Mathematical Walkthrough:
Exact-Stroke Correlation Engine with Acoustic Cooling

EchoKey Asks: Correlated Engine Demo

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Abstract

These notes give a detailed, equation-level walkthrough of the correlation-fueled engine/refrigerator realized by the accompanying Python implementation `ek_engine_acoustic.py`. All strokes are exact (piecewise-constant Hamiltonians with unitary propagators). We define heat/work at the cycle boundary, construct an entropic resource ledger, derive consistency checks, and relate resource spending to heat balances. Units: $\hbar = k_B = 1$.

0) Hilbert Space, Ordering, and Conventions

Composite space

$$\mathcal{H} = \mathcal{H}_{S_h} \otimes \mathcal{H}_{S_c} \otimes \mathcal{H}_{B_{h1}} \otimes \mathcal{H}_{B_{h2}} \otimes \mathcal{H}_M \otimes \mathcal{H}_D,$$

with dimensions $(2, 2, 2, 2, n_M, n_D)$ and ordering $(S_h, S_c, B_{h1}, B_{h2}, M, D)$. Partial trace: $\rho_X = \mathrm{Tr}_{\bar{X}}\, \rho$. Frequencies from CLI in GHz are converted internally to angular frequencies $\omega = 2\pi f$.

1) Bare Hamiltonians and Couplings (Embedded)

1.1 Bare terms

$$H_S = \tfrac{\omega_h}{2}\sigma_z^{(S_h)} + \tfrac{\omega_c}{2}\sigma_z^{(S_c)}, \tag{1}$$

$$H_{B_h} = \tfrac{\omega_{h1}}{2}\sigma_z^{(B_{h1})} + \tfrac{\omega_{h2}}{2}\sigma_z^{(B_{h2})}, \tag{2}$$

$$H_M = \omega_M\left(a_M^\dagger a_M + \tfrac{1}{2}\right), \qquad H_D = \omega_D\left(a_D^\dagger a_D + \tfrac{1}{2}\right), \tag{3}$$

$$H_{B_c} = H_M + H_D, \qquad H_0 = H_S + H_{B_h} + H_{B_c}. \tag{4}$$

1.2 Work, contact, and cooling couplings

Work channel (XY) on S_h–S_c:

$$H_g = \tfrac{1}{2}\left(\sigma_x^{(S_h)}\sigma_x^{(S_c)} + \sigma_y^{(S_h)}\sigma_y^{(S_c)}\right). \tag{5}$$

Contacts:

$$H_{SB}^{(h)} = \sigma_x^{(S_h)}\sigma_x^{(B_{h1})} + \sigma_x^{(S_h)}\sigma_x^{(B_{h2})}, \tag{6}$$

$$H_{SB}^{(c)} = \sigma_x^{(S_c)}\, X^{(M)}, \quad X^{(M)} = a_M + a_M^\dagger. \tag{7}$$

Cooling leg:

$$H_{\rm bs} = \sigma_-^{(S_c)} a_M^\dagger + \sigma_+^{(S_c)} a_M \quad \text{(beam-splitter)}, \tag{8}$$

$$H_{\rm md} = a_M a_D^\dagger + a_M^\dagger a_D \quad \text{(mode–mode)}. \tag{9}$$

2) Stroke Hamiltonians and Exact Evolution

With couplings $(\kappa_h, \kappa_c, g_{\rm on}, \chi, g_{\rm bs}, g_{\rm md})$ and durations $(t_{\rm th}, t_{\rm work}, t_{\rm cool}, t_{\rm rlx})$,

$$H_{\rm th} = H_0 + \kappa_h\, H_{SB}^{(h)} + \kappa_c\, H_{SB}^{(c)}, \tag{10}$$

$$H_{\rm work} = H_0 + g_{\rm on}\, H_g + \chi \left(H_{SB}^{(h)} + H_{SB}^{(c)} \right), \tag{11}$$

$$H_{\rm cool} = H_0 + g_{\rm bs} H_{\rm bs} + g_{\rm md} H_{\rm md}, \tag{12}$$

$$H_{\rm rlx} = H_0 + \tfrac{1}{2}\kappa_h\, H_{SB}^{(h)} + \tfrac{1}{2}\kappa_c\, H_{SB}^{(c)}. \tag{13}$$

Each stroke is unitary: $U = \exp(-iHt)$ via spectral decomposition.

