From Gunslinger Continuum to Navier–Stokes: A Full Nondimensional Derivation via the EchoKey Transform

CC0 Notes

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Figure 1: Steady Hand

Abstract

We construct a parent *Gunslinger Continuum* (GC) for anisotropic, intent-modulated flow with alignment and readiness fields. We carry out a full nondimensionalization, identify the governing dimensionless groups, and prove positive-definite dissipation. Finally, we define an EchoKey operator transform that collapses the GC to the isotropic, memoryless Navier–Stokes (NS) equations as a symmetry-restored limit.

1 Field Content and Kinematics

Let $\rho(x,t) > 0$ be mass density, $\boldsymbol{u}(x,t) \in \mathbb{R}^3$ the velocity, $\boldsymbol{a}(x,t) \in \mathbb{R}^3$ a unit alignment field with $|\boldsymbol{a}| = 1$, and $\chi(x,t) \in [0,1]$ a readiness (suppression) scalar. Define

$$\boldsymbol{P}_\perp := \boldsymbol{I} - \boldsymbol{a}\boldsymbol{a}^\top, \qquad \boldsymbol{S} := \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top), \qquad \boldsymbol{\Omega} := \frac{1}{2}(\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^\top). \tag{1}$$

The constitutive viscosity tensor is

$$\boldsymbol{\mu}(\boldsymbol{a}) = \mu_\perp \boldsymbol{I} + (\mu_\parallel - \mu_\perp)\boldsymbol{a}\boldsymbol{a}^\top, \qquad \mu_\parallel, \mu_\perp > 0. \tag{2}$$

2 Balance Laws (Dimensional)

Mass conservation:

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0. \tag{3}$$

Momentum balance with anisotropic viscosity and readiness drag:

$$\rho(\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) = -\nabla p + \nabla \cdot \big(2\boldsymbol{\mu}(\boldsymbol{a})\boldsymbol{S}\big) - \kappa\chi \boldsymbol{P}_\perp \boldsymbol{u} + \boldsymbol{f}. \tag{4}$$

Alignment transport (frame-indifferent Jeffery/Leslie-like evolution):

$$\partial_t \boldsymbol{a} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{a} = \boldsymbol{\Omega}\boldsymbol{a} + \alpha \boldsymbol{P}_\perp(\boldsymbol{S}\boldsymbol{a}) - \frac{1}{\tau_a}\boldsymbol{P}_\perp \nabla\Phi(x,t,\rho), \qquad |\boldsymbol{a}| = 1. \tag{5}$$

Readiness advection–relaxation–diffusion:

$$\partial_t \chi + (\boldsymbol{u} \cdot \nabla)\chi = -\frac{\chi - \chi_0(x,t)}{\tau_\chi} + D_\chi \nabla^2 \chi. \tag{6}$$

Boundary data. On a solid boundary $\partial\Omega$ with outward normal $\boldsymbol{n}$: no-penetration/no-slip $\boldsymbol{u} = \boldsymbol{u}_b(t)$; natural choices for $\boldsymbol{a}$ include anchored ($\boldsymbol{a} = \boldsymbol{a}_b$) or free-rotating ($\boldsymbol{n} \cdot \nabla\boldsymbol{a} = 0$). For χ: Dirichlet $\chi = \chi_b$ or Neumann $\boldsymbol{n} \cdot \nabla\chi = 0$.

3 Energetics and Dissipation

Define kinetic and readiness functionals:

$$\mathcal{K} = \int_\Omega \frac{1}{2}\rho|\boldsymbol{u}|^2\,dx, \qquad \mathcal{V}_{\text{ready}} = \int_\Omega \frac{\kappa}{2}\chi|\boldsymbol{P}_\perp\boldsymbol{u}|^2\,dx. \tag{7}$$

Proposition 1 (Positive Dissipation)**.** Assuming $\mu_\parallel, \mu_\perp, \kappa \geq 0$ and compatible boundary work, the total dissipation rate is

$$\mathcal{D} = \int_\Omega 2\boldsymbol{S} : \boldsymbol{\mu}(\boldsymbol{a})\boldsymbol{S}\,dx + \int_\Omega \kappa\chi|\boldsymbol{P}_\perp\boldsymbol{u}|^2\,dx \geq 0. \tag{8}$$

