EchoKey 7-Operator Toy: "Ghetto Math" & Emergent Pauli

EchoKey Project

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Abstract

This note formalizes the Day–1 "ghetto math" for the seven local EchoKey operators used in the toy model (`echokey_toy_model.py`). We specify how the operators are parameterized from a site–dependent $A \in \mathbb{R}^{7\times 3}$, how the Pauli generators $(\sigma_x, \sigma_y, \sigma_z)$ *emerge* from the EchoKey span via a right inverse $B \in \mathbb{R}^{3\times 7}$, and how this predicts circuit–level equivalence for single– and two–qubit evolution. The document also records practical diagnostics (rank, condition numbers, commutator residuals) and the expected outputs of each step.

1 Pauli preliminaries

Let $\{\sigma_x, \sigma_y, \sigma_z\}$ be Pauli matrices, I_2 identity. Standard identities:

$$\frac{1}{2}\operatorname{Tr}(\sigma_a\sigma_b) = \delta_{ab}, \quad [\sigma_a, \sigma_b] = 2i\,\varepsilon_{abc}\,\sigma_c, \quad \{\sigma_a, \sigma_b\} = 2\delta_{ab} I_2. \tag{1}$$

Any 2×2 Hermitian H decomposes as $H = a_0 I_2 + \boldsymbol{a}\cdot\boldsymbol{\sigma}$ with $a_0 \in \mathbb{R}$ and $\boldsymbol{a} \in \mathbb{R}^3$. We denote the *traceless* part by $H^\circ := H - \frac{1}{2}\operatorname{Tr}(H)\, I_2 = \boldsymbol{a}\cdot\boldsymbol{\sigma}$. For any unit vector $\hat{\boldsymbol{n}}$ and angle θ we have the exact axis–angle exponential

$$e^{-i\theta\,\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}} = \cos\theta\, I_2 - i\sin\theta\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}), \tag{2}$$

which corresponds to a Bloch rotation by physical angle 2θ about $\hat{\boldsymbol{n}}$.

2 Seven EchoKey directions at a site

At each site i we define seven traceless local operators

$$E_k^{(i)\circ} \;:=\; \boldsymbol{a}_k^{(i)}\cdot\boldsymbol{\sigma}, \qquad k = 1,\ldots,7, \tag{3}$$

from a site–specific real matrix $A^{(i)} \in \mathbb{R}^{7\times 3}$ whose rows are unit vectors $\boldsymbol{a}_k^{(i)\top}$. In the toy code the directions are constructed from the cube coordinates $\boldsymbol{c}^{(i)} = (x, y, z)$ (with $x + y + z = 0$) derived from axial hex coordinates:

$$\begin{aligned}
&\boldsymbol{a}_1 = \pm\hat{\boldsymbol{e}}_x, && \boldsymbol{a}_2 = \pm\hat{\boldsymbol{e}}_y, && \boldsymbol{a}_3 = \pm\hat{\boldsymbol{e}}_z,\\
&\boldsymbol{a}_4 = \operatorname{norm}(\boldsymbol{a}_1+\boldsymbol{a}_2), \quad \boldsymbol{a}_5 = \operatorname{norm}(\boldsymbol{a}_2+\boldsymbol{a}_3), && \boldsymbol{a}_6 = \operatorname{norm}(\boldsymbol{a}_1+\boldsymbol{a}_3), && \boldsymbol{a}_7 = \operatorname{norm}(\boldsymbol{a}_1+\boldsymbol{a}_2+\boldsymbol{a}_3),
\end{aligned} \tag{4}$$

where $\operatorname{norm}(\boldsymbol{v}) = \boldsymbol{v}/\|\boldsymbol{v}\|$ and the $\pm$ signs are taken from $\operatorname{sign}(x)$ etc. A tiny "identity leakage" $a_0 \sim \mathcal{U}[-\epsilon, \epsilon]$ may be added in the raw construction, but it is removed immediately by $E_k^\circ = E_k - \frac{1}{2}\operatorname{Tr}(E_k) I_2$.

