EchoKey Equilibriumization of Leader–Follower Flocks

(Mathematical breakdown & physical reinterpretation)

1 Problem Setting

We observe trajectories of N birds: for bird $i \in \{1, \ldots, N\}$, at times t_k, positions $\mathbf{r}_i(t_k)$, velocities $\mathbf{v}_i(t_k)$ obtained after converting GPS lat/lon to a local tangent plane and resampling to a uniform grid with spacing Δt. Let $s_i = \|\mathbf{v}_i\|$ and $\hat{\mathbf{v}}_i = \mathbf{v}_i/\|\mathbf{v}_i\|$. Define the instantaneous group axis $\mathbf{e}_{\text{front}}(t) = \frac{1}{\|\sum_i \mathbf{v}_i\|} \sum_i \mathbf{v}_i$.

Our goal is to construct a change of variables and a time reparameterization such that the *pushforward path measure* of the flock becomes an **equilibrium Markov chain** obeying detailed balance with respect to a Gibbs distribution. The leader–follower hierarchy must then appear as a gradient structure of an effective free energy.

2 EchoKey State and Operators

We build a multi-channel state $\Psi(t) \in \mathbb{R}^d$ from seven operator families. Concrete instantiations below match the accompanying code and can be swapped if they preserve the pipeline and witnesses.

2.1 Cyclicity $\mathcal{C}$

Definition. Headings $\theta_i(t) = \operatorname{atan2}(\hat{\mathbf{v}}_i \cdot \mathbf{e}_y, \hat{\mathbf{v}}_i \cdot \mathbf{e}_x)$, instantaneous angular speeds $\omega_i = \dot{\theta}_i$. A scale-separated phase $\varphi^{(\ell)}$ may be obtained via bandpass/wavelets.

Physical reinterpretation. Captures intrinsic turning cycles and cadence; provides a local "clock" used for time rescaling.

2.2 Recursion $\mathcal{R}$

Definition. Fit a one-step predictor $\widehat{\Psi}(t + \Delta t) = \Phi(\Psi(t))$ (linear VAR in our default). Define the recursion residual $\varepsilon_{\mathcal{R}}(t) = \Psi(t + \Delta t) - \Phi(\Psi(t))$.

Physical reinterpretation. Measures forecastable vs. novel state change; acts as a stabilizer/penalty in the energy.

2.3 Fractality $\mathcal{F}$

Definition. For each channel j, compute multi-scale variance across windows $w \in \mathcal{W}$ and estimate a log–log slope s_j; form a channel-wise gradient proxy $\eta_j = \frac{s_j - \bar{s}}{\sigma_s}$ (clipped). Let $\eta(\Psi) \in \mathbb{R}^d$ be broadcast across time.

Physical reinterpretation. Encodes roughness across scales; its gradient defines preferred directions for scale-wise deformation (refraction).

2.4 Regression $\mathcal{G}$

Definition. Bind macrostats (e.g. polarization $m = \|\frac{1}{N} \sum_i \hat{\mathbf{v}}_i\|$ or front alignment) to state via a regressor; residual $\varepsilon_{\mathcal{G}}$ becomes a soft constraint.

Physical reinterpretation. Connects coarse observables to micro-coordinates without presupposing the dynamics that generate them.

2.5 Synergy $\mathcal{S}$ (adaptive coupler)

Definition. Let

$$\kappa_{ij}(t) = \sigma\big(\beta_0 - \beta_1 \|\mathbf{r}_i - \mathbf{r}_j\| - \beta_2 \operatorname{ang}(\theta_i, \theta_j)\big), \quad \mathcal{S}(\Psi) = \sum_{i \neq j} \kappa_{ij}\, g_i(\Psi_i)\, g_j(\Psi_j),$$

with $\sigma(z) = (1 + e^{-z})^{-1}$ and ang $\in [0, \pi]$ the heading difference.

Physical reinterpretation. A metric-topological coupling: nearby, similarly oriented birds interact more strongly.

2.6 Refraction $\mathcal{R}f$ (fractal-layer mapping)

Definition. For layer index $L \in \mathbb{Z}$,

$$\Psi_r(t, L) = R(\Psi(t), L) := \Psi(t) \odot \Big(1 + \mu\, L\, \eta(\Psi(t))\Big), \tag{1}$$

elementwise $\odot$. In practice we apply R to the base channels and leave appended extras (residuals, outliers, $\bar{\kappa}$) untouched.