One cycle from $\rho_{\rm in}$:

$$\rho_1 = U_{\rm th}\rho_{\rm in}U_{\rm th}^\dagger, \quad U_{\rm th} = e^{-iH_{\rm th}t_{\rm th}}, \tag{14}$$

$$\rho_2 = U_{\rm work}\rho_1 U_{\rm work}^\dagger, \quad U_{\rm work} = e^{-iH_{\rm work}t_{\rm work}}, \tag{15}$$

$$\rho_3 = U_{\rm cool}\rho_2 U_{\rm cool}^\dagger, \quad U_{\rm cool} = e^{-iH_{\rm cool}t_{\rm cool}}, \tag{16}$$

$$\rho_4 = U_{\rm rlx}\rho_3 U_{\rm rlx}^\dagger, \quad U_{\rm rlx} = e^{-iH_{\rm rlx}t_{\rm rlx}}. \tag{17}$$

Set $\rho_{\rm out} = \rho_4$ (faithful *paper* path). The optional *standard* control replaces $\rho_{\rm out} \mapsto \rho_{S_hS_c} \otimes \rho_{B_h} \otimes \rho_{B_c}$ at the boundary (non-unitary).

3) Heat and Work (Cycle Boundary)

3.1 Heat definitions

$$Q_h = -\Delta\langle H_{B_h}\rangle = -\big(\mathrm{Tr}[\rho_{\rm out}H_{B_h}] - \mathrm{Tr}[\rho_{\rm in}H_{B_h}]\big), \quad \text{(+ means energy from hot to device)}, \tag{18}$$

$$Q_c = +\Delta\langle H_{B_c}\rangle = Q_M + Q_D, \tag{19}$$

$$Q_M = +\Delta\langle H_M\rangle\,, \qquad Q_D = +\Delta\langle H_D\rangle\,. \tag{20}$$

$Q_M < 0$ indicates refrigeration of the mechanical mode.

3.2 Quench work identity (exact)

At a sudden change $H_{\rm before} \to H_{\rm after}$ the state is unchanged instantaneously, so

$$W_q = \mathrm{Tr}\big[\rho\,(H_{\rm after} - H_{\rm before})\big]. \tag{21}$$

For our four boundaries:

$$W_q^{(1)} = \mathrm{Tr}[\rho_1(H_{\rm work} - H_{\rm th})], \quad W_q^{(2)} = \mathrm{Tr}[\rho_2(H_{\rm cool} - H_{\rm work})], \tag{22}$$

$$W_q^{(3)} = \mathrm{Tr}[\rho_3(H_{\rm rlx} - H_{\rm cool})], \quad W_q^{(4)} = \mathrm{Tr}[\rho_4(H_{\rm th} - H_{\rm rlx})]. \tag{23}$$

Total work done *on* the device: $W_{\rm on} = \sum_k W_q^{(k)}$. We report

$$W_{\rm out} = -W_{\rm on}. \tag{24}$$

Optional reset charge (fair control comparison). If the control decorrelates at temperature T_{reset}, the minimum erasure work is

$$W_{\text{reset}} \;\geq\; T_{\text{reset}}\big(\sigma_{\text{pre}} - \sigma_{\text{post}}\big), \tag{25}$$

and we include it via $W_{\text{on}} \mapsto W_{\text{on}} + W_{\text{reset}}$.

3.3 Consistency checks

Evaluated at H_{th}:

$$\underbrace{\big(\langle H_{\text{th}}\rangle_f - \langle H_{\text{th}}\rangle_i\big) + W_{\text{out}}}_{\text{same-}H\text{ ledger}} \approx 0, \tag{26}$$

$$\underbrace{W_{\text{out}} - (Q_h - Q_c) + (\Delta E_S + \Delta E_{\text{int}})}_{\text{split ledger}} \approx 0, \tag{27}$$

where $H_{\text{int}} = H_{\text{th}} - H_S - H_{B_h} - H_{B_c}$, $\Delta E_S = \Delta\langle H_S\rangle$, and $\Delta E_{\text{int}} = \Delta\langle H_{\text{int}}\rangle$. In the faithful path, both residuals are at machine precision in simulation.

