EchoKey v2: EchoKey Asks
A Question-First Mathematical Specification

EchoKey Team

May 2025

Abstract

EchoKey asks: can a small set of reusable operators act on heterogeneous equations so that composition is well-posed, structurally transparent, and mathematically tractable? This document records questions, definitions, and derivation attempts for EchoKey v2, avoiding assertions, applications, statistics, or code. Objects are defined for clarity; all substantive points are formulated as questions to guide inquiry.

Contents

1 Orientation: EchoKey v2 as Questions **2**

2 Notation and Space Choices **2**

3 Cyclicity $\mathcal{C}$: periodic structure **2**

4 Recursion $\mathcal{R}$: self-reference **3**

5 Fractality $\mathcal{F}$: scale structure **3**

6 Regression $\mathcal{G}$: stability/relaxation **3**

7 Synergy $\mathcal{S}$: coupling/composition **3**

8 Refraction ?? $: layer/domaintransform$ **4**

9 Outliers $\mathcal{O}$: singular measures/jumps **4**

10 The v2 Composite $\mathcal{H}_{\mathrm{v2}}$ **4**

11 Well-Posedness as Questions **4**

12 Structural Questions: Invariants and Energies **5**

13 Discretization and Approximation **5**

14 Compatibility Questions Between Blocks **5**

15 Minimal Assumption Sets (as Questions) **5**

16 Identifiability and Observability (Question-Only) **5**

17 Open List: What EchoKey v2 Asks Next **5**

1 Orientation: EchoKey v2 as Questions

EchoKey v2 treats equations as objects on which a small palette of operators act. The seven operators—C (Cyclicity), R (Recursion), F (Fractality), G (Regression/Stability), S (Synergy/Coupling), **??**(Refraction/Layer Transform), and O (Outliers/Measures)—are intended to be composed. The central theme is to replace declarative claims with concrete questions that can be answered (or falsified) by formal derivation.

Question 1.1 (Minimal state model). Is it sufficient to model a system by a state $\Psi(\cdot, t) \in \mathcal{H}$ and a coefficient vector $\boldsymbol{c}(t) \in \mathbb{C}^K$ such that

$$\Psi(x,t) = \sum_{k=1}^{K} c_k(t)\, \phi_k(x), \tag{1.1}$$

for a fixed orthonormal family $\{\phi_k\}_{k=1}^K \subset L^2$; and is there a choice of $\mathcal{H}$ for which all seven operators are closed?

Question 1.2 (Composite evolution). If each operator acts on $\boldsymbol{c}$ (or on Ψ) as a map with a common domain, does the v2 composite

$$\mathcal{H}_{\mathrm{v2}} \;=\; \mathcal{C} \circ \mathcal{R} \circ \mathcal{F} \circ \mathcal{G} \circ \mathcal{S} \circ \circ \mathbf{??} \mathcal{O} \tag{1.2}$$

define an evolution $\dot{\boldsymbol{c}}(t) \;=\; \mathcal{H}_{\mathrm{v2}}[\boldsymbol{c}(t)]$ that is well-posed (existence, uniqueness, continuous dependence) on a specified time interval?

2 Notation and Space Choices

Definition 2.1 (Ambient spaces and pairings)**.** Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $\boldsymbol{c}(t) \in \mathbb{C}^K$ collect the coordinates of $\Psi(\cdot, t)$ in the basis $\{\phi_k\}$. When convenient, operators act on $\boldsymbol{c}$ and are extended to Ψ via the basis.

Question 2.1 (Choice of topology). Which topology on $\mathcal{H}$ (e.g. L^2, Sobolev H^s, sequence space ℓ^2 for $\boldsymbol{c}$) yields the strongest closure properties for all seven operators simultaneously?

3 Cyclicity $\mathcal{C}$: periodic structure

Definition 3.1 (Fourier-like projector)**.** For $T > 0$, define

$$\mathcal{C}_T[f](t) \;=\; \sum_{n \in \mathbb{Z}} \left(\frac{1}{T} \int_0^T f(\tau)\, e^{-i\omega_n \tau}\, d\tau \right) e^{i\omega_n t}, \quad \omega_n = \frac{2\pi n}{T}. \tag{3.1}$$

On coordinates, $\mathcal{C}_T$ acts coefficientwise if f is replaced by $\boldsymbol{c}$.

