8 Days of EchoKey — Day 3: Fractality

Z–Axis Fast Path and Layout–Aware ZYZ Synthesis

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Abstract

This Day 3 note introduces the *Fractality* generator in the EchoKey 7–operator frame and derives a compiler rewrite with a **Z–axis fast path**. When the local axis is (approximately) $\pm\hat{z}$, the symbolic gate ek_frac$(\theta) = e^{-i\theta(\mathbf{a}_3\cdot\boldsymbol{\sigma})}$ collapses to a single native rotation RZ$(\pm 2\theta)$. Otherwise we synthesize an exact ZYZ Euler product from the SU(2) matrix. The pass is *layout–aware*: per–wire axes are read from the physical mapping. We state the rule, prove correctness up to global phase, and give the fidelity metric used in verification.

1 Background and Notation

Let $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ be the Pauli vector and $\mathbf{A} \in \mathbb{R}^{7\times 3}$ the EchoKey frame whose unit–norm rows are $\mathbf{a}_k^\top$. As in Day 1 and Day 2, a traceless 1–qubit EchoKey Hamiltonian is

$$H_{\mathrm{EK}}(\mathbf{c}) = \sum_{k=1}^{7} c_k\, E_k^\circ, \qquad E_k^\circ := \mathbf{a}_k \cdot \boldsymbol{\sigma}, \qquad H_{\mathrm{EK}}(\mathbf{c}) = \boldsymbol{\alpha}\cdot\boldsymbol{\sigma} \text{ with } \boldsymbol{\alpha} = \mathbf{A}^\top \mathbf{c}. \tag{1}$$

We use 1–based indexing in the math ($\mathbf{a}_3$ is the third row), while the reference implementation uses 0–based (`A[2]`).

Day 3 choice (Fractality). Select $\mathbf{a}_3$ and define the gate

$$\text{ek\_frac}(\theta) \stackrel{\text{def}}{=} e^{-i\theta\,(\mathbf{a}_3\cdot\boldsymbol{\sigma})}. \tag{2}$$

2 Axis–Angle Form and Z–Axis Fast Path

Any $U \in \mathrm{SU}(2)$ admits the axis–angle parametrization

$$U(\varphi, \hat{\mathbf{n}}) = \cos\frac{\varphi}{2}\,\mathbb{I} - i\sin\frac{\varphi}{2}\,(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}), \qquad \hat{\mathbf{n}} \in \mathbb{S}^2. \tag{3}$$

Comparing with (2) yields $\hat{\mathbf{n}} = \mathbf{a}_3$ and $\varphi = 2\theta$. If $\mathbf{a}_3 = \pm\hat{z} = (0, 0, \pm 1)$ then

$$\text{ek\_frac}(\theta) = e^{-i\theta(\pm\sigma_z)} \;\doteq\; \mathrm{RZ}(\pm 2\theta), \tag{4}$$

where $\doteq$ denotes equality up to a global phase.

Numerical predicate (fast path). With unit axis $\hat{\mathbf{n}} = (n_x, n_y, n_z)$ and tolerance $\varepsilon > 0$, detect Z–alignment by

$$|n_x| < \varepsilon, \quad |n_y| < \varepsilon, \quad \big|\,|n_z| - 1\,\big| < \varepsilon. \tag{5}$$

If (5) holds, emit $\mathrm{RZ}(\mathrm{sign}(n_z)\, 2\theta)$. Otherwise, synthesize a ZYZ Euler product from $U(2\theta, \hat{\mathbf{n}})$.

3 ZYZ Euler Decomposition (General Case)

For arbitrary axis $\hat{\mathbf{n}}$, we form the exact matrix

$$U \equiv U(2\theta, \hat{\mathbf{n}}) = \cos(\theta)\, \mathbb{I} - i \sin(\theta)\, (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z), \tag{6}$$

then decompose it to ZYZ angles (α, β, γ) to obtain

$$\text{ek\_frac}(\theta) \;\doteq\; \mathrm{RZ}(\alpha)\, \mathrm{RY}(\beta)\, \mathrm{RZ}(\gamma). \tag{7}$$

Any consistent branch of Euler angles is acceptable since global phase is ignored.