4) Entropic Resource Ledger

Let $R = B_{h1}B_{h2}MD$ and $B_c = MD$. Define

$$I(S_hS_c : R) = S(\rho_{S_hS_c}) + S(\rho_R) - S(\rho), \tag{28}$$

$$D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]. \tag{29}$$

With thermal references $\rho^{\text{th}}_{B_h}$ at T_h and $\rho^{\text{th}}_{B_c}$ at T_c,

$$\sigma \equiv I(S_hS_c : R) + D\Big(\rho_{B_h}\Big\|\rho^{\text{th}}_{B_h}\Big) + D\Big(\rho_{B_c}\Big\|\rho^{\text{th}}_{B_c}\Big), \qquad \Delta\sigma = \sigma_{\text{out}} - \sigma_{\text{in}}. \tag{30}$$

$\Delta\sigma < 0$ flags *resource consumption* in that cycle.

5) Linking $\Delta\sigma$ to Heat/Entropy Balances

For a fixed bath Hamiltonian H_B and Gibbs state $\tau_B = \mathrm{e}^{-\beta_B H_B}/Z_B$:

$$D(\rho_B\|\tau_B) = \beta_B\ \mathrm{Tr}[\rho_B H_B] - \log Z_B - S(\rho_B) \quad\Rightarrow\quad \Delta D_B = \beta_B\,\Delta E_B - \Delta S_B. \tag{31}$$

With our heat signs ($Q_h = -\Delta E_{B_h}$, $Q_c = +\Delta E_{B_c}$),

$$\beta_h Q_h + \beta_c Q_c = -\big(\Delta S_{B_h} + \Delta S_{B_c}\big) - \big(\Delta D_{B_h} + \Delta D_{B_c}\big). \tag{32}$$

Global unitary dynamics conserve $S(\rho)$, so

$$\Delta I(S_hS_c : R) = \Delta S_{S_hS_c} + \Delta S_R. \tag{33}$$

Approximating $\Delta S_R \simeq \Delta S_{B_h} + \Delta S_{B_c}$ (bath-internal correlations absorbed in the resource term) gives

$$\boxed{\Delta\sigma \;\simeq\; \Delta S_{S_hS_c} \;-\; \beta_h Q_h \;-\; \beta_c Q_c} \tag{$\star$}$$

Hence, with no net resource spending ($\Delta\sigma \geq 0$) the Clausius-like inequality is recovered; when $\Delta\sigma < 0$ a cycle can transiently offset the heat-entropy ledger by consuming correlations/athermality.

6) First Law and Efficiency

At the boundary (same reference $H_{\rm th}$):

$$\Delta E_{\rm tot} + W_{\rm out} = 0, \qquad \Delta E_{\rm tot} = \Delta \langle H_{\rm th} \rangle . \tag{34}$$

Split form:

$$W_{\rm out} = Q_h - Q_c - \big(\Delta E_S + \Delta E_{\rm int}\big). \tag{35}$$

In steady operation, $\Delta E_S + \Delta E_{\rm int} \approx 0$ and $W_{\rm out} \approx Q_h - Q_c$.

Efficiency (engine mode only).

$$\eta = \frac{W_{\rm out}}{\max(Q_h, 0)} \quad \text{if } Q_h > 0 \text{ and } W_{\rm out} > 0, \qquad \eta_C = 1 - \frac{T_c}{T_h}. \tag{36}$$

We print "n/a" outside genuine engine mode to avoid ill-conditioned ratios when Q_h is tiny.

7) Modes of Operation (Sign Logic)

Using $(Q_M, W_{\rm out}, \Delta\sigma)$:

- **Engine-burst:** $W_{\rm out} > 0$ and $\Delta\sigma < 0$ (resource was spent to deliver work).
- **Refrigerator:** $Q_M < 0$ and $W_{\rm out} < 0$ (work invested to cool M; typically $\Delta\sigma \geq 0$).
- **Heat-dump:** $Q_D \gg 0$, $Q_M \approx 0$, $W_{\rm out} < 0$ (energy pumped mainly to the dump).

8) Thermal References and Exact Quantities

Thermal state on (H, β): $\rho^{\rm th} = \mathrm{e}^{-\beta H}/Z$, $Z = \mathrm{Tr}\big[\mathrm{e}^{-\beta H}\big]$. Von Neumann entropy: $S(\rho) = -\mathrm{Tr}[\rho \log \rho]$. Relative entropy: $D(\rho \| \sigma) = \mathrm{Tr}[\rho(\log \rho - \log \sigma)]$. Mutual information: $I(S:R) = S(\rho_S) + S(\rho_R) - S(\rho_{SR})$.

