8 Days of EchoKey — Day 2: Recursion

Axis–Angle to ZYZ Synthesis with Layout Awareness

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Abstract

This Day 2 note introduces the *Recursion* generator in the EchoKey 7–operator frame and shows a complete, implementation–ready derivation for rewriting the symbolic gate $\text{ek\_rec}(\theta) = e^{-i\theta\,(\mathbf{a}_2\cdot\boldsymbol{\sigma})}$ into the native ZYZ Euler basis $\mathrm{RZ}(\alpha)\,\mathrm{RY}(\beta)\,\mathrm{RZ}(\gamma)$. We formalize: (i) the operator and axis–angle mapping, (ii) a layout–aware per–wire frame $\mathbf{A}^{(q)} \in \mathbb{R}^{7\times 3}$, (iii) a compiler rewrite rule with correctness, and (iv) a fidelity metric used to validate exactness.

1 Background and Notation

Let $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ denote the Pauli 3–vector

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In the EchoKey frame we fix a matrix $\mathbf{A} \in \mathbb{R}^{7\times 3}$ whose rows $\mathbf{a}_k^\top$ are unit vectors in $\mathbb{R}^3$ (one row per generator). For any coefficient vector $\mathbf{c} \in \mathbb{R}^7$ the traceless Hamiltonian is

$$H_{\text{EK}}(\mathbf{c}) = \sum_{k=1}^{7} c_k\, E_k^\circ, \qquad E_k^\circ \equiv \mathbf{a}_k \cdot \boldsymbol{\sigma}, \tag{1}$$

with equivalent Pauli vector $\boldsymbol{\alpha} = \mathbf{A}^\top \mathbf{c}$ so that $H_{\text{EK}}(\mathbf{c}) = \boldsymbol{\alpha}\cdot\boldsymbol{\sigma}$.

Day 2 choice (Recursion). We select the second row $\mathbf{a}_2 \equiv \mathbf{A}[2]$ and define the single–qubit unitary gate

$$\text{ek\_rec}(\theta) \;\stackrel{\text{def}}{=}\; e^{-i\theta\,(\mathbf{a}_2\cdot\boldsymbol{\sigma})}. \tag{2}$$

Convention: all rows of $\mathbf{A}$ are normalized, so $\|\mathbf{a}_k\| = 1$.

2 Axis–Angle Form of $\text{ek\_rec}(\theta)$

Any single–qubit SU(2) rotation has the *axis–angle* form

$$U(\varphi, \hat{\mathbf{n}}) \;=\; \cos\frac{\varphi}{2}\,\mathbb{I} \;-\, i\sin\frac{\varphi}{2}\,(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}), \qquad \hat{\mathbf{n}} \in \mathbb{S}^2. \tag{3}$$

Comparing (2) and (3) gives the identification

$$\hat{\mathbf{n}} = \mathbf{a}_2, \qquad \varphi = 2\theta. \tag{4}$$

Thus $\text{ek\_rec}(\theta)$ is a Bloch–sphere rotation by angle $\varphi = 2\theta$ about the fixed axis $\hat{\mathbf{n}} = \mathbf{a}_2$.

3 ZYZ Euler Decomposition

Every $U \in \mathrm{SU}(2)$ can be written (up to a global phase) as a ZYZ Euler product

$$U \doteq \mathrm{RZ}(\alpha)\,\mathrm{RY}(\beta)\,\mathrm{RZ}(\gamma)\,, \tag{5}$$

for some $(\alpha, \beta, \gamma) \in \mathbb{R}^3$. A useful set of relations between the Euler angles and the axis–angle parameters $(\varphi, \hat{\mathbf{n}} = (n_x, n_y, n_z))$ is (up to branch choices):

$$\cos\tfrac{\varphi}{2} = \cos\tfrac{\beta}{2}\,\cos\tfrac{\alpha+\gamma}{2}, \tag{6}$$

$$n_z \sin\tfrac{\varphi}{2} = \cos\tfrac{\beta}{2}\,\sin\tfrac{\alpha+\gamma}{2}, \tag{7}$$

$$n_x \sin\tfrac{\varphi}{2} = \sin\tfrac{\beta}{2}\,\cos\tfrac{\alpha-\gamma}{2}, \tag{8}$$

$$n_y \sin\tfrac{\varphi}{2} = \sin\tfrac{\beta}{2}\,\sin\tfrac{\alpha-\gamma}{2}. \tag{9}$$

Given $(\varphi, \hat{\mathbf{n}})$ one can recover (α, β, γ) by any numerically stable inversion of (6)–(9) or, equivalently, by synthesizing the exact 2×2 unitary[1] and decomposing it with a ZYZ Euler routine. In practice we take $(\varphi, \hat{\mathbf{n}}) = (2\theta, \mathbf{a}_2)$ from (4) and use a stable decomposer to obtain (α, β, γ).

Rewrite rule (mathematical statement). For all $\theta \in \mathbb{R}$,

$$\text{ek\_rec}(\theta) \doteq \mathrm{RZ}(\alpha(\theta))\,\mathrm{RY}(\beta(\theta))\,\mathrm{RZ}(\gamma(\theta)), \tag{10}$$

where (α, β, γ) are the ZYZ Euler angles of $U(2\theta, \mathbf{a}_2)$.