Question 3.1 (Boundedness and orthogonality). Is $\mathcal{C}_T$ an idempotent orthogonal projector on $L^2([0,T])$ (i.e. $\mathcal{C}_T^2 = \mathcal{C}_T$ and $\langle \mathcal{C}_T f,\, g \rangle = \langle f,\, \mathcal{C}_T g \rangle$), and how does this interact with the basis $\{\phi_k\}$ for Ψ?

Question 3.2 (Commutation with time differentiation). For what regularity classes does $\frac{d}{dt}\mathcal{C}_T f = \mathcal{C}_T \frac{d}{dt} f$ hold?

4 Recursion $\mathcal{R}$: self-reference

Definition 4.1 (Iterated functional transform)**.** Given F on $\mathcal{H}$, define finite iterates $\mathcal{R}^{(n)}[f] = F(\mathcal{R}^{(n-1)}[f])$ with $\mathcal{R}^{(0)}[f] = f$. If the limit exists, set

$$\mathcal{R}[f] \;=\; \lim_{n\to\infty} \mathcal{R}^{(n)}[f]. \tag{4.1}$$

Question 4.1 (Existence via contraction). Under which metrics on $\mathcal{H}$ (or on a subset) does F become a contraction, ensuring existence of $\mathcal{R}[f]$ by the Banach fixed-point theorem?

Question 4.2 (Nonlinear stability). If F is nonlinear, are there Lyapunov-like functionals V for which $V(\mathcal{R}^{(n)}[f])$ is monotone in n?

5 Fractality $\mathcal{F}$: scale structure

Definition 5.1 (Multiscale superposition)**.** Fix $\lambda > 1$ and a weight sequence $\{\alpha_k\}_{k\geq 0}$. For a map E on $\mathcal{H}$ and a dilation operator D_λ, define

$$\mathcal{F}[E](\Psi) \;=\; \sum_{k=0}^{\infty} \alpha_k\, E(\mathsf{D}_{\lambda^k}\Psi)\,, \qquad \mathsf{D}_\lambda\phi(x) = \phi(\lambda x). \tag{5.1}$$

Question 5.1 (Absolute convergence). What decay on α_k (e.g. $\alpha_k = \lambda^{-kD}$) suffices for unconditional convergence in $\mathcal{H}$?

Question 5.2 (Spectral footprint). If E is linear with spectrum $\sigma(E)$, how is $\sigma(\mathcal{F}[E])$ related to $\sigma(E)$ and $\{\alpha_k\}$?

6 Regression $\mathcal{G}$: stability/relaxation

Definition 6.1 (Mean-reverting template)**.** For a reference state $\Psi^\star \in \mathcal{H}$ and rate $\lambda > 0$,

$$\mathcal{G}_\lambda[\Psi](t) \;=\; \Psi^\star + e^{-\lambda t}\,(\Psi(0) - \Psi^\star). \tag{6.1}$$

On coordinates: $\mathcal{G}_\lambda[\boldsymbol{c}](t) = \boldsymbol{c}^\star + e^{-\lambda t}(\boldsymbol{c}(0) - \boldsymbol{c}^\star)$.

Question 6.1 (Generator form). Does there exist a (possibly time-dependent) linear generator $A_\lambda(t)$ such that $\partial_t \mathcal{G}_\lambda[\Psi](t) = A_\lambda(t)\,\mathcal{G}_\lambda[\Psi](t)$, and under what conditions is $\{U(t,s)\}$ a contraction family?

Question 6.2 (Compatibility with other operators). When does $\mathcal{G}_\lambda$ commute (or approximately commute) with $\mathcal{C}_T$ and $\mathcal{F}$?

7 Synergy $\mathcal{S}$: coupling/composition

Definition 7.1 (Pairwise coupling operator)**.** Let $\Psi = (\Psi_1, \ldots, \Psi_n)$ denote components in $\mathcal{H}^n$. With coefficients κ_{ij} and a bilinear form $B(\cdot,\cdot)$,

$$\mathcal{S}[\Psi] \;=\; \sum_{1\leq i<j\leq n} \kappa_{ij}\, B(\Psi_i, \Psi_j). \tag{7.1}$$

Question 7.1 (Boundedness via kernels). If $B(u,v) = \langle Ku,\, v\rangle$ for a bounded operator K, what norms on $\mathcal{H}$ guarantee $\|\mathcal{S}[\Psi]\| \leq C \sum_{i<j} |\kappa_{ij}|\;\|\Psi_i\|\;\|\Psi_j\|$?