Physical reinterpretation. A smooth, invertible deformation that "bends" coordinates along fractal gradients, moving between coarse and fine descriptions without changing identities.

2.7 Outliers $\mathcal{O}$

Definition. Detect sparse maneuver events from robust z-scores of group tangential acceleration and mean curvature; $O(t) \in \{0, 1\}$ (or weighted).

Physical reinterpretation. Captures bursty corrections and sharp turns; used to modulate synergy via adaptive coupling.

3 Time Reparameterization

Define a positive scalar field $\alpha(\Xi)$ in transformed coordinates $\Xi(t) = \Psi_r(t, L^{)}$:

$$d\tau = \alpha(\Xi)\, dt, \qquad \alpha(\Xi) = c_1\, \alpha_{\mathcal{C}}(\Xi) + c_2\, \overline{\kappa}(\Xi), \tag{2}$$

where $\alpha_{\mathcal{C}}$ is a normalized cyclicity speed (e.g. $|\langle \omega_i \rangle_i|$) and $\overline{\kappa}$ is the mean pairwise coupling. This defines a rescaled time τ that equalizes cycle phases and downweights weakly interacting epochs.

Physical reinterpretation. Moves the clock faster in phases with strong coordinated turning and strong coupling; slower otherwise.

4 Discretization and Empirical Path Measure

Partition the manifold of Ξ into K cells $\{x\}$ using PCA→k-means. Let $x_t \in \{1, \ldots, K\}$ be the cell at time index t. Accumulate τ-weighted transition counts

$$C_{xy} = \sum_t \mathbf{1}[x_t = x,\, x_{t+1} = y]\, \Delta\tau_t, \qquad \Delta\tau_t = \int_t^{t+\Delta t} \alpha(\Xi(u))\, du. \tag{3}$$

The raw empirical chain has row-normalized transitions $M_{xy}^{\mathrm{raw}} = \frac{C_{xy}}{\sum_z C_{xz}}$ and stationary weights $\pi_x^{\mathrm{raw}} \propto \sum_z C_{xz}$ (not generally reversible).

5 Equilibriumization via Reversible-Flux Projection

Define the symmetrized flux

$$F^{\mathrm{sym}}_{xy} = \tfrac{1}{2}\,(C_{xy} + C_{yx}), \tag{4}$$

and set

$$\pi^{x\propto\sum_z F^{\mathrm{sym}}_{xz},} \qquad M^{xy=\frac{F^{\mathrm{sym}}_{xy}}{\sum_z F^{\mathrm{sym}}_{xz}}.}(5)$$

Proposition 1 (Detailed balance)**.** *π and M satisfy detailed balance for all x, y:*

$$\pi^{x}M^{xy=\pi^{y}M^{yx}.}$$

Proof. $\pi^{x}M^{xy\propto(\sum_z F^{\mathrm{sym}}_{xz})\cdot\frac{F^{\mathrm{sym}}_{xy}}{\sum_z F^{\mathrm{sym}}_{xz}}=F^{\mathrm{sym}}_{xy}=F^{\mathrm{sym}}_{yx}=\pi^{y}M^{yx}}$. □

Proposition 2 (Max-entropy / KL projection)**.** *Among all Markov chains with detailed balance, $(\pi^{,M)}$ minimizes the empirical path KL divergence to $(\pi^{\mathrm{raw}}, M^{\mathrm{raw}})$ subject to preserving the symmetric pairwise flux constraints F^{sym}_{xy}. Equivalently, it is the maximum-entropy reversible model consistent with F^{sym}.*

6 Gibbs Energy and Temperature

Define an effective energy on states by

$$U(x) = -T_{\mathrm{eff}}\,\log \pi^{x+\mathrm{const.}}(6)$$

Interpolating U from cells to the continuous Ξ yields $U(\Xi)$. The pushforward stationary law is Gibbs: $P(\Xi) \propto e^{-U(\Xi)/T_{\mathrm{eff}}}$.

Physical reinterpretation. U is the potential whose level sets follow occupancy of the equilibriumized chain; T_{eff} fixes an energy scale (we set $T_{\mathrm{eff}} = 1$ without loss).

7 Equilibrium Witnesses

Cycle affinities. For any directed cycle $\Gamma = (x_1 \to x_2 \to \cdots \to x_m \to x_1)$,

$$\mathcal{A}(\Gamma) = \sum_{k=1}^{m} \log \frac{\pi^{x_k}M^{x_k x_{k+1}}}{\pi^{x_{k+1}}M^{x_{k+1}x_k=0}}$$

exactly for the reversible chain; numerically ≈ 0.

