EchoKey Day 8: Exact QASM→EchoKey QASM (ZYZ Fusion, Bitstring-Preserving)

EchoKey Project

September 13, 2025

Abstract

Day 8 introduces a *bitstring-preserving* converter from OpenQASM 2.0 circuits to an *EchoKey QASM* that replaces every per-wire chain of 1-qubit gates by at most three EchoKey gates:

$$R_z(\alpha)R_y(\beta)R_z(\gamma) \;\equiv\; \exp\!\left(-\tfrac{i\alpha}{2}\sigma_z\right)\,\exp\!\left(-\tfrac{i\beta}{2}\sigma_y\right)\,\exp\!\left(-\tfrac{i\gamma}{2}\sigma_z\right) \;\equiv\; \texttt{ek\_frac}(\alpha/2)\,\texttt{ek\_rec}(\beta/2)\,\texttt{ek\_frac}(\gamma/2).$$

We prove correctness (up to global phase), explain the block-flush semantics around non-unitary and multi-qubit boundaries, and justify that the final measurement *bitstrings* are identical to the input circuit for all inputs (no approximations).

1 Preliminaries: SU(2), Euler ZYZ, and QASM gate semantics

Let $\{\sigma_x, \sigma_y, \sigma_z\}$ be Pauli matrices. Any $U \in \mathrm{SU}(2)$ admits an axis–angle form

$$U(\varphi, \hat{\boldsymbol{n}}) \;=\; \cos\frac{\varphi}{2}\, I_2 - i \sin\frac{\varphi}{2} (\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}), \qquad \hat{\boldsymbol{n}} \in \mathbb{S}^2, \tag{1}$$

and an Euler ZYZ factorization

$$U \;\doteq\; R_z(\alpha)R_y(\beta)R_z(\gamma), \tag{2}$$

where $\doteq$ denotes equality up to a *global phase*. In OpenQASM 2 (and Qiskit),

$$R_z(\lambda) = e^{-i\frac{\lambda}{2}\sigma_z}, \qquad R_y(\beta) = e^{-i\frac{\beta}{2}\sigma_y}. \tag{3}$$

EchoKey's canonical single-axis gates use the same Lie generators:

$$\texttt{ek\_frac}(\theta) \;:=\; e^{-i\theta\,\sigma_z}, \qquad \texttt{ek\_rec}(\theta) \;:=\; e^{-i\theta\,\sigma_y}. \tag{4}$$

Comparing (3) and (4) gives the *half-angle* map

$$R_z(\lambda) \;\equiv\; \texttt{ek\_frac}(\lambda/2), \qquad R_y(\beta) \;\equiv\; \texttt{ek\_rec}(\beta/2). \tag{5}$$

2 Problem statement and converter semantics

Given an input QASM 2 circuit $\mathcal{C}_{\mathrm{in}}$, the converter partitions it into *per-wire* maximal 1-qubit blocks separated by **boundaries**:

$$\text{Boundary} \;:=\; \{\text{any multi-qubit instruction}\} \cup \{\texttt{barrier}, \texttt{measure}, \texttt{reset}\} \cup \{\text{classically conditioned op}\}. \tag{6}$$

For each qubit q and each block B (a temporally contiguous subsequence of 1q gates on q with no boundaries in between), form its exact 2×2 unitary U_B (latest gates on the *left*). The converter replaces B by the unique ZYZ triple of U_B:

$$U_B \;\doteq\; R_z(\alpha_B)R_y(\beta_B)R_z(\gamma_B) \;\mapsto\; \texttt{ek\_frac}(\alpha_B/2)\,\texttt{ek\_rec}(\beta_B/2)\,\texttt{ek\_frac}(\gamma_B/2). \tag{7}$$

Blocks are *flushed* at each boundary, and all non-1q instructions are emitted verbatim.

Key invariants.

- **Exactness.** U_B is computed from the exact matrix product of the block's gates; Euler ZYZ angles $(\alpha_B, \beta_B, \gamma_B)$ are extracted from U_B (global phase ignored).
- **Ordering.** Within each wire, the sequence of block unitaries and boundaries is preserved.
- **Conditionals.** A conditional instruction is a boundary. Pending 1q work on the wire(s) is flushed *before* the condition is evaluated and the instruction emitted. Thus the classical control flow is untouched.
- **Non-unitaries.** `reset`/`measure` are boundaries; pending 1q work is applied before them, exactly as in the input.

3 Correctness: unitary and measurement equivalence

3.1 Per-block equality

Let U_B be the exact block unitary on a given wire. By (2) there exist $(\alpha_B, \beta_B, \gamma_B)$ s.t. $U_B \doteq R_z(\alpha_B)R_y(\beta_B)R_z(\gamma_B)$. Using (5), the EchoKey triple in (7) is exactly the same unitary up to global phase. Hence the *per-block* replacement is unitary-equivalent.