Question 7.2 (Higher-order terms). How does the analysis change if $\mathcal{S}$ includes m-ary terms (hypergraph coupling) with weights $\kappa_{i_1\cdots i_m}$?

8 Refraction ?? : $layer/domaintransform$

Definition 8.1 (Layer transform)**.** For a layer index $L \in \mathbb{N}$ and a transform density η,

$$??\Psi; L] \;=\; \Psi \;+\; \mu\, L\, \eta(\Psi), \tag{8.1}$$

interpreted as a small deformation across an interface or domain change.

Question 8.1 (Regularity transfer). If $\eta : \mathcal{H} \to \mathcal{H}$ is (Fréchet) differentiable, when is $??\cdot; L]$ locally Lipschitz, and how does Lipschitz dependence scale with L?

Question 8.2 (Interface composition). For two layers L_1, L_2, does $????\Psi; L_1]; L_2]$ admit a Baker–Campbell–Hausdorff-type expansion if η is nonlinear?

9 Outliers $\mathcal{O}$: singular measures/jumps

Definition 9.1 (Impulse superposition)**.** Let $\{(t_k, w_k)\}_{k\in\mathbb{N}}$ be times and weights. Define

$$\mathcal{O}[\Psi](t) \;=\; \Psi(t) \;+\; \sum_k w_k\, \delta(t - t_k)\, \Xi_k, \tag{9.1}$$

where $\Xi_k \in \mathcal{H}$ are jump directions. In integrated form against a smooth test φ, this contributes $\sum_k w_k\, \varphi(t_k)\, \Xi_k$.

Question 9.1 (Well-posed impulses). What admissibility conditions on $\{(t_k, w_k, \Xi_k)\}$ ensure that $\mathcal{O}$ defines a bounded map from $C([0,T]; \mathcal{H})$ to distributions valued in $\mathcal{H}$?

Question 9.2 (Compatibility with $\mathcal{C}$). If impulses are T-periodic, can $\mathcal{C}_T$ regularize $\mathcal{O}$ in a distributional sense?

10 The v2 Composite $\mathcal{H}_{\mathrm{v2}}$

Definition 10.1 (Composite map)**.** Given the seven operator blocks, define the v2 composite as in (1.2). The intended evolution is

$$\frac{d\boldsymbol{c}}{dt} \;=\; \mathcal{H}_{\mathrm{v2}}[\boldsymbol{c}(t)]. \tag{10.1}$$

Question 10.1 (Common domain). What is a natural common domain $\mathcal{D} \subseteq \mathcal{H}$ such that each block is well-defined and $\mathcal{H}_{\mathrm{v2}} : \mathcal{D} \to \mathcal{H}$ is (i) measurable, (ii) locally Lipschitz, or (iii) accretive?

Question 10.2 (Operator ordering). How sensitive is (10.1) to the ordering of the seven blocks? Are there pairs that provably commute, or families whose commutators are small in a quantified sense?

Question 10.3 (Semigroup generation). Does $t \mapsto \boldsymbol{c}(t)$ define a (nonlinear) semigroup on $\mathcal{H}$? If so, under what dissipativity or monotonicity conditions?

11 Well-Posedness as Questions

Question 11.1 (Local existence). Under what hypotheses on $(\mathcal{C}, \mathcal{R}, \mathcal{F}, \mathcal{G}, \mathcal{S}, ??\mathcal{O})$ does Picard–Lindelöf (or Carathéodory) apply to (10.1) for $\boldsymbol{c}(0)$ in a ball $B_\rho \subset \mathcal{H}$?

Question 11.2 (Uniqueness). If $\mathcal{H}_{\mathrm{v2}}$ is only one-sided Lipschitz, $\langle \mathcal{H}_{\mathrm{v2}}[u] - \mathcal{H}_{\mathrm{v2}}[v],\, u - v\rangle \le L\,\|u - v\|^2$, is uniqueness retained?

Question 11.3 (Continuation criterion). What blow-up alternatives (e.g. norm inflation in a specific block) govern the maximal interval of existence?

12 Structural Questions: Invariants and Energies

Question 12.1 (Conserved quantity?). Is there a functional $\mathsf{I}(\boldsymbol{c})$ left invariant by some sub-composition (e.g. $\mathcal{C} \circ \mathcal{F}$) that can survive in the full composite?