Sketch. Multiply (4) by $\boldsymbol{u}$, integrate by parts, use $\boldsymbol{S} : \boldsymbol{\mu}\boldsymbol{S} = \operatorname{tr}(\boldsymbol{S}^\top\boldsymbol{\mu}\boldsymbol{S}) \geq 0$ since $\boldsymbol{\mu}$ is SPD for the stated parameters, and $\boldsymbol{P}_\perp$ is an orthogonal projector, so $\boldsymbol{u} \cdot \boldsymbol{P}_\perp\boldsymbol{u} = |\boldsymbol{P}_\perp\boldsymbol{u}|^2 \geq 0$. □

4 Scaling and Nondimensionalization

Choose characteristic scales: length L, speed U, time $T = L/U$, density ρ_0, viscosity $\mu_\perp$, readiness baseline $\chi_* \in (0,1]$, and potential scale Φ_*. Define nondimensional variables (denoted by $\hat{\cdot}$):

$$x = L\hat{x}, \quad t = T\hat{t}, \quad \boldsymbol{u} = U\hat{\boldsymbol{u}}, \quad \rho = \rho_0\hat{\rho}, \quad p = \rho_0 U^2 \hat{p}, \quad \chi = \chi_*\hat{\chi}, \quad \Phi = \Phi_*\hat{\Phi}. \tag{9}$$

The alignment $\boldsymbol{a}$ is already unitless. Using $\nabla = (1/L)\hat{\nabla}$, $\boldsymbol{S} = (U/L)\hat{\boldsymbol{S}}$, and $\nabla\cdot = (1/L)\hat{\nabla}\cdot$, substitute into (3)–(6).

Dimensionless Mass

$$\partial_{\hat{t}}\hat{\rho} + \hat{\nabla}\cdot(\hat{\rho}\hat{\boldsymbol{u}}) = 0. \tag{10}$$

Dimensionless Momentum

The nondimensional viscosity tensor is $\hat{\boldsymbol{\mu}}(\boldsymbol{a}) = \boldsymbol{I} + \mathrm{An}\,\boldsymbol{a}\boldsymbol{a}^\top$, with the anisotropy number $\mathrm{An} := (\mu_\parallel - \mu_\perp)/\mu_\perp$. This leads to:

$$\hat{\rho}\left(\partial_{\hat{t}}\hat{\boldsymbol{u}} + (\hat{\boldsymbol{u}}\cdot\hat{\nabla})\hat{\boldsymbol{u}}\right) = -\hat{\nabla}\hat{p} + \frac{1}{\mathrm{Re}}\hat{\nabla}\cdot\big(2\hat{\boldsymbol{\mu}}(\boldsymbol{a})\hat{\boldsymbol{S}}\big) - \mathrm{Rd}\,\hat{\chi}\boldsymbol{P}_\perp\hat{\boldsymbol{u}} + \hat{\boldsymbol{f}}, \tag{11}$$

where the dimensionless groups are the Reynolds number, Readiness number, and scaled force:

$$\mathrm{Re} := \frac{\rho_0 U L}{\mu_\perp}, \qquad \mathrm{Rd} := \frac{\kappa\chi_* L}{\rho_0 U}, \qquad \hat{\boldsymbol{f}} := \frac{L}{\rho_0 U^2}\boldsymbol{f}. \tag{12}$$

Dimensionless Alignment

$$\partial_{\hat{t}}\boldsymbol{a} + (\hat{\boldsymbol{u}}\cdot\hat{\nabla})\boldsymbol{a} = \hat{\boldsymbol{\Omega}}\boldsymbol{a} + \alpha\boldsymbol{P}_\perp(\hat{\boldsymbol{S}}\boldsymbol{a}) - \frac{1}{\mathrm{Me}_a}\boldsymbol{P}_\perp\hat{\nabla}\hat{\Phi}, \tag{13}$$

with the memory number $\mathrm{Me}_a := U\tau_a/L$ and the flow-alignment coupling α.