9) Why All Mechanical Work is Quench Work Here

Within a stroke $\dot{H} = 0$ and $\dot{\rho} = -i[H, \rho]$ gives $\mathrm{d}\langle H \rangle/\mathrm{d}t = \mathrm{Tr}[\dot{\rho} H] + \mathrm{Tr}\Big[\rho \dot{H}\Big] = 0$. Thus internal stroke evolution contributes no external work; all mechanical work is at parameter quenches, exactly the W_q terms.

10) Resource–Heat Inequality and Bounds (Intuition)

From $(\star)$:

$$\Delta S_{S_h S_c} - \beta_h Q_h - \beta_c Q_c \;\approx\; \Delta\sigma. \tag{37}$$

If $\Delta\sigma \geq 0$, a Clausius-like bound holds; if $\Delta\sigma < 0$, the resource offsets the ledger and can transiently enable $W_{\rm out} > 0$ or enhanced cooling. Sustained super-Carnot averages require sustained resource spending and replenishment by nonthermal means.

11) Map to Experiment (Symbols $\leftrightarrow$ Knobs)

- H_{work}: XY exchange via tunable coupler or CR; tune g_{on}, t_{work}. Small χ during work allows $I(S_hS_c\!:\!R)$ to *drop* (spend resource) where it converts to W_{out}.
- H_{cool}: red-sideband $\mathrm{S}_c \leftrightarrow M$ sets g_{bs}; parametric bridge $M \leftrightarrow D$ sets g_{md}. Larger values and longer t_{cool} drive refrigeration ($Q_M < 0$, $W_{\text{out}} < 0$).
- $H_{\text{th}}, H_{\text{rlx}}$: soft contacts to build correlations and settle; overly strong contacts wash out coherence; overly weak slow resource charging.

12) Parameter Effects (Qualitative)

- Increase g_{on}, t_{work}, $|\chi| \Rightarrow$ higher chance of *athermal burst* ($W_{\text{out}} > 0$, $\Delta\sigma < 0$).
- Increase g_{bs}, g_{md}, $t_{\text{cool}} \Rightarrow$ stronger refrigeration ($Q_M < 0$, $n_M \downarrow$), typically $W_{\text{out}} < 0$.
- Long cycles + strong cooling $\Rightarrow$ commonly $\Delta\sigma > 0$ per cycle (resource charging).

13) Efficiency Guard (Why Sometimes "n/a")

Reporting $\eta = W_{\text{out}}/Q_h$ is ill-conditioned when Q_h is tiny; we only report η when $Q_h > 0$ and $W_{\text{out}} > 0$.

14) Optional Control and Fairness

The diagnostic control performs $\rho \mapsto \rho_{S_hS_c} \otimes \rho_{B_h} \otimes \rho_{B_c}$ at the boundary. This is non-unitary and not free. To compare fairly, charge the minimal erasure work $W_{\text{reset}} \geq T_{\text{reset}}(\sigma_{\text{pre}} - \sigma_{\text{post}})$ (Landauer), and, if desired, include the state-change work at the same reference Hamiltonian to close the ledger.

15) What to Check in Data

Per cycle:

- Energy checks $\sim 10^{-13}$ (simulation) / within error bars (experiment).
- **Engine-burst:** $W_{\text{out}} > 0$ and $\Delta\sigma < 0$.
- **Refrigeration:** $Q_M < 0$, n_M decreases, typically $W_{\text{out}} < 0$.
- Persist logs of $Q_h, Q_M, Q_D, Q_c, W_{\text{out}}, I(S_hS_c\!:\!R), D_{B_h}, D_{B_c}, n_M$ with calibration metadata.

16) Minimal Identities Used

- **Relative-entropy identity:** $D(\rho\|\tau_\beta) = \beta\,\mathrm{Tr}[\rho H] - \log Z - S(\rho)$, hence $\Delta D = \beta\,\Delta E - \Delta S$.
- **Mutual information under global unitary:** $\Delta I(S:R) = \Delta S_S + \Delta S_R$ (since S_{SR} is constant).
- **Sudden quench work:** $W_q = \mathrm{Tr}[\rho\,(H_{\text{after}} - H_{\text{before}})]$.

Companion files: `ek_engine_acoustic.py` (exact-stroke simulator) and `ek_engine_acoustic_experiment.tex` (build/measurement guide).