3.2 Whole-circuit equality

Write the full circuit as an alternating product of single-wire blocks and boundaries (multi-qubit or non-unitary instructions):

$$\mathcal{C}_{\text{in}} \;=\; \cdots\; U_{B_1}\, G_1\, U_{B_2}\, G_2\, \cdots$$

where each G_k is emitted verbatim and each U_{B_j} is replaced by its EchoKey triple. Since the converter preserves the temporal order and wire targets, and every U_{B_j} is unitary-equivalent (up to a phase) to its replacement, the overall unitary U_{in} and the EchoKey unitary U_{ek} satisfy

$$U_{\text{ek}} \;=\; e^{i\phi}\, U_{\text{in}} \qquad \text{for some (wire-independent) global phase } \phi. \tag{8}$$

3.3 Bitstring identity

Let $|\psi\rangle$ be any input state (possibly mixed via a purification argument). Measurement probabilities in the computational basis are $p(b) = \|\langle b|\, U_{\text{in}}\, |\psi\rangle\|^2$. By (8), $U_{\text{ek}} = e^{i\phi} U_{\text{in}}$ so $\|\langle b|\, U_{\text{ek}}\, |\psi\rangle\|^2 = \left\|e^{i\phi}\, \langle b|\, U_{\text{in}}\, |\psi\rangle\right\|^2 = p(b)$. Therefore the output *bitstring distribution* is identical. In particular, when the input is a computational basis state and the circuit is classical-reversible on measured wires, the distribution is a Kronecker delta (deterministic) and is preserved exactly.

4 Angle extraction and numerical notes

Angles are obtained from the exact 2×2 block matrix U_B using a ZYZ Euler routine (equivalently, from the axis–angle parameters in (1)). Two practical points:

Branch handling. Euler angles are not unique; any ZYZ triple in the same equivalence class (global-phase-equivalent) suffices. The converter always uses a canonical branch from the decomposer.

Near-identity blocks. If $\beta_B \approx 0$ and $\alpha_B \approx -\gamma_B$ within tolerance, the emitted EchoKey triple may reduce to ≤ 2 gates (and ultimately 0 if $U_B \approx I$). The converter naturally emits whatever the ZYZ routine returns.

5 Boundaries and non-unitary semantics

5.1 Multi-qubit gates

Any instruction with arity ≥ 2 (e.g. `cx`, `cz`, `swap`, `ccx`, `cswap`, `ch`) constitutes a boundary. All pending 1q work on the touched wires is flushed *before* emitting the multi-qubit instruction, preserving the original causal structure.

5.2 Resets and measurements

`reset` and `measure` are non-unitary. The converter flushes each wire's 1q block *prior* to these operations, ensuring that the quantum state presented to the non-unitary matches the input circuit exactly. Since (8) still holds for the pre-measurement unitary, the post-measurement classical outcomes are identical.

5.3 Classical control

Any classically conditioned instruction (`if(c==v) op`) is treated as a boundary. Pending 1q work is flushed first; then the conditional is emitted unchanged. Because we neither alter the classical register nor the guard condition, and we preserve the quantum state prior to the conditional, the branch behavior is identical.

6 Cost model: why it is cheaper

Let a single wire have L 1q gates inside a contiguous block. The converter emits at most three EchoKey gates for that block. If a circuit on n wires has m such blocks in total, the 1q gate count reduces from $\sum_\ell L_\ell$ to at most $3m$. Entanglers and non-unitaries are unchanged. In practice, long alternating chains (e.g. RZ/RY/RZ/$\cdots$) collapse aggressively.

7 Verification metric (optional)

For a purely unitary prefix (before the first `measure/reset`), we can compute the global-phase-invariant proxy

$$\mathcal{F} \;=\; \frac{1}{2}\left|\mathrm{Tr}\big(U_{\mathrm{in}}^{\dagger} U_{\mathrm{ek}}\big)\right| \;\approx\; 1, \tag{9}$$

for each 1q block or for the tensor product over all wires. Our reference script reconstructs a native circuit by expanding `ek_frac`$(\theta) \mapsto R_z(2\theta)$ and `ek_rec`$(\theta) \mapsto R_y(2\theta)$ and checks $\mathcal{F} \approx 1$.

8 Edge cases and guarantees

- **Global phase.** All claims are modulo a wire-independent global phase, which never affects measurement probabilities or classical post-processing.
- **Opaque EchoKey gates.** The output QASM declares `opaque ek_rec(theta) q;` and `opaque ek_frac(theta) q;`. Back-ends that know EchoKey can exploit them directly; otherwise, for simulation we expand them back to R_y/R_z via (5).
- **Conditionals in QASM2.** Conditioned ops remain conditioned *verbatim* and form flush boundaries; since we neither add nor remove condition checks, logical behavior is preserved.
- **Numerical stability.** Blocks are exactly 2×2; the ZYZ extraction is stable. Near-identity blocks may return angles close to machine precision; emitting them is semantically safe.

Appendix A: Code–math correspondences (Day 8)

- `ACCUMULATABLE_1Q` $\leftrightarrow$ the allowable 1q gate set whose exact product defines U_B.
- `_flush_qubit` $\leftrightarrow$ block map (7) using ZYZ (2) and half-angle (5).
- `is_boundary` and walk over `qc.data` $\leftrightarrow$ boundary definition (6) and block partition.
- `verify_unitary_prefix` $\leftrightarrow$ fidelity proxy $\mathcal{F}$ on the unitary prefix.
- Runner's `expand_echokey` $\leftrightarrow$ inverse of (5) for simulation: `ek_frac`$(\theta) \mapsto R_z(2\theta)$, `ek_rec`$(\theta) \mapsto R_y(2\theta)$.

Appendix B: Practical reproducibility checklist

1. **Parse** input QASM2 to a circuit; **walk** instructions and build per-wire U_B.
2. On boundary, **flush** $U_B \mapsto$ ZYZ $\mapsto$ EchoKey triple via (7).
3. **Emit** non-1q instructions unchanged (including conditionals/resets/measures).
4. (Optional) **Verify** unitary prefix via $\mathcal{F} \approx 1$.
5. For simulation, **expand** EchoKey to R_y/R_z and run on Aer (normal or MPS).