} dxdy$$

Since the flow is irrotational the circulation around C is zero

$$0 = \iint_R \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} dxdy$$
$$\iint_R \frac{\partial u_2}{\partial x} dxdy = \iint_R \frac{\partial u_1}{\partial y} dxdy$$

Thus

$$\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0$$

Where $v_2 = u_2$ and $v_1 = u_1$.

(d) Inviscid: Fluid is nonviscious (has zero friction)

B Code For Streamlines Around a Joukowski Airfoil

```
theta = 0:pi/100:2*pi;
r = 11;
x0 = -1;
y0 = 1;
V_i = 1;

x_circle = x0 + r*cos(theta);
y_circle = y0 + r*sin(theta);


[x, y] = meshgrid(-r*2:r*2, -r*2:r*2);
z = x + 1i*y;

for i=1:length(x);
   for k=1:length(x);
      if sqrt((x(i,k) - x0).^2+(y(i,k) - y0).^2)<r;
         x(i,k)=x0;
         y(i,k)=y0;
      end
   end
end

z_circle = x_circle + 1i*y_circle;
c1 = 10;
zeta_circle = z_circle + (c1.^2)./z_circle;

psi = V_i.*(y-y0).*(1-(r^2./((x-x0).^2 + (y-y0).^2)));
```

```
epsilon = real(zeta_circle);
nu = imag(zeta_circle);

hold on
figure(1)
plot(x_circle, y_circle, 'k')
contour(x, y, psi, 25);
axis square
title('Stream Lines Around Circle')
grid off
hold off;

zeta = z + (c1.^2)./z;
x_transform = real(zeta);
y_transform = imag(zeta);

figure(2)
hold on;
plot(epsilon, nu, 'k')
contour(x_transform, y_transform, psi, 25);
axis square
title('Stream Lines Around Airfoil')
grid off
xlim([-r*2 r*2])
ylim([-r*2 r*2])
```

C Code For Streamlines Around a Cylinder with Circulation

```
theta = 0:pi/100:2*pi;
r = 11;
x0 = -1;
y0 = 1;
V_i = 1;

x_circle = r*cos(theta);
y_circle = r*sin(theta);


[x, y] = meshgrid(-r*2:r*2, -r*2:r*2);

for i=1:length(x)
   for k=1:length(x)
      if sqrt(x(i,k).^2 + y(i,k).^2)<r
         x(i,k)=0;
         y(i,k)=0;
      end
   end
end

z = x + 1i*y;
```

```
theta_flow = angle(z);
r_flow = abs(z);

z_circle = x_circle + 1i*y_circle;
c1 = 10;
zeta_circle = z_circle + (c1.^2)./z_circle;

psi = V_i.*sin(theta_flow).*(r_flow - (r.^2./r_flow)) + 100./(2*pi).*log(r_flow./r);

epsilon = real(zeta_circle);
nu = imag(zeta_circle);

hold on
figure(1)
plot(x_circle, y_circle, 'k')
contour(x, y, psi, 15);
axis square
title('Stream Lines Around Cylinder with Circulation \Gamma = 100')
grid off
hold off;
```

D Plot of Flow Around Cylinder with Circulation

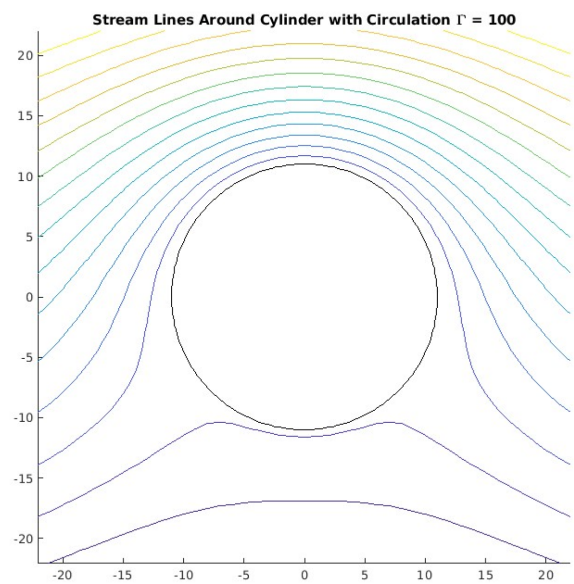


Figure 2: Streamlines around a cylinder with circulation