Dimensionless Readiness

$$\partial_{\hat{t}}\hat{\chi} + (\hat{\boldsymbol{u}}\cdot\hat{\nabla})\hat{\chi} = -\frac{1}{\mathrm{Me}_\chi}(\hat{\chi} - \hat{\chi}_0) + \frac{1}{\mathrm{Pe}_\chi}\hat{\nabla}^2\hat{\chi}, \tag{14}$$

where $\mathrm{Me}_\chi := U\tau_\chi/L$ and the Péclet number is $\mathrm{Pe}_\chi := UL/D_\chi$.

Dimensionless Summary

The GC equations depend on the set of dimensionless groups:

$$\boxed{\mathrm{Re}, \mathrm{An}, \mathrm{Rd}, \alpha, \mathrm{Me}_a, \mathrm{Me}_\chi, \mathrm{Pe}_\chi} \tag{15}$$

plus any prescribed sources $\hat{\boldsymbol{f}}$ and $\hat{\chi}_0$.

5 EchoKey Transform $\mathcal{T}_{EK}$ and the NS Limit

We formalize the reduction as a composition of operators $\mathcal{T}_{EK} := \mathcal{A}\circ\mathcal{F}\circ\mathcal{O}\circ\mathcal{I}\circ\mathcal{R}\circ\mathcal{C}$ on fields and parameters:

$\mathcal{C}$ **(Cycle Average)** Average over a characteristic period: $(\cdot) \mapsto \langle\cdot\rangle_T$.

$\mathcal{R}$ **(Recursion/Fixed-Point)** Evolve on slow manifolds: $\partial_t \mapsto 0$.

$\mathcal{I}$ **(Isotropization)** Remove directional dependence: $\hat{\boldsymbol{\mu}}(\boldsymbol{a}) \mapsto \hat{\mu}\boldsymbol{I}$.

$\mathcal{O}$ **(Outlier Removal)** Replace anisotropic spikes with bulk averages.

$\mathcal{F}$ **(Fractality Renormalization)** Model subgrid effects as an eddy viscosity ν_t. (Adaptivity Quench)] Neutralize active fields: $(\hat{\chi}, \boldsymbol{a}) \mapsto (0, \mathrm{const})$.

Applying this transform to the GC system yields the steady, incompressible Navier–Stokes equations:

$$\hat{\rho}\big((\hat{\boldsymbol{u}}\cdot\hat{\nabla})\hat{\boldsymbol{u}}\big) = -\hat{\nabla}\hat{p} + \frac{1}{\mathrm{Re}_{\mathrm{eff}}}\hat{\nabla}^2\hat{\boldsymbol{u}} + \hat{\boldsymbol{f}}, \qquad \hat{\nabla}\cdot\hat{\boldsymbol{u}} = 0, \tag{16}$$

where $\mathrm{Re}_{\mathrm{eff}}$ absorbs molecular and eddy viscosities. In dimensional variables:

$$\rho(\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) = -\nabla p + \mu_{\mathrm{eff}}\nabla^2\boldsymbol{u} + \boldsymbol{f}, \qquad \nabla\cdot\boldsymbol{u} = 0. \tag{17}$$

6 Symmetries and Invariance

- **Galilean Invariance:** GC terms depend on relative velocities; $\boldsymbol{P}_\perp$ is frame-indifferent.
- **Objectivity:** The use of $\boldsymbol{\Omega}$ and $\boldsymbol{S}$ in the $\boldsymbol{a}$-transport equation ensures frame indifference.
- **Energy:** Dissipation is positive definite (Prop. 1).

7 Diagnostics and Regimes

Define a domain-averaged gunslinger index, $0 \le \mathrm{GI} \le 1$:

$$\mathrm{GI} = \left\langle \frac{|\boldsymbol{P}_\perp \hat{\boldsymbol{u}}|}{|\hat{\boldsymbol{u}}| + \varepsilon} \right\rangle_\Omega . \tag{18}$$

NS-like Regime: $\mathrm{An} \to 0, \mathrm{Rd} \to 0, \mathrm{Me}_a, \mathrm{Me}_\chi \to 0$, which implies $\mathrm{GI} \to 0$.