Entropy production (EP). With stationary currents $J_{xy} = \pi_x M_{xy}$,

$$\dot{S}_{\mathrm{prod}} = \tfrac{1}{2}\sum_{x,y}(J_{xy} - J_{yx})\log\frac{J_{xy}}{J_{yx}}.$$

EP>0 for the raw chain; EP$=0$ (up to floating point) for $(\pi^{,M)}$.

Crooks/Jarzynski (discrete). Small parametric perturbations of U lead to forward/backward work distributions obeying $P_F(W)/P_R(-W) = e^{W/T_{\mathrm{eff}}}$ for the reversible chain.

Together these certify that in EchoKey coordinates and τ-time the dynamics admit an equilibrium representation.

8 Mapping Back to Leader–Follower

Let $\mathbf{e}_{\mathrm{front}}$ be the group axis. Define a per-bird *front-alignment feature* $\mathrm{align\_front}_i = \hat{\mathbf{v}}_i \cdot \mathbf{e}_{\mathrm{front}}$. Using local linear fits around each state x we estimate ∇U and transport it back to the original feature space. The per-bird *leadership field* is

$$\lambda_i(t) \equiv - \frac{\partial U}{\partial(\mathrm{align\_front}_i)}\Big|_{\Xi(t)}. \tag{7}$$

Time-averaged $\bar{\lambda}_i$ yields a ranking; higher $\bar{\lambda}_i$ indicates a stronger downhill pull of U along the group's front axis.

Physical reinterpretation. "Leadership" is the energy-gradient responsibility for moving the flock forward in the equilibrium landscape, not a lagged cross-correlation definition.

9 End-to-End Algorithm

1. **Ingest & preprocess.** Convert lat/lon→meters; resample uniformly; compute θ_i, ω_i, curvature, group axis.

2. **Build EchoKey state.** Concatenate per-bird channels (cyclicity, density, alignment, etc.), append recursion residual, outlier flag, mean coupling $\bar{\kappa}$.

3. **Refraction.** Apply $\Psi \mapsto \Psi_r(\cdot, L^)$ to base channels; extras left unchanged.

4. **Time reparameterization.** Form $\alpha(\Xi)$ and τ-weights $\Delta\tau_t$.

5. **Discretize.** PCA $\to$ K-means; label states; accumulate C_{xy} with $\Delta\tau$ weights.

6. **Equilibriumization.** Symmetrize flux F^{sym}; compute $(\pi^{,M)}$; set $U = -\log \pi$.

7. **Witnesses.** Compute EP and cycle affinities; require EP≈ 0, affinities ≈ 0.

8. **Leadership.** Estimate ∇U locally; read $-\partial U/\partial(\mathrm{align\_front}_i)$; average in time for ranking.

10 Theoretical Notes and Invariances

Existence (by construction). The reversible-flux projection always yields a detailed-balance chain; hence an equilibrium representation exists for the τ-reweighted symbolic path.

Coordinate choice. Different EchoKey instantiations (filters, kernels) that leave the symmetrized flux nearly unchanged will lead to close $(\pi^{,M)}$ and thus similar U.

Temperature scale. Multiplying U by a constant rescales T_{eff}; rankings and witnesses are invariant.

Limitations. The construction does not claim the raw $(\mathbf{r}, \mathbf{v}, t)$ dynamics are equilibrium; it proves existence of a *coordinate/time representation* that is.

11 Defaults Used in Experiments

Sampling $\Delta t = 0.2$s (5 Hz), Savitzky–Golay smoothing (win=9, poly=3). Refraction: $L^{=1}$, $\mu = 0.12$. Synergy kernel $\beta_0 = 6.0$, $\beta_1 = 0.12\,\mathrm{m}^{-1}$, $\beta_2 = 4.0\,\mathrm{rad}^{-1}$. Time rescale $\alpha(\Xi) = 0.7\,\alpha_{\mathcal{C}} + 0.3\,\overline{\kappa}$, with floor 10^{-3}. Discretization: PCA to 12 dims, $K = 80$ clusters. Temperature $T_{\text{eff}} = 1$.

12 Reporting

We report: (i) witness metrics (EP of raw vs. reversible chain; mean absolute cycle affinity), (ii) energy table per cluster $\{\pi^{x,U(x)}\}$, (iii) per-time leadership scores $\lambda_i(t)$ and mean ranking $\bar{\lambda}_i$, (iv) macrostats (e.g. polarization) for face validity.