Matrix view. Stacking the $\boldsymbol{a}_k$ as rows gives $A \in \mathbb{R}^{7\times 3}$ and the EchoKey basis $E^\circ = (E_1^\circ, \dots, E_7^\circ)$ with

$$E_k^\circ = \boldsymbol{a}_k \cdot \boldsymbol{\sigma}, \qquad A = \begin{bmatrix} - \\ \boldsymbol{a}_1^\top \\ \vdots \\ \boldsymbol{a}_7^\top \\ - \end{bmatrix}, \qquad \|\boldsymbol{a}_k\| = 1. \tag{5}$$

3 Forward map and right inverse

Given coefficients $\boldsymbol{c} \in \mathbb{R}^7$, the EchoKey Hamiltonian is

$$H_{\mathrm{EK}}(\boldsymbol{c}) \;=\; \sum_{k=1}^{7} c_k E_k^\circ \;=\; \underbrace{\Big(\sum_k c_k \boldsymbol{a}_k\Big)}_{\boldsymbol{\alpha}(\boldsymbol{c})} \cdot \boldsymbol{\sigma} \;=\; (A^\top \boldsymbol{c}) \cdot \boldsymbol{\sigma}. \tag{6}$$

Thus the *forward map* from EchoKey coefficients to Pauli coordinates is simply

$$\boldsymbol{\alpha} \;=\; A^\top \boldsymbol{c} \in \mathbb{R}^3. \tag{7}$$

If $\operatorname{rank}(A) = 3$, a right inverse exists. The minimum–norm choice is the (regularized) pseudoinverse

$$B \;\coloneqq\; (A^\top A + \lambda I_3)^{-1} A^\top \in \mathbb{R}^{3\times 7}, \tag{8}$$

with $\lambda \geq 0$ (set $\lambda = 0$ when $\kappa(A)$ is modest). Define the linear combinations

$$S_i \;\coloneqq\; \sum_{k=1}^{7} B_{ik} E_k^\circ, \qquad i \in \{x, y, z\}. \tag{9}$$

Then

$$S_i \;=\; \Big(\sum_k B_{ik} \boldsymbol{a}_k\Big) \cdot \boldsymbol{\sigma} \;=\; (BA)_{i\cdot} \cdot \boldsymbol{\sigma} \;=\; \sigma_i, \tag{10}$$

so that the Pauli generators *emerge* from the EchoKey family.

Diagnostics. Let $s_1 \geq s_2 \geq s_3 > 0$ be the singular values of A. We report

- $\operatorname{rank}(A) = 3$ (requirement for emergence),
- $\operatorname{cond}(A) = s_1/s_3$ (numerical stability),
- orthonormality matrix $O_{ij} = \frac{1}{2}\operatorname{Tr}(S_i S_j) \overset{!}{=} \delta_{ij}$,
- commutator residuals $\|[S_i, S_j] - 2i\,\varepsilon_{ijk} S_k\|_F$ (should be $\ll 1$).

4 Single–qubit evolution equivalence

For any step with EchoKey coefficients $\boldsymbol{c}$ and angle θ the unitary is

$$U_{\mathrm{EK}}(\boldsymbol{c}, \theta) \;=\; \exp\big(-i\,\theta\, H_{\mathrm{EK}}(\boldsymbol{c})\big) \;=\; \exp\big(-i\,\theta\,(A^\top \boldsymbol{c}) \cdot \boldsymbol{\sigma}\big). \tag{11}$$

Let $\boldsymbol{\alpha} = A^\top \boldsymbol{c}$. The Pauli reference step is

$$U_{\mathrm{P}}(\boldsymbol{\alpha}, \theta) \;=\; \exp\big(-i\,\theta\,\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}\big). \tag{12}$$

Because $H_{\mathrm{EK}}(\boldsymbol{c})$ *equals* $\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}$ by (6), we have exact equality $U_{\mathrm{EK}}(\boldsymbol{c}, \theta) = U_{\mathrm{P}}(\boldsymbol{\alpha}, \theta)$. For a sequence of L steps, multiply them in order. The code reports the global–phase–invariant proxy

$$\mathcal{F}_1 \;=\; \frac{1}{2}\left|\operatorname{Tr}\Big(U_{\mathrm{EK}}^\dagger U_{\mathrm{P}}\Big)\right| \;\approx\; 1. \tag{13}$$

5 Two–qubit XYZ equivalence (one edge)

At two sites i and j, define the emergent single–site Pauli operators $S_a^{(i)} = \sum_k B_{ak}^{(i)} E_k^{(i)\circ}$ and $S_b^{(j)}$ analogously. Form the (EchoKey) two–qubit XYZ Hamiltonian

$$H_{\mathrm{EK}}^{(i,j)} \;=\; J_x\, S_x^{(i)} \otimes S_x^{(j)} + J_y\, S_y^{(i)} \otimes S_y^{(j)} + J_z\, S_z^{(i)} \otimes S_z^{(j)}. \tag{14}$$