Gunslinger Regime: $\mathrm{An} \sim O(1), \mathrm{Rd} \sim O(1 - 10^2), \mathrm{Me}_a, \mathrm{Me}_\chi \sim O(1)$, which implies $\mathrm{GI} \sim O(1)$.

8 Worked Boundary-Layer Estimate

Consider a 2D shear flow with $\boldsymbol{a} = \boldsymbol{e}_x$, steady $\hat{\boldsymbol{u}} = (u(y), 0)$, and constant $\hat{\chi} = \bar{\chi}$. The momentum equation (11) reduces to a balance between the pressure gradient and viscous stress:

$$0 = -\frac{d\hat{p}}{dx} + \frac{1}{\mathrm{Re}} \frac{d}{dy} \left(2(1 + \mathrm{An}) \frac{du}{dy} \right) . \tag{19}$$

The term $-\mathrm{Rd}\, \bar{\chi} \boldsymbol{P}_\perp \hat{\boldsymbol{u}}$ vanishes because $\boldsymbol{P}_\perp \hat{\boldsymbol{u}} = (\boldsymbol{I} - \boldsymbol{e}_x \boldsymbol{e}_x^\top)(u(y), 0)^\top = \boldsymbol{0}$. This shows the effective viscosity is $(1 + \mathrm{An})/\mathrm{Re}$ and clarifies how to distinguish the effects of An versus Rd.

Three-Dimensional Extension and Hypothetical Framing

Full 3D GC Equations

Let $\boldsymbol{u} = (u, v, w)$ and $\boldsymbol{a} \in \mathbb{S}^2$. With $\boldsymbol{P}_\perp = \boldsymbol{I} - \boldsymbol{a}\boldsymbol{a}^\top \in \mathbb{R}^{3\times 3}$, $\boldsymbol{S} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top)$, $\boldsymbol{\Omega} = \frac{1}{2}(\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^\top)$, and $\boldsymbol{\mu}(\boldsymbol{a}) = \mu_\perp \boldsymbol{I} + (\mu_\parallel - \mu_\perp)\, \boldsymbol{a}\boldsymbol{a}^\top$, the dimensional GC system in 3D is

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{20}$$

$$\rho(\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u}) = -\nabla p + \nabla \cdot (2 \boldsymbol{\mu}(\boldsymbol{a}) \boldsymbol{S}) - \kappa\, \chi\, \boldsymbol{P}_\perp\, \boldsymbol{u} + f, \tag{21}$$

$$\partial_t \boldsymbol{a} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{a} = \boldsymbol{\Omega}\, \boldsymbol{a} + \alpha\, \boldsymbol{P}_\perp (\boldsymbol{S}\, \boldsymbol{a}) - \tfrac{1}{\tau_a}\, \boldsymbol{P}_\perp \nabla \Phi, \qquad |\boldsymbol{a}| = 1, \tag{22}$$

$$\partial_t \chi + (\boldsymbol{u} \cdot \nabla) \chi = -\tfrac{\chi - \chi_0}{\tau_\chi} + D_\chi \nabla^2 \chi. \tag{23}$$

All nondimensional groups and the dissipation proof carry verbatim to 3D with $\boldsymbol{S} : \boldsymbol{\mu}\, \boldsymbol{S} \ge 0$ and $\boldsymbol{u} \cdot \boldsymbol{P}_\perp \boldsymbol{u} = |\boldsymbol{P}_\perp \boldsymbol{u}|^2 \ge 0$.

2D Slice as a Controlled Simplification

The 2D solver is a z-invariant slice with $w \equiv 0$, in-plane constant $\boldsymbol{a}$, and periodic boundaries. It is didactic; the parent theory is inherently 3D.

Hypothetical Relation to 3D NS Smoothness

We do *not* claim a proof of the Clay NS problem. We outline a program: if the GC system is globally well-posed in 3D for admissible data and bounded parameters, then under the EchoKey transform $\mathcal{T}_{EK}$ (isotropize, quench readiness, collapse memories) the isotropic, memoryless limit inherits well-posedness:

$$\text{GC well-posed} \quad \xRightarrow{\mathcal{T}_{EK}} \quad \text{NS well-posed (limit)}.$$

This requires (i) GC well-posedness with structural coercivity, (ii) uniform a-priori estimates independent of $(\mathrm{An}, \mathrm{Rd}, \mathrm{Me}_a, \mathrm{Me}_\chi)$ near zero, and (iii) a stable limit passage. Our numerics illustrate parameter-to-solution continuity in simplified settings.