4 Layout–Aware Per–Wire Frames

On a multi–qubit device, each *physical* wire $p \in \{0, \ldots, n-1\}$ may carry its own local frame $\mathbf{A}^{(p)} \in \mathbb{R}^{7\times 3}$ (e.g., different $\mathbf{a}_2$ orientations). Denote by $\mathrm{phys} : \{\text{logical qubits}\} \to \{0, \ldots, n-1\}$ the layout mapping chosen by placement/routing.

$$\boxed{\textbf{Axis resolution: } \hat{\mathbf{n}} \text{ (for gate on logical } q) = \mathbf{a}_2^{(\,\mathrm{phys}(q)\,)}, \quad \varphi = 2\theta.} \tag{11}$$

The rewrite must therefore be performed *after* the layout is known (or using the current mapping in the pass property set) to ensure the correct per–wire axis is used.

5 Compiler Rewrite and Correctness

Traversing a circuit DAG, replace every instance of ek_rec(θ) by the ZYZ Euler product computed from (3)–(10) with axis resolved by (11). Formally, writing $\mathcal{R}$ for the transformation:

$$\mathcal{R}\big[\,\text{ek\_rec}(\theta)\,\big] = \mathrm{RZ}\big(\alpha(\theta)\big)\,\mathrm{RY}\big(\beta(\theta)\big)\,\mathrm{RZ}\big(\gamma(\theta)\big)\,. \tag{12}$$

Correctness. For each gate, $\mathcal{R}$ uses the exact SU(2) matrix $U(2\theta, \hat{\mathbf{n}})$ with $\hat{\mathbf{n}}$ chosen per (11). Since ZYZ Euler synthesis is complete for SU(2), there exists (α, β, γ) with $\mathrm{RZ}(\alpha)\mathrm{RY}(\beta)\mathrm{RZ}(\gamma) \doteq U$. Therefore every local replacement preserves the unitary up to a global phase, and so does the full circuit.

[1] From (3): $U = \cos(\varphi/2)\,\mathbb{I} - i\sin(\varphi/2)\,(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)$

6 Validation Metric

For small n we compare the exact unitaries U_{in} (materialized by replacing ek_rec with its 2×2 matrix) and U_{out} (after rewriting) using the global–phase–insensitive overlap

$$\mathcal{F}(U_{\text{in}}, U_{\text{out}}) \;=\; \frac{\left|\mathrm{Tr}\left(U_{\text{in}}^{\dagger} U_{\text{out}}\right)\right|}{2^n} \in [0,1]. \tag{13}$$

Exact synthesis yields $\mathcal{F} \approx 1.000\,000\,000\,000$ numerically.

7 Worked Examples

Below $\mathbf{A}$ is the shared frame unless specified; all rows are unit–norm.

Ex 1: 1q simple: ek_rec(0.37) then H. The first gate rewrites to ZYZ with axis $\mathbf{a}_2$; H is native.

Ex 2: 1q sequence: RZ(0.11) ek_rec(-0.42) RY(0.23) ek_rec(0.80) RX(-0.31). Both ek_rec gates rewrite independently.

Ex 3: 2q: H_0 ek_rec$^{(0)}$(0.5) CX$_{0\to 1}$ ek_rec$^{(1)}$(-0.25) RZ$^{(1)}$(0.40). Axes are resolved per wire, potentially distinct if $\mathbf{A}^{(0)} \neq \mathbf{A}^{(1)}$.

Ex 4: n–q chain with per–site frames: each wire p has its own $\mathbf{A}^{(p)}$ with distinct $\mathbf{a}_2^{(p)}$; insert ek_rec$^{(p)}$ $\big(0.17(p+1)\big)$ on each site and CNOTs between neighbors. The rewrite uses (11) at each site.

8 Optional: Recovering the Pauli Basis

When rank($\mathbf{A}$) = 3 the right inverse $\mathbf{B} = (\mathbf{A}^{\top}\mathbf{A})^{-1}\mathbf{A}^{\top}$ satisfies

$$S_i \;=\; \sum_{k=1}^{7} B_{ik}\, E_k^{\circ} \;=\; \sigma_i\,, \qquad i \in \{x, y, z\}, \tag{14}$$

so the chosen EchoKey generators span the Pauli basis exactly. This justifies treating $\mathbf{a}_2 \cdot \boldsymbol{\sigma}$ as a physical rotation axis.

9 Edge Cases and Numerics

- **Degenerate axis:** if $\|\mathbf{a}_2\| \approx 0$ or NaN, the gate is ill–posed (reject).
- **Branch cuts:** Euler angles are not unique; any consistent branch yields identical unitaries up to global phase.
- **Placement order:** apply the rewrite after layout is available to honor per–wire frames.

Complexity. The pass is linear in the number of ek_rec gates. Each ZYZ synthesis is $\mathcal{O}(1)$ for 2×2 matrices.

10 Repro Checklist

1. Choose frames $\{\mathbf{A}^{(p)}\}_{p=0}^{n-1}$ (unit rows).
2. Build the circuit with symbolic ek_rec(θ) gates.

3. Resolve axis $\hat{\mathbf{n}} = \mathbf{a}_2^{(\,\mathrm{phys}(q)\,)}$ and angle $\varphi = 2\theta$ for each gate.
4. Compute U via (3) and synthesize ZYZ angles to replace the gate.
5. Validate with (13); expect $\mathcal{F} \approx 1$.