Question 12.2 (Candidate Lyapunov). Consider $V(\boldsymbol{c}) = \frac{1}{2}\|\boldsymbol{c} - \boldsymbol{c}^{\star}\|^2 + \sum_{i<j} \alpha_{ij} \langle K\boldsymbol{c}_i, \boldsymbol{c}_j\rangle$ with K bounded. Are there parameter regimes where $\dot{V} \le 0$ along (10.1)?

13 Discretization and Approximation

Question 13.1 (Time discretization). For an explicit step $\boldsymbol{c}^{n+1} = \boldsymbol{c}^n + \Delta t\, \mathcal{H}_{\mathrm{v2}}[\boldsymbol{c}^n]$, which blocks impose the strongest stability restriction on Δt?

Question 13.2 (Operator splitting). If $e^{t(\mathsf{A}+\mathsf{B})} \approx e^{t\mathsf{A}}e^{t\mathsf{B}}$ heuristics are used, which partitions of $\mathcal{H}_{\mathrm{v2}}$ minimize splitting error at second order?

14 Compatibility Questions Between Blocks

Question 14.1 ($\mathcal{C}$ vs. $\mathcal{G}$). Does $\mathcal{C}_T$ preserve the decay profile introduced by $\mathcal{G}_\lambda$, in the sense that $\|\mathcal{C}_T\mathcal{G}_\lambda[\Psi] - \mathcal{G}_\lambda[\mathcal{C}_T\Psi]\|$ admits a uniform bound as $t \to \infty$?

Question 14.2 ($\mathcal{F}$ vs. ??). Under scaling, can **??***berecastasarenormalizationinthe*F*hierarchy, i.e. doestheree* such that $\circ$**??**$\mathcal{F} \approx \widetilde{\mathcal{F}}$?

Question 14.3 ($\mathcal{S}$ vs. $\mathcal{O}$). Do impulsive events interfere with bilinear couplings in a way that preserves bounded variation in time for the composite trajectory?

15 Minimal Assumption Sets (as Questions)

Question 15.1 (Candidate minimal set). Is the following sufficient for local well-posedness?

- $\mathcal{C}_T$ orthogonal projector on a closed subspace of $\mathcal{H}$;
- $\mathcal{R}$ defined by a contraction F on a convex subset of $\mathcal{H}$;
- $\sum_k |\alpha_k| < \infty$ in $\mathcal{F}$ and dilations bounded on $\mathcal{H}$;
- $\mathcal{G}_\lambda$ generated by a bounded (or m-dissipative) operator;
- $\mathcal{S}$ built from a bounded bilinear form on $\mathcal{H}$;
- **??***locallyLipschitzwithsmallparameter*μL;
- $\mathcal{O}$ with admissible impulse set producing a Radon measure in time.

16 Identifiability and Observability (Question-Only)

Question 16.1 (Identifiability of block parameters). Given an observed trajectory $t \mapsto \boldsymbol{c}(t)$, are parameters of $\mathcal{H}_{\mathrm{v2}}$ (e.g. λ in $\mathcal{G}$, $\{\alpha_k\}$ in $\mathcal{F}$, κ_{ij} in $\mathcal{S}$) identifiable from finite-time data?

Question 16.2 (Observability under projection). If only a projection $P\boldsymbol{c}(t)$ is observed, which blocks remain distinguishable?

17 Open List: What EchoKey v2 Asks Next

Question 17.1 (Closure under limits). If a sequence of composites $\mathcal{H}_{\mathrm{v2}}^{(m)}$ converges (in operator sense) to $\mathcal{H}_{\mathrm{v2}}$, does well-posedness persist in the limit?

Question 17.2 (Robustness to misspecification). If one block is misspecified but small in operator norm, how does the solution map deviate (stability of the data-to-solution map)?

Question 17.3 (Invariant manifolds). Do specific subspaces (e.g. band-limited states for $\mathcal{C}$) form invariant or approximately invariant manifolds for the full composite?

Question 17.4 (Complexity bounds (analytical)). Is there an a priori bound on the number of active scales (effective support in k for $\mathcal{F}$) over any finite time window?

Closing Note

This manuscript deliberately avoids claims. Each entry is a question whose answer should emerge from explicit derivation or counterexample. The intent is to keep EchoKey v2 resolutely inquiry-first: *EchoKey asks.*