A Appendix: Detailed Nondimensional Steps

The terms in the momentum equation (4) scale as follows:

$$\begin{aligned}
\text{Inertia:} \quad & \rho(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) \to \frac{\rho_0 U^2}{L}\,\hat\rho(\partial_{\hat t}\hat{\boldsymbol{u}} + (\hat{\boldsymbol{u}}\cdot\hat\nabla)\hat{\boldsymbol{u}}) \\
\text{Pressure:} \quad & -\nabla p \to -\frac{\rho_0 U^2}{L}\,\hat\nabla\hat p \\
\text{Viscous:} \quad & \nabla\cdot(2\boldsymbol{\mu}\boldsymbol{S}) \to \frac{\mu_\perp U}{L^2}\,\hat\nabla\cdot\big(2\hat{\boldsymbol{\mu}}\hat{\boldsymbol{S}}\big) = \frac{\rho_0 U^2}{L}\frac{1}{\mathrm{Re}}\,\hat\nabla\cdot\big(2\hat{\boldsymbol{\mu}}\hat{\boldsymbol{S}}\big) \\
\text{Readiness:} \quad & -\kappa\chi\boldsymbol{P}_\perp\boldsymbol{u} \to -\kappa\chi_* U\,\hat\chi\boldsymbol{P}_\perp\hat{\boldsymbol{u}} = -\frac{\rho_0 U^2}{L}\underbrace{\frac{\kappa\chi_* L}{\rho_0 U}}_{\mathrm{Rd}}\,\hat\chi\boldsymbol{P}_\perp\hat{\boldsymbol{u}}
\end{aligned}$$

Dividing by the inertial scale $\rho_0 U^2/L$ yields the dimensionless equation (11).

B Appendix: Alternative Closures

The readiness penalty may be absorbed into an effective eddy-viscosity tensor via fractality renormalization ($\mathcal{F}$):

$$-\mathrm{Rd}\,\hat\chi\boldsymbol{P}_\perp\hat{\boldsymbol{u}} \quad \rightsquigarrow \quad \frac{1}{\mathrm{Re}}\hat\nabla\cdot\big(2\hat{\boldsymbol{\mu}}_{\mathrm{eff}}(\boldsymbol{a},\hat\chi)\hat{\boldsymbol{S}}\big), \tag{24}$$

where $\hat{\boldsymbol{\mu}}_{\mathrm{eff}} = \hat{\boldsymbol{\mu}} + \nu_t(\boldsymbol{a},\hat\chi)\boldsymbol{I}$. Under further operators, this collapses to a scalar μ_{eff}.

C Appendix: Discrete Observables

Given tracked angles $\theta_L(t), \theta_R(t)$, we can define empirical observables:

$$\begin{aligned}
A_i &:= \text{RMS swing amplitude} \\
\Delta\phi &:= \text{Phase lag} \\
\mathrm{GI}_{\mathrm{obs}} &:= 1 - \frac{A_R}{A_L} + \lambda\left(1 - \frac{\overline{|\dot\theta_R|}}{\overline{|\dot\theta_L|}}\right)
\end{aligned}$$

These can be mapped to the model parameters $(\mathrm{An}, \mathrm{Rd})$ by fitting the GC model to data via least squares.

D Conclusion

The Gunslinger Continuum elevates alignment and readiness to first-class continuum variables. The classic Navier–Stokes equations emerge as the EchoKey-reduced, symmetry-restored image of the GC when anisotropy and readiness are neutralized. The dimensionless set $(\mathrm{Re}, \mathrm{An}, \mathrm{Rd}, \alpha, \mathrm{Me}_a, \mathrm{Me}_\chi, \mathrm{Pe}_\chi)$ fully parametrizes the regimes of fluid behavior, from classical isotropic flow to strongly asymmetric, adaptive "gunslinger" dynamics.