4 Layout–Aware Axis Resolution

Let $\mathrm{phys} : \{\text{logical wires}\} \to \{0, \ldots, n-1\}$ be the placement mapping. Each physical wire p carries its own frame $\mathbf{A}^{(p)}$. The axis used for a gate placed on logical q is

$$\boxed{\hat{\mathbf{n}} = \mathbf{a}_3^{(\,\mathrm{phys}(q)\,)}, \qquad \varphi = 2\theta.} \tag{8}$$

Thus the rewrite should run *after* the layout is available (or read it from the pass property set).

5 Correctness

Each local replacement uses the exact SU(2) unitary $U(2\theta, \hat{\mathbf{n}})$ computed with the per–wire axis (8). Either the fast path (4) is applied, or the ZYZ factorization (7). Because ZYZ covers all of SU(2) (up to phase) and RZ is native, the circuit unitary is preserved up to a global phase. Composition of such replacements over the DAG preserves the full circuit unitary (up to global phase).

6 Validation Metric

For small n we compare unitaries before and after the rewrite with the phase–insensitive overlap

$$\mathcal{F}(U_{\mathrm{in}}, U_{\mathrm{out}}) = \frac{\left|\mathrm{Tr}\left(U_{\mathrm{in}}^{\dagger} U_{\mathrm{out}}\right)\right|}{2^n} \in [0, 1]. \tag{9}$$

Exact synthesis yields $\mathcal{F} \approx 1.000\,000\,000\,000$ in numerical tests.

7 Worked Examples

Ex 1: Z fast path: ek_frac(0.40) then H. With $\mathbf{a}_3 = \hat{z}$ the rewrite emits a single RZ(0.8).

Ex 2: 1q sequence: RX(0.11) ek_frac(-0.42) RY(0.23) ek_frac(0.80) RZ(-0.31). Each ek_frac rewrites independently (fast or general based on its axis).

Ex 3: 2q with entangler: H_0 ek_frac$^{(0)}$(0.50) $\mathrm{CX}_{0\to1}$ ek_frac$^{(1)}$(-0.25) $\mathrm{RY}^{(1)}$(0.40). Per–wire frames may tilt $\mathbf{a}_3^{(p)}$; the pass uses (8).

Ex 4: Multi–qubit mixed axes: give each site its own $\mathbf{A}^{(p)}$ and tilt every second $\mathbf{a}_3^{(p)}$ slightly off $\hat{z}$ to exercise both paths; insert nearest–neighbor CNOTs.

8 Edge Cases and Numerics

- **Degenerate/NaN axis:** if $\|\mathbf{a}_3\| \approx 0$ or contains NaNs, reject.
- **Tolerance ε:** choose ε conservatively (default 10^{-12} in the code) to avoid misclassifying near–Z axes; the general ZYZ path is exact and safe.
- **Branch cuts:** Euler angles are not unique; any consistent choice is phase–equivalent.
- **Ordering:** run after placement so that (8) uses *physical* indices.

Complexity. The pass is linear in the number of ek_frac gates. Z fast path takes $\mathcal{O}(1)$; ZYZ synthesis is $\mathcal{O}(1)$ for 2×2 matrices.

9 Repro Checklist

1. Choose per–wire frames $\{\mathbf{A}^{(p)}\}_{p=0}^{n-1}$ (unit rows).
2. Build the circuit with symbolic ek_frac(θ) gates.
3. Resolve $\hat{\mathbf{n}} = \mathbf{a}_3^{(\,\mathrm{phys}(q)\,)}$ and $\varphi = 2\theta$ for each gate.
4. If (5) holds, emit RZ($\mathrm{sign}(n_z)\, 2\theta$); else synthesize ZYZ from $U(2\theta, \hat{\mathbf{n}})$.
5. Validate with the fidelity metric; expect $\mathcal{F} \approx 1$.