8 Days of EchoKey — Day 5: Diagonality (YZ)

Layout–Aware ZYZ Euler Synthesis

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Abstract

This Day 5 note introduces the *Diagonality (YZ)* generator in the EchoKey 7–operator frame and derives a compiler rewrite to native ZYZ Euler rotations. The symbolic gate ek_diagyz(θ) = $e^{-i\theta(\mathbf{a}_5\cdot\boldsymbol{\sigma})}$ rotates by physical angle 2θ about the YZ–diagonal axis $\mathbf{a}_5 \propto (0,1,1)$. The pass is *layout–aware*: the per–wire axis is read from the *physical* qubit after placement. We state the rule, argue correctness up to global phase, and list the verification metric used in the examples.

1 Background and Notation

Let $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ be the Pauli vector and $\mathbf{A} \in \mathbb{R}^{7\times 3}$ the EchoKey frame whose unit–norm rows are $\mathbf{a}_k^\top$. As in earlier days, define traceless local generators

$$E_k^\circ := \mathbf{a}_k \cdot \boldsymbol{\sigma}, \qquad k = 1, \ldots, 7. \tag{1}$$

We use 1–based indexing in the text; the implementation uses 0–based (so row index 4 in code corresponds to $\mathbf{a}_5$ here).

Day 5 choice (Diagonality YZ). Select the YZ diagonal

$$\mathbf{a}_5 \propto (0, 1, 1), \qquad \|\mathbf{a}_5\| = 1, \tag{2}$$

with possible small per–site tilts (see Section 4). The Day 5 gate is

$$\text{ek\_diagyz}(\theta) \overset{\text{def}}{=} e^{-i\theta\,(\mathbf{a}_5\cdot\boldsymbol{\sigma})}. \tag{3}$$

2 Axis–Angle Form

Every $U \in \mathrm{SU}(2)$ admits the axis–angle representation

$$U(\varphi, \hat{\mathbf{n}}) = \cos\frac{\varphi}{2}\,\mathbb{I} - i\sin\frac{\varphi}{2}\,(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}), \qquad \hat{\mathbf{n}} \in \mathbb{S}^2. \tag{4}$$

Matching (3) to (4) yields

$$\hat{\mathbf{n}} = \mathbf{a}_5, \qquad \varphi = 2\theta. \tag{5}$$

Thus ek_diagyz(θ) is a Bloch rotation about a YZ–plane axis.

3 ZYZ Euler Decomposition

Any single–qubit unitary is (up to a global phase) a ZYZ product,

$$U \doteq \mathrm{RZ}(\alpha)\,\mathrm{RY}(\beta)\,\mathrm{RZ}(\gamma). \tag{6}$$

We form the exact 2×2 matrix $U(2\theta, \mathbf{a}_5)$ via (4) and decompose it to obtain Euler angles (α, β, γ), giving the local rewrite

$$\text{ek\_diagyz}(\theta) \doteq \mathrm{RZ}(\alpha(\theta))\,\mathrm{RY}(\beta(\theta))\,\mathrm{RZ}(\gamma(\theta)). \tag{7}$$

Unlike Day 3 (Z–axis), there is no special one–gate fast path here; we always synthesize ZYZ exactly.

4 Layout–Aware Axis Resolution

Let phys : {logical wires} $\to \{0, \ldots, n-1\}$ be the placement mapping. Each physical wire p has its own local frame $\mathbf{A}^{(p)}$. The axis used when rewriting a gate on logical wire q is

$$\boxed{\hat{\mathbf{n}} = \mathbf{a}_5^{(\,\mathrm{phys}(q)\,)}, \qquad \varphi = 2\theta.} \tag{8}$$

Therefore the pass should run *after* placement or receive the final layout through the property set.

5 Correctness

For each gate occurrence, we compute the exact SU(2) unitary $U(2\theta, \hat{\mathbf{n}})$ from the per–wire axis (8) and factor it as in (6). Since ZYZ covers all of SU(2) up to a phase, the substitution (7) preserves the circuit unitary up to a global phase. Composing these local substitutions over the DAG preserves the total unitary.

6 Validation Metric

We compare the materialized input unitary (symbolic echo gates replaced by their exact 2×2 matrices) and the output unitary (after the pass) using the phase–insensitive overlap

$$\mathcal{F}(U_{\mathrm{in}}, U_{\mathrm{out}}) = \frac{\left|\mathrm{Tr}\left(U_{\mathrm{in}}^{\dagger} U_{\mathrm{out}}\right)\right|}{2^n} \in [0, 1]. \tag{9}$$

Exact synthesis yields $\mathcal{F} \approx 1.000\,000\,000\,000$ across the included examples.

7 Worked Examples

Ex 1: 1q simple: ek_diagyz(0.40) then H. With $\mathbf{a}_5 = (0, 1, 1)/\sqrt{2}$ the pass emits a native ZYZ triple.

Ex 2: 1q sequence: RX(0.11) ek_diagyz(−0.42) RY(0.23) ek_diagyz(0.80) RZ(−0.31). Each echo gate rewrites independently.

Ex 3: 2q with entangler: H_0 ek_diagyz$^{(0)}$(0.50) $\mathrm{CX}_{0\to1}$ ek_diagyz$^{(1)}$(−0.25) $\mathrm{RY}^{(1)}$(0.40). Per–wire frames may tilt the YZ diagonal; the pass uses (8).

Ex 4: Multi–qubit per–site frames: assign distinct $\mathbf{A}^{(p)}$ and tilt every second $\mathbf{a}_5^{(p)}$ slightly toward $+X$ to ensure the general path is exercised; add nearest–neighbor CNOTs.

8 Edge Cases and Numerics

- **Degenerate/NaN axis:** if $\|\mathbf{a}_5\| \approx 0$ or contains NaNs, reject the gate.
- **Branch cuts:** Euler angles are not unique; any consistent branch yields phase–equivalent unitaries.
- **Ordering:** run the rewrite after placement so (8) uses *physical* indices.

Complexity. The pass is linear in the number of ek_diagyz gates. Each ZYZ synthesis is $\mathcal{O}(1)$ for 2×2 matrices.

9 Repro Checklist

1. Choose per–wire frames $\{\mathbf{A}^{(p)}\}_{p=0}^{n-1}$ with unit rows; set $\mathbf{a}_5^{(p)}$.
2. Build the circuit with symbolic ek_diagyz(θ) gates.
3. Resolve $\hat{\mathbf{n}} = \mathbf{a}_5^{(\,\mathrm{phys}(q)\,)}$ and $\varphi = 2\theta$ for each gate.
4. Materialize $U(2\theta, \hat{\mathbf{n}})$ and decompose to ZYZ; replace each echo gate by RZ RY RZ.
5. Validate with the fidelity metric; expect $\mathcal{F} \approx 1$.