8 Days of EchoKey — Day 4: Diagonality (XY)

Layout–Aware ZYZ Euler Synthesis

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Abstract

This Day 4 note introduces the *Diagonality (XY)* generator in the EchoKey 7–operator frame and develops a complete compiler rewrite that maps the symbolic gate $\mathrm{ek\_diagxy}(\theta) = e^{-i\theta(\mathbf{a}_4\cdot\boldsymbol{\sigma})}$ to a native ZYZ Euler product $\mathrm{RZ}(\alpha)\,\mathrm{RY}(\beta)\,\mathrm{RZ}(\gamma)$. The pass is *layout–aware*: the rotation axis is resolved from the *physical* wire according to the placement mapping. We state the rule, prove correctness up to global phase, and give the fidelity metric used for verification.

1 Background and Notation

Let $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ be the Pauli vector, and let $\mathbf{A} \in \mathbb{R}^{7\times 3}$ be the EchoKey frame with unit–norm rows $\mathbf{a}_k^\top$. A traceless EchoKey Hamiltonian is

$$H_{\mathrm{EK}}(\mathbf{c}) = \sum_{k=1}^{7} c_k\, E_k^\circ, \qquad E_k^\circ := \mathbf{a}_k \cdot \boldsymbol{\sigma}, \quad H_{\mathrm{EK}}(\mathbf{c}) = \boldsymbol{\alpha}\cdot\boldsymbol{\sigma} \text{ with } \boldsymbol{\alpha} = \mathbf{A}^\top\mathbf{c}. \tag{1}$$

We use 1–based indexing in the math ($\mathbf{a}_4$ is the fourth row), while the code uses 0–based (`A[3]`).

Day 4 choice (Diagonality XY). We pick the XY diagonal direction

$$\mathbf{a}_4 \propto (1,1,0), \qquad \|\mathbf{a}_4\| = 1, \tag{2}$$

under the shared frame $\mathbf{A}$. (Per–site frames may tilt this row; see Section 4.) The gate is

$$\mathrm{ek\_diagxy}(\theta) \overset{\mathrm{def}}{=} e^{-i\theta\,(\mathbf{a}_4\cdot\boldsymbol{\sigma})}. \tag{3}$$

2 Axis–Angle Form

Every $U \in \mathrm{SU}(2)$ has the axis–angle representation

$$U(\varphi, \hat{\mathbf{n}}) = \cos\frac{\varphi}{2}\,\mathbb{I} - i\sin\frac{\varphi}{2}\,(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}), \qquad \hat{\mathbf{n}} \in \mathbb{S}^2. \tag{4}$$

Comparing (3) and (4) gives

$$\hat{\mathbf{n}} = \mathbf{a}_4, \qquad \varphi = 2\theta. \tag{5}$$

Thus $\mathrm{ek\_diagxy}(\theta)$ is a Bloch–sphere rotation about an axis lying in the XY plane (up to per–site tilts).

3 ZYZ Euler Decomposition

As in Days 1–3, any single–qubit unitary is (up to global phase) a ZYZ product,

$$U \doteq \mathrm{RZ}(\alpha)\,\mathrm{RY}(\beta)\,\mathrm{RZ}(\gamma). \tag{6}$$

In practice we synthesize the exact 2×2 unitary $U(2\theta, \mathbf{a}_4)$ using (4) and decompose it to obtain (α, β, γ), yielding the rewrite rule

$$\text{ek\_diagxy}(\theta) \doteq \mathrm{RZ}(\alpha(\theta))\,\mathrm{RY}(\beta(\theta))\,\mathrm{RZ}(\gamma(\theta)). \tag{7}$$

4 Layout–Aware Axis Resolution

On multi–qubit devices, each *physical* wire p can carry a distinct local frame $\mathbf{A}^{(p)}$. If phys : $\{\text{logical wires}\} \to \{0, \ldots, n-1\}$ is the placement map, the axis for a gate acting on logical q is

$$\boxed{\hat{\mathbf{n}} = \mathbf{a}_4^{(\,\mathrm{phys}(q)\,)}, \qquad \varphi = 2\theta.} \tag{8}$$

Hence the rewrite should be run *after* placement (or read the layout from the pass property set).

5 Correctness

Each local substitution uses the exact SU(2) matrix $U(2\theta, \hat{\mathbf{n}})$ computed from the per–wire axis (8). Because ZYZ decomposition is complete for SU(2) up to a global phase, there exist (α, β, γ) such that (7) holds. Replacing all occurrences in the circuit preserves the full unitary up to global phase.

6 Validation Metric

We compare U_{in} (materialized by substituting the exact 2×2 matrix for each echo gate) and U_{out} (after the pass) with the phase–insensitive overlap

$$\mathcal{F}(U_{\text{in}}, U_{\text{out}}) = \frac{\left|\mathrm{Tr}\left(U_{\text{in}}^{\dagger} U_{\text{out}}\right)\right|}{2^n} \in [0, 1]. \tag{9}$$

Exact synthesis yields $\mathcal{F} \approx 1.000\,000\,000\,000$ numerically across the included examples.

7 Worked Examples

Ex 1: 1q simple: ek_diagxy(0.40) then H. With $\mathbf{a}_4 = (1, 1, 0)/\sqrt{2}$ the rewrite emits a native ZYZ triple.

Ex 2: 1q sequence: RX(0.11) ek_diagxy(−0.42) RY(0.23) ek_diagxy(0.80) RZ(−0.31). Each echo gate rewrites independently.

Ex 3: 2q with entangler: H_0 ek_diagxy$^{(0)}$(0.50) $\mathrm{CX}_{0\to1}$ ek_diagxy$^{(1)}$(−0.25) $\mathrm{RY}^{(1)}$(0.40). Per–wire frames may tilt the XY diagonal; the pass uses (8).

Ex 4: Multi–qubit per–site frames: assign a distinct $\mathbf{A}^{(p)}$ to each wire and tilt every second $\mathbf{a}_4^{(p)}$ slightly out of the plane to exercise the general path; insert nearest–neighbor CNOTs.

8 Edge Cases and Numerics

- **Degenerate/NaN axis:** if $\|\mathbf{a}_4\| \approx 0$ or contains NaNs, reject the gate.
- **Branch cuts:** Euler angles are not unique; any consistent branch yields phase–equivalent unitaries.
- **Ordering:** run the rewrite after placement so (8) uses *physical* indices.

Complexity. The pass is linear in the number of ek_diagxy gates; each ZYZ synthesis is $\mathcal{O}(1)$ for 2×2 matrices.

9 Repro Checklist

1. Choose per–wire frames $\{\mathbf{A}^{(p)}\}_{p=0}^{n-1}$ with unit rows; set $\mathbf{a}_4^{(p)}$.
2. Build the circuit with symbolic ek_diagxy(θ) gates.
3. Resolve $\hat{\mathbf{n}} = \mathbf{a}_4^{(\,\mathrm{phys}(q)\,)}$ and $\varphi = 2\theta$ for each gate.
4. Materialize $U(2\theta, \hat{\mathbf{n}})$ and decompose to ZYZ; replace the echo gate by RZ RY RZ.
5. Validate with the fidelity metric; expect $\mathcal{F} \approx 1$.