Because $S_a^{(i)} = \sigma_a^{(i)}$ by (10), we have

$$H_{\mathrm{EK}}^{(i,j)} \;=\; J_x\, \sigma_x^{(i)} \otimes \sigma_x^{(j)} + J_y\, \sigma_y^{(i)} \otimes \sigma_y^{(j)} + J_z\, \sigma_z^{(i)} \otimes \sigma_z^{(j)} \;=\; H_{\mathrm{P}}^{(i,j)}, \tag{15}$$

and hence $U_{\mathrm{EK}}^{(i,j)}(t) = e^{-itH_{\mathrm{EK}}^{(i,j)}} = e^{-itH_{\mathrm{P}}^{(i,j)}} = U_{\mathrm{P}}^{(i,j)}(t)$. The code prints the normalized trace proxy for the 4×4 case,

$$\mathcal{F}_2 \;=\; \frac{1}{4}\left|\mathrm{Tr}\big((U_{\mathrm{EK}}^{(i,j)})^{\dagger} U_{\mathrm{P}}^{(i,j)}\big)\right| \;\approx\; 1. \tag{16}$$

6 Graph, orientation, and per–site A

The hex patch graph (radius R) is used only for site indexing and an interpretable per–site orientation. Axial coordinates (q, r) are mapped to cube coordinates $(x, y, z) = (q, r, -q-r)$ with $x + y + z = 0$. The sign pattern in (x, y, z) fixes the local axes $\pm\hat{\boldsymbol{e}}_x, \pm\hat{\boldsymbol{e}}_y, \pm\hat{\boldsymbol{e}}_z$ from which the seven rows of A are assembled and normalized as above. Any other site–dependent frame can be substituted without affecting the emergence math provided $\mathrm{rank}(A) = 3$.

7 Regularization and numerical notes

When A is ill–conditioned, we use the Tikhonov–regularized right inverse (8) with small λ. This preserves emergence to within $O(\lambda)$ while stabilizing the inversion. Diagnostics to monitor:

- singular values of A (report $s(A)$, $\mathrm{rank}(A)$, $\mathrm{cond}(A)$);
- orthonormality matrix O (should be $\approx I_3$ within numerical tolerance);
- commutator residuals $\|[S_i, S_j] - 2i\,\varepsilon_{ijk} S_k\|_F$ (should be $\ll 1$).

8 Outputs you should expect (default settings)

- **Emergence per site:** all sites pass with (rank $= 3$), $O \approx I$, residuals $\leq 10^{-12}$.
- **Single–qubit sequence:** $\mathcal{F}_1 \approx 1.0$ for a random 6–step program.
- **Two–qubit XYZ:** $\mathcal{F}_2 \approx 1.0$ on a random edge.
- **Export:** JSON file of $B^{(i)}$ (shape 3×7 per site) for downstream compiler injection.

9 Interpretation and implications

1. The seven EchoKey directions provide a convenient *authoring basis* that is independent of device geometry yet compiles exactly to Pauli space via the forward map (6).
2. The right inverse B furnishes an explicit, constructive proof that Pauli generators emerge from EchoKey operators and can be *recovered per site.*
3. Because single– and two–qubit unitaries formed from EchoKey operators equal their Pauli counterparts, we can safely inject EchoKey gates into compilers and rewrite them to native rotations without loss.

Appendix: Code–math correspondences

- `ek7_directions_from_cube` $\leftrightarrow$ construction of A rows $\boldsymbol{a}_k$.
- `make_echokey7_ops` $\leftrightarrow$ raw $E_k = a_0 I + \boldsymbol{a}_k \cdot \sigma$ then trace removal.
- `build_A_from_ops` $\leftrightarrow$ recover A and traceless E_k° by Pauli decomposition.
- `right_inverse` $\leftrightarrow$ formula (8) for B.
- `emergence_report` $\leftrightarrow$ computes S_i, orthonormality O, commutator residuals.
- `ek_single_qubit_unitary`, `pauli_single_qubit_unitary` $\leftrightarrow$ equality of single–qubit steps.
- `ek_two_qubit_xyz` $\leftrightarrow$ equality of two–qubit XYZ unitaries.
- `export_json` flag $\leftrightarrow$ per–site B weights